

# **Efficient Operation of Natural Gas Transmission Systems: A Network-Based Heuristic for Cyclic Structures**

**Roger Z. Ríos-Mercado<sup>1</sup>**

Graduate Program in Systems Engineering  
Universidad Autónoma de Nuevo León  
AP 111 – F, Cd. Universitaria  
San Nicolás de los Garza, NL 66450, México  
E-mail: *roger@uanl.mx*

**Seongbae Kim**

Institute of Information Technology, Inc.  
2203 Timberloch Place, Suite 100  
The Woodlands, TX 77380, USA  
E-mail: *skim@e-iit.com*

**E. Andrew Boyd**

PROS Revenue Management  
3100 Main Street, Suite 900  
Houston, TX 77002, USA  
E-mail: *aboyd@prosr.com*

July 2003

<sup>1</sup>Corresponding author

## **Abstract**

Gas transmission network problems differ from traditional network flow problems in some fundamental aspects. First, in addition to the flow variables for each arc, which in this case represent mass flow rates, a pressure variable is defined at every node. Second, besides the mass balance constraints, there exist two other types of constraints: (i) a nonlinear equality constraint on each pipe, which represents the relationship between the pressure drop and the flow; and (ii) a nonlinear non-convex set which represents the feasible operating limits for pressure and flow within each compressor station. The objective function is given by a nonlinear function of flow rates and pressures. In the real world, these type of instances are very large both in terms of the number of decision variables and the number of constraints, and very complex due to the presence of non-linearity and non-convexity in both the set of feasible solutions and the objective function.

In this paper we propose a heuristic solution procedure for fuel cost minimization on gas transmission systems with a cyclic network topology, that is, networks with at least one cycle containing two or more compressor station arcs. Our heuristic solution methodology is based on a two-stage iterative procedure. In a particular iteration, at a first stage, gas flow variables are fixed and optimal pressure variables are found via dynamic programming. At a second stage, pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. Empirical evidence supports the effectiveness of the proposed procedure.

*Keywords:* natural gas, pipeline optimization, transmission networks, heuristic, dynamic programming

# 1 Introduction

As natural gas pipeline systems have grown larger and more complex, the importance of optimum operation and planning of these facilities has increased. The investment costs and operation expenses of pipeline networks are so large that even small improvements in system utilization can involve substantial amounts of money.

The natural gas industry services include producing, moving, and selling gas. Our main interest in this study is focused on the transportation of gas through a pipeline network. Moving gas is divided into two classes: transmission and distribution. Transmission of gas means moving a large volume of gas at high pressures over long distances from a gas source to distribution centers. In contrast, gas distribution is the process of routing gas to individual customers. For both transmission and distribution networks, the gas flows through various devices including pipes, regulators, valves, and compressors. In a transmission network, gas pressure is reduced due to friction with the pipe wall as the gas travels through the pipe. Some of this pressure is added back at compressor stations, which raises the pressure of the gas passing through them.

In a gas transmission network, the overall operating cost of the system is highly dependent upon the operating cost of the compressor stations in a network. A compressor station's operating cost, however, is generally measured by the fuel consumed at the compressor station. According to Luongo, Gilmour, and Schroeder [12], the operating cost of running the compressor stations represents between 25% and 50% of the total company's operating budget. Hence, the objective for a transmission network is to minimize the total fuel consumption of the compressor stations while satisfying specified delivery flow rates and minimum pressure requirements at the delivery terminals.

Depending on how the gas flow changes with respect to time, we distinguish between systems in steady state and transient state. A system is said to be in steady state when the values characterizing the flow of gas in the system are independent of time. In this case, the system constraints, particularly the ones describing the gas flow through the pipes, can be described using algebraic non-linear equations. In contrast, transient analysis requires the use of partial differential equations to describe such relationships. This makes the problem considerably harder to solve from the optimization perspective. In fact, optimization of transient models is one of the most challenging areas of opportunity for future research. In the case of transient optimization, variables of the system, such as pressures and flows, are functions of time. In this work, we focus on steady-state gas transmission network problems with the objective of minimizing the operational costs.

Gas transmission network problems differ from traditional network flow problems in some fundamental aspects. First, in addition to the flow variables for each arc, which in this case represent mass flow rates, a pressure variable is defined at every node. Second, besides the mass balance constraints, there exist two other types of constraints: (i) a nonlinear equality constraint on each pipe, which represents the relationship between the pressure drop and the flow; and (ii) a nonlinear non-convex set

which represents the feasible operating limits for pressure and flow within each compressor station. The objective function is given by a nonlinear function of flow rates and pressures.

The problem is very difficult due to the presence of a non-convex objective function and non-convex feasible region. Optimization algorithms (most of them based on dynamic programming) for non-cyclic gas network topologies are in a relatively well developed stage. However, effective algorithms for cyclic topologies are practically non-existent. So our work focuses on addressing gas transmission problems on cyclic topologies. A cyclic topology is a network with at least one cycle containing two or more compressor station arcs.

In this paper we propose a heuristic solution procedure for fuel cost minimization on gas transmission systems with a cyclic network topology. Our heuristic solution methodology is based on a two-stage iterative procedure. In a particular iteration, at a first stage, gas flow variables are fixed and optimal pressure variables are found via dynamic programming. At a second stage, the pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. Empirical evidence supports the effectiveness of this proposed procedure.

The organization of this paper is as follows. In Section 2 we introduce the gas transmission network problem and present the mathematical model. Section 3 presents a survey of previous related work. In Section 4, we present the network formulation of the gas transmission system and discuss the network decomposition. In Section 5, we describe the proposed algorithm. Implementation, numerical examples and computational experiments are reported in Section 6. This is followed by our conclusions and directions for future research in Section 7.

## **2 Problem Definition**

A gas transmission network is composed of pipelines, junction nodes, including supply and delivery nodes, and compressor stations. The existence of compressor stations in the network is one of the key characteristics of a gas transmission network. The transmission segment operates systems of pipes and compressors and attempts to move large quantities of gas over long distances. When traveling through the pipes, the gas pressure is reduced by friction with the pipe walls. Some of this pressure is added back at the compressor stations.

Compressor stations in a transmission network are complex entities typically involving a number of compressor units arranged in parallel or in serial at one location. The operating cost of the gas transmission network is usually ruled by the operating cost of the compressor stations. Therefore, representing compressor stations within a network configuration is quite an important issue and the way of representing them varies according to the solution methodology. We now present the model we use for this problem and discuss the most important assumptions.

## 2.1 Model for Steady-State Problem

Let  $G = (\mathcal{N}, \mathcal{L}, \mathcal{M})$  be a directed network defined by a set  $\mathcal{N}$  of  $n$  nodes, a set  $\mathcal{L}$  of  $l$  pipes, and a set  $\mathcal{M}$  of  $m$  compressor stations. Note that the set of arcs  $\mathcal{A}$  in  $G$  is defined as  $\mathcal{A} = \mathcal{L} \cup \mathcal{M}$ , with  $\mathcal{L} \cap \mathcal{M} = \emptyset$ . The decision variables are  $v_{ij}$ , the mass flow rate at arc  $(i, j) \in \mathcal{A}$  and  $p_i$ , the gas pressure at node  $i \in \mathcal{N}$ . At each node  $i \in \mathcal{N}$ , there is a known parameter  $s_i$  called the net flow through that node. Clearly,  $s_i > 0$  ( $s_i < 0$ ) implies node  $i$  is a source (delivery) node, whereas  $s_i = 0$  means node  $i$  is just a transshipment node. In addition, pressure limits  $p_i^L$  and  $p_i^U$  are given at every node. For each pipe  $(i, j)$  the pipe resistance  $t_{ij}$ , which is computed from the pipe physical properties, is assumed to be known.

Note also that for each arc  $(i, j) \in \mathcal{A}$  there are three associated variables:  $u_{ij}$ ,  $p_i$  and  $p_j$ . In particular, in a compressor station arc,  $p_i$  and  $p_j$  are called suction and discharge pressures, respectively.

The goal of the problem is to minimize the total fuel consumption used by the stations while satisfying specified delivery requirements throughout the system. We assume that there is no transportation cost associated with ordinary pipes. The cost function is only defined at each compressor station, which is given by a function  $g(u_{ij}, p_i, p_j)$  described below.

The mathematical formulation of the gas transmission network problem (GTN) is given by

$$\text{(GTN) minimize} \quad \sum_{(i,j) \in \mathcal{M}} \bar{g}_{ij}(v_{ij}, p_i, p_j) \quad (1a)$$

$$\text{subject to} \quad \sum_{j:(i,j) \in \mathcal{A}} w_{ij} - \sum_{j:(j,i) \in \mathcal{A}} w_{ji} = s_i \quad i \in \mathcal{N} \quad (1b)$$

$$p_i^2 - p_j^2 = t_{ij} u_{ij}^2 \quad (i, j) \in \mathcal{L} \quad (1c)$$

$$p_i \in [p_i^L, p_i^U] \quad i \in \mathcal{N} \quad (1d)$$

$$(v_{ij}, p_i, p_j) \in D_{ij} \subset R^3 \quad (i, j) \in \mathcal{M} \quad (1e)$$

where  $D_{ij}$  is the feasible operating domain of compressor station  $(i, j)$  and  $\bar{g}_{ij}(v_{ij}, p_i, p_j)$  is the fuel consumed at station  $(i, j)$ .

Flow balance conservation is given by Equation (1b). Constraint (1c) represents the dynamics of the pipe flow under isothermal and steady-state assumptions. The compressor station constraints (1e) are not expressed explicitly here. It suffices to know that the feasible set  $D_{ij}$  is a non-convex set. The details can be found in Wu et al. [28].

For measuring fuel consumption, we use a function  $\bar{g}$  which is given by

$$\bar{g}(v, p_s, p_d) = \frac{a' Z R T_s v}{m \eta'} \left\{ \left( \frac{p_d}{p_s} \right)^m - 1 \right\}, \quad (v, p_s, p_d) \in D,$$

where the adiabatic efficiency  $\eta'$  is assumed constant and is fixed at its maximum value. A more detailed study on the nature of both the compressor station domain and the fuel consumption function is given in [28].

The problem is very difficult due to the non-convex nature of both the objective function and the feasible region. Furthermore, the feasible domain  $D_{ij}$  of the compressor station is not represented in algebraic form, but as a result of curve fitting methods based on empirical data for the compressor units used. Also, the type of underlying network topology becomes a crucial issue. That is, depending on the underlying network configuration, the problem can be more difficult to solve. We will discuss this issue in more detail in Section 4.

### 3 Literature Review

Dynamic programming (DP) has been by far the most popular technique for solving many classes of natural gas pipeline networks since the late 1960s. One of the main reasons is that, in a DP framework, it is relatively easy to satisfy the pipeline constraints and to handle the non-convexity of the feasible domain. Other techniques such as mathematical programming and hierarchical control methods have been applied as well with modest degrees of success. Mathematical programming is usually used for cyclic systems. Hierarchical control techniques can be more effective when the model of the compressor station is fairly complicated. In the following sections we present an overview of the most relevant work done on the solution methodologies used for steady state gas transmission networks.

#### 3.1 Mathematical Programming Approaches

As we have seen in Section 2, problem GTN has a non-convex feasible region, and a non-convex objective function. In fact, it has been reported by Wu et al. [28] that this objective function typically has many local optima. These problem features make it very difficult to solve using classical techniques from mathematical programming. Several researchers have tried to apply mathematical programming techniques, but their approaches are based on inaccurate or oversimplified models of the compressor stations. Next we review the most significant works related with mathematical programming, which has been rather limited.

Pratt and Wilson [19] propose a successive mixed integer linear programming method. Their algorithm solves the nonlinear optimization problem iteratively by linearizing the pressure drop-flow equations (1c). Integer variables are included in the formulation for compressor unit selection, and the problem is solved using branch and bound. Percell and Ryan [18] propose an algorithm using a generalized reduced gradient method for minimizing the fuel consumption problem for a gas transmission network. We reemphasize the difficulties in handling the nonlinear equality constraints and the complex nature of the compressor stations in the gas transmission network as the key factors in the relatively poor success of mathematical programming approaches.

### 3.2 Hierarchical Control Approaches

Difficulty in solving a gas transmission network problem in an integrated way calls for other techniques, such as hierarchical structure in the solution process, which in turn demands an efficient method for decomposing of the problem. In a hierarchical control approach (Singh [22]), the overall network is decomposed into two levels: (i) the network state level and (ii) the compressor station level. The compressor station problem is the lowest level and the network state problem is the highest level. The local optimization of the compressor stations at the lowest level is the firm basis for an optimization procedure for the global minimization of total cost. Optimization of the compressor station subproblem has been studied previously by Osiadacz [13], Percell and Van Reet [17], and Wu, Boyd, and Scott [27].

Larson and Wismer [11] propose a hierarchical control approach for a transient operation of a gun-barrel pipeline system. Osiadacz and Bell [15] suggest a simplified algorithm for the optimization of the transient gas transmission network, which is based on a hierarchical control approach. The hierarchical control approach for transient models can be found in Anglard and David [2], Osiadacz [14], and Osiadacz and Swierczewski [16]. Some degree of success has been reported from these approaches as far as optimizing the compressor station subproblem. However, these approaches have limitations in globally optimizing the minimum cost. As mentioned in Carter, Schroeder, and Harbick [6], numerical simulation of the behavior of the gas transmission network is quite widespread, and with a successful compressor station optimizer, these simulations are quite accurate. However, little work has been done or even attempted in optimizing the gas transmission network. After the introduction of the hierarchical control approach, in which detailed compressor station optimization is attached as a lower level subproblem, several approaches have focused on the optimization of the higher level problem, the network state problem.

### 3.3 Dynamic Programming Approaches

Dynamic programming (DP) (Dreyfus and Law [8]) was first used for a steady state gas transmission system in Wong and Larson [24, 25]. The authors apply DP to the gunbarrel and diverging branch tree systems to solve the network state problem. The gunbarrel system, which is basically a single-path topology, possesses an appropriate serial structure so it can be solved via DP. For the diverging branch problem, the overall problem is decomposed into a sequence of several one-dimensional DP problems, each of which deals with a single branch.

There are, however, some limitations to the DP method described by Wong and Larson. First, its application is limited only to tree networks. Second, the method assumes that there is only a single compressor unit installed within each compressor station. In addition, the feasible domain of the compressor station is oversimplified in order to make an easier solution process. A successful commercial optimizer for tree-structured gas transmission networks using DP was developed by Zimmer [29], and Lall and Percell [10].

When applying DP, the underlying network configuration of the given problem can enhance the solution process. Note that for these non-cyclic systems, the DP formulation is one-dimensional and can be easily applied because it has been shown [21] that the flow variables can be uniquely determined beforehand and thus eliminated from the problem, so we only deal with the pressure variables. That is, in a tree network, once both supply and delivery flow rates are given, the steady state flows can be uniquely determined. This nice property does not hold for a cyclic system. The existence of cycles breaks the serial structure, so the flow variables must be explicitly handled, which makes the underlying DP multidimensional.

The DP approach for cyclic systems has been limited. There were some efforts in applying DP on the nonsequential structure for applications in chemical engineering problems. Two of the earliest works include Wilde [23] and Aris, Nemhauser, and Wilde [3]. Both groups treated cycles via “cuts”, i.e., they cut one end of a cycle in the system and treat the resulting system using DP for tree structures. More general issues of the nonsequential DP can be found in Bertele and Brioschi [4].

Although several researchers over the years have addressed branched systems in the gas transmission problem, cyclic systems were not addressed until Luongo, Gilmour, and Schroeder [12]. In their study, the authors apply DP with the assumption that the flow rates through pipes and at the compressor stations are fixed. After solving the DP problem with prefixed flow rates, they use a direct search method with multiple restarts on different flow settings. Thus their approach is a hybrid of DP and either brute-force enumeration or simulated annealing, depending on problem size. Recently Carter [5] proposed a DP approach on more general structures with flow rates being fixed. A more detailed description of DP approaches to gas transmission networks can be found in Ríos-Mercado [20].

The principal obstacle to date using DP is that its application is limited in practice to simple structures. Otherwise, the computational effort becomes too large to be practical. This leads to an interesting question of how to find the optimal setting of the flow variables and how to modify the current flow setting to obtain a better objective value. This study focuses on these issues.

## 4 Natural Gas Network Decomposition

In our solution procedure, we first decompose the original graph into several subgraphs (not containing compressor stations), and we further contract each subgraph, yielding a reduced network. It turns out that the network configuration for each subgraph is not crucial, while the network configuration of the reduced network is critical. In this section we will see how the underlying network topology becomes a key issue, and plays a very important role in developing solution procedures for this problem.

## 4.1 Modeling Assumptions

In this section we consider the general network configuration of the gas transmission network and discuss several assumptions about the underlying network configuration. Gas pipeline networks are very complex entities. To make our mathematical model simple enough, several assumptions are made through our discussion.

**Assumption 1** *A gas transmission network is composed of only nodes, pipes, and compressor stations.*

Under Assumption 1, the only control variables over the network are the ones in the compressor stations  $((v, p_s, p_d))$ . A compressor station may be in one of three states: active, bypassed, or closed. In our network model these states are not explicitly represented in the formulation, but are decided in the context of the optimization problem. If the discharge pressure value is greater than the suction pressure value at the compressor station, then the compressor station is said to be active. If the discharge pressure value equals to the suction pressure value, then the compressor station is bypassed. When the flow rate through the compressor station is zero, then the compressor station is closed. For both bypassed and closed compressor stations, the operating costs are zero.

**Assumption 2** *There are no self-loops in the system.*

**Assumption 3** *Bypass over any compressor station is not allowed.*

Any directed sequence of pipes connecting node  $i$  to node  $j$ , for which there is an alternate path containing at least one compressor station, is considered a bypass. In Figure 1, the dotted arcs  $(3, 5)$  and  $(4, 10)$  bypass compressor station  $(3, 5)$  and  $(8, 9)$ , respectively, so they are not allowed in our model.

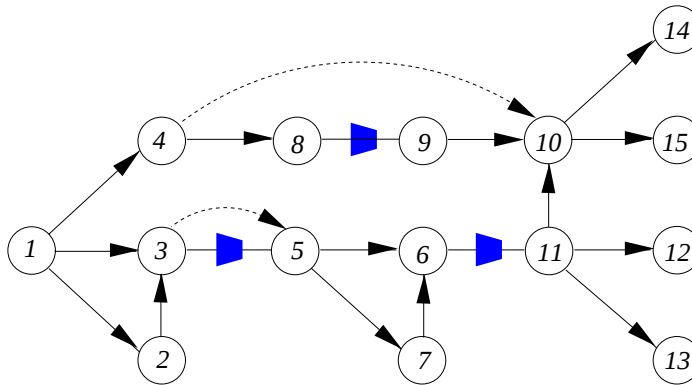


Figure 1: Example of bypasses in a gas transmission network.

Under assumptions 2 and 3, the gas transmission network problem can be viewed as a directed network whose components are nodes, pipeline arcs, and compressor station arcs.

**Assumption 4** At each node  $i \in \mathcal{N}$  of  $G$ , the net inflow  $s_i$  is assumed to be known with certainty. Moreover, at each delivery node, minimum requirement of the pressure is assumed to be known.

**Assumption 5** A gas transmission network can have multiple source nodes from which gas enters the system. We assume, however, that there is only one source node (super source), which regulates the input pressure value of the whole system.

By assumption 5, we have only one reference pressure with which our solution procedure works. The purpose and the effect of these assumptions will be explained in detail in a later section. There are other assumptions on the configuration of the gas transmission network, and they will be discussed in the relevant sections.

## 4.2 Decomposition

Now consider the gas transmission network  $G = (\mathcal{N}, \mathcal{L}, \mathcal{M})$ , with  $n$  nodes,  $l$  pipes, and  $m$  compressor stations, and its mathematical model GTN defined in (1). Let  $\mathbf{A}_{\mathcal{L}}$  be the  $n \times l$  node-pipe incidence matrix,  $\mathbf{A}_{\mathcal{M}}$  be the  $n \times m$  node-compressor station incidence matrix, and  $\mathbf{A} = (\mathbf{A}_{\mathcal{L}} \mid \mathbf{A}_{\mathcal{M}})$ . The equality constraint set of the problem GTN can be represented in a vector form by

$$\begin{cases} \mathbf{A} \mathbf{w} = \mathbf{s}, \\ \mathbf{A}_{\mathcal{L}}^T \mathbf{p}^2 = \phi(\mathbf{u}), \end{cases} \quad (2)$$

where  $\mathbf{p}^2 = (p_1^2, \dots, p_n^2)^T$  and  $\phi(\mathbf{u})$  is the vector of  $\phi_{ij}(u_{ij})$ 's,  $(i, j) \in \mathcal{L}$ , in which  $\phi_{ij}(u_{ij}) = t_{ij}u_{ij}^2$ .

Since we assume that there is no bypass, if we delete every compressor station from the entire system, then we have disconnected subgraphs, each one of which being composed only of pipes and nodes. Figure 2 shows an example of the subgraphs created by deleting compressor stations.

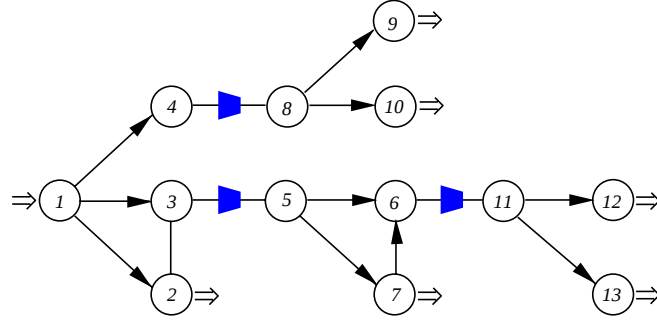
Each node or pipe of the  $G$  belongs to exactly one subgraph. It follows that the subgraphs of  $G$  determine a unique partition of its nodes and pipes. Assume that there are  $r$  subgraphs. Let  $G_i = (\mathcal{N}_i, \mathcal{L}_i)$ ,  $i = 1, \dots, r$ , be the subgraph defined by a set of nodes,  $\mathcal{N}_i$ , and a set of pipes,  $\mathcal{L}_i$ . We assume  $G_i$  has  $n_i$  nodes and  $l_i$  pipes. Note that the nodes in  $G$  can be renumbered in such a way that its node-pipe incidence matrix takes the block diagonal form

$$\mathbf{A}_{\mathcal{L}} = \begin{pmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_r \end{pmatrix}, \quad (3)$$

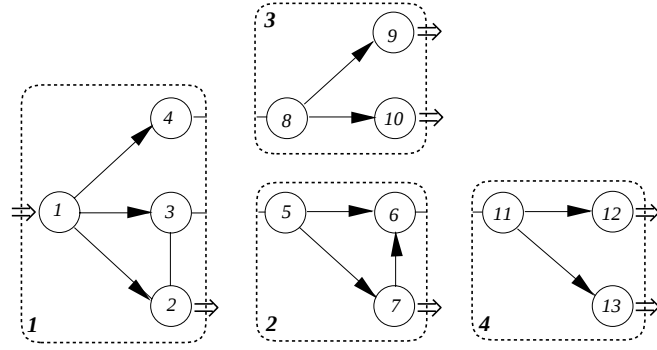
where  $\mathbf{A}_i$  is the node-pipe incidence matrix for  $G_i$ ,  $i = 1, \dots, r$ .

Let  $\mathbf{u}_i$  be the vector of mass flow rates through the pipes of  $G_i$ , and  $\mathbf{p}_i$  be the pressure vector for each node of  $G_i$ , i.e.,

$$\mathbf{u}_i = \{u_{jk} \mid (j, k) \in \mathcal{L}_i\}, \quad \mathbf{p}_i = \{p_j \mid j \in \mathcal{N}_i\}.$$



(a) Network  $G$



(b) Subgraphs  $G_1, G_2, G_3$  and  $G_4$

Figure 2: Example of the subgraphs after removal of compressor arcs.

Let  $\mathbf{b}_i$  be the vector of net inflow at nodes located in subgraph  $G_i$ , that is,  $\mathbf{b}_i = \{s_j \mid j \in \mathcal{N}_i\}$ . Since  $\mathbf{A} = (\mathbf{A}_{\mathcal{L}} \mid \mathbf{A}_{\mathcal{M}})$  and  $\mathbf{w} = (\mathbf{u}, \mathbf{v})^T$ , flow balance equations of system (2) become

$$\mathbf{A}_{\mathcal{L}} \mathbf{u} + \mathbf{A}_{\mathcal{M}} \mathbf{v} = \mathbf{s}. \quad (4)$$

Then, by using (3), equation (4) can be rewritten as

$$\begin{pmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_r \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_r \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{\mathcal{M}_1} \\ \mathbf{A}_{\mathcal{M}_2} \\ \vdots \\ \mathbf{A}_{\mathcal{M}_r} \end{pmatrix} \mathbf{v} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_r \end{pmatrix},$$

where  $\mathbf{A}_{\mathcal{M}_i}$  is the rearranged node-station incidence matrix corresponding to  $G_i, i = 1, \dots, r$ . That is, we decompose the set of flow balance equation constraints to each subgraph  $G_i$  by

$$\mathbf{A}_i \mathbf{u}_i + \mathbf{A}_{\mathcal{M}_i} \mathbf{v} = \mathbf{b}_i, \quad i = 1, \dots, r.$$

Since the sets  $\mathcal{L}_i$  of pipes in  $G_i, i = 1, \dots, r$  are disjoint, the nonlinear pressure drop constraint set

in (2) can be naturally decomposed as

$$\mathbf{A}_i^T \mathbf{p}_i^2 = \phi(\mathbf{u}_i), \quad i = 1, \dots, r, \quad (5)$$

where  $\mathbf{p}_i^2$  is the vector of  $p_j^2$ 's,  $j \in \mathcal{N}_i$ , and  $\phi(\mathbf{u}_i)$  is the vector of  $\phi_{jk}(u_{jk})$ 's,  $(j, k) \in \mathcal{L}_i$ , in which  $\phi_{jk}(u_{jk}) = t_{jk} u_{jk}^2$ . Therefore system (2) can be decomposed for each subgraph as

$$\begin{cases} \mathbf{A}_i \mathbf{u}_i + \mathbf{A}_{\mathcal{M}_i} \mathbf{v} = \mathbf{b}_i, \\ \mathbf{A}_i^T \mathbf{p}_i^2 = \phi(\mathbf{u}_i), \end{cases} \quad i = 1, \dots, r. \quad (6)$$

Since  $\mathbf{A}_i$  is the node-pipe incidence matrix for the subgraph  $G_i$ , we have (by Proposition 3 in Kim [9])

$$\mathbf{A}_i \mathbf{C}_{T_i}^T = \mathbf{C}_{T_i} \mathbf{A}_i^T = \mathbf{0}, \quad i = 1, \dots, r,$$

where  $\mathbf{C}_{T_i}$  is the cycle matrix of  $G_i$  with respect to the spanning tree  $T_i$  of  $G_i$ . If we multiply  $\mathbf{C}_{T_i}$  on both sides of (5), we have

$$\mathbf{C}_{T_i} \phi(\mathbf{u}_i) = \mathbf{0}, \quad i = 1, \dots, r.$$

Hence for each  $G_i, i = 1, \dots, r$ , we have the following system

$$\begin{cases} \mathbf{A}_i \mathbf{u}_i + \mathbf{A}_{\mathcal{M}_i} \mathbf{v} = \mathbf{b}_i, \\ \mathbf{C}_{T_i} \phi(\mathbf{u}_i) = \mathbf{0}, \\ \mathbf{A}_i^T \mathbf{p}_i^2 = \phi(\mathbf{u}_i). \end{cases} \quad (7)$$

The advantage of system (7) over (6) is that if the mass flow rates through the compressor stations  $\mathbf{v}$  are known, then the flow variables,  $\mathbf{u}_i$ , can be solved separately from the  $n_i$  pressure variables. That is, the first two equations in (7) can be used for solving  $\mathbf{u}_i$  if  $\mathbf{v}$  is fixed. The system of equations for solving  $\mathbf{u}_i$  for the  $i$ -th subnetwork  $G_i$  becomes

$$\begin{cases} \mathbf{A}_i \mathbf{u}_i = \mathbf{b}'_i, \\ \mathbf{C}_{T_i} \phi(\mathbf{u}_i) = \mathbf{0}, \end{cases} \quad (8)$$

where  $\mathbf{b}'_i = \mathbf{b}_i - \mathbf{A}_{\mathcal{M}_i} \mathbf{v}$ . System (8) is not of full rank, since  $\text{rank}(\mathbf{A}_i) = n_i - 1$ . Let  $\bar{\mathbf{A}}_i$  be any  $(n_i - 1) \times m_i$  submatrix of  $\mathbf{A}_i$ . The matrix  $\bar{\mathbf{A}}_i$  could be any submatrix created by deleting an arbitrary row from  $\mathbf{A}_i$ . Using this full row rank matrix  $\bar{\mathbf{A}}_i$ , we have the following square system

$$\begin{cases} \bar{\mathbf{A}}_i \mathbf{u}_i = \mathbf{b}'_i, \\ \mathbf{C}_{T_i} \phi(\mathbf{u}_i) = \mathbf{0}. \end{cases} \quad (9)$$

Since  $\bar{\mathbf{A}}_i$  and  $\mathbf{C}_{T_i}$  have  $n_i - 1$  and  $m_i - n_i + 1$  independent rows, respectively, and there are  $m_i$  unknown flow variables, the above system is a square system and yields a unique solution. Formal

proof of the uniqueness and existence of the solution of the above system (9) can be found in Ríos-Mercado et al. [21]. If the configuration of  $G_i$  is a tree, then no cycle matrix is defined, and we have only  $\bar{\mathbf{A}}_i \mathbf{u}_i = \mathbf{b}'_i$ , which is trivial to solve. If  $G_i$  has cycles, a method such as the modified Newton method can be used to solve the nonlinear system (9).

Decomposing the given network into subgraphs gives us insight into the structure and motivation for the development of a solution procedure. That is, once we fix the flow rates  $\mathbf{v}$  at the compressor stations, the rest of the flow variables are calculated at each subgraph by solving system (9). Then we are left with the pressure variables  $\mathbf{p}$  as unknown variables, and with pressure drop equations  $\mathbf{A}_i^T \mathbf{p}_i^2 = \phi(\hat{\mathbf{u}}_i)$ , where  $\hat{\mathbf{u}}_i$  is the known flow vector for  $G_i$ . We need to address two issues: (i) how to solve the rest of the problem which contains only pressure variables, and (ii) how to modify the flow rates through the compressor stations. The first issue will be handled by dynamic programming. To answer the second question, we further consider the network configuration of the entire system, which is the purpose of the following two subsections.

### 4.3 The Associated Reduced Network

As we have seen in the previous section, once the flow variables  $\mathbf{v}$  through the compressor stations are fixed, then at each subgraph  $G_i, i = 1, \dots, r$ , regardless of its configuration, the flow variables through the pipes are uniquely determined. Hence, we focus on analyzing how the compressors participate in the entire network structure, rather than analyzing the individual subgraphs.

For this purpose, we introduce the concept of a *reduced network*. Let  $G' = (\mathcal{V}, \mathcal{E})$  be the reduced network of  $G = (\mathcal{N}, \mathcal{L}, \mathcal{M})$ , where  $\mathcal{V}, \mathcal{E}$  are the set of nodes and the set of arcs of  $G'$ . Each node of  $G'$  corresponds to exactly one subgraph of  $G$ . It follows that the subgraphs of a graph determine a unique partition of its nodes. Suppose we contract all those arcs, i.e., pipes, which lies in each subgraph. Then the resulting contraction network has the appearance of the network in Figure 3 which is the reduced network  $G'$  of the example network  $G$  in Figure 2(a).

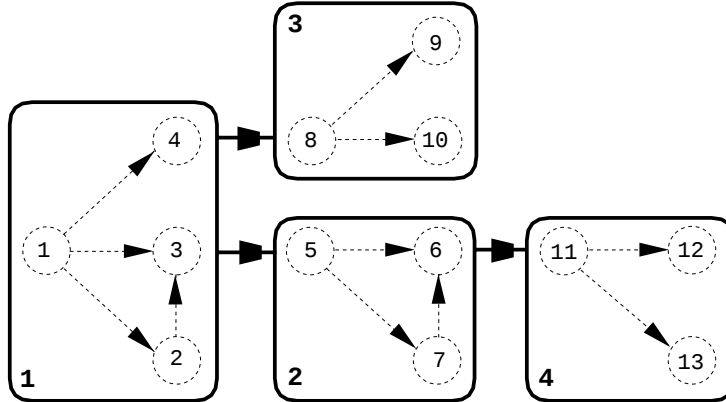


Figure 3: Example of the reduced network  $G'$ .

Note that each node of the reduced network  $G'$  is identified with a subgraph  $G_i$  of  $G$ . The set of arcs  $\mathcal{E}$  of  $G'$  is the same as  $\mathcal{M}$ , the set of compressor stations of  $G$ . That is, an arc in  $G'$  corresponds to a compressor station in  $G$ . Note that the reduced network  $G'$  is a connected directed network with no self-loops. Hence, it could have various configurations, which we divide into three classes: a gunbarrel system, a tree structured system, and a cyclic system. Throughout the following chapters, when we mention the network configuration of a gas transmission network, we mean the configuration of the reduced network  $G'$ .

In the reduced network  $G'$ , since we contract a whole subgraph into a node, the configuration of each subgraph does not affect the configuration of a reduced network. For example, in Figure 3, even though the original network example has cycles, its reduced network is acyclic, and thus is considered a tree structure. Similarly, in Figure 4 below, the reduced networks of Figure 4(a) and 4(b) are the same, and they both are gunbarrel systems, even though the network in Figure 4 (b) has a cycle in subgraphs  $G_1$ ,  $G_2$ , and  $G_3$ .

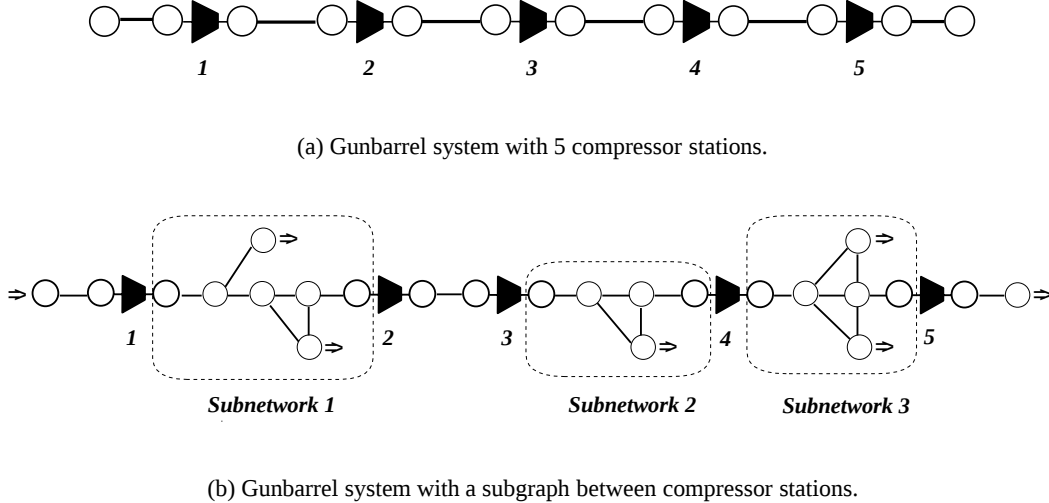


Figure 4: Example of a gunbarrel system.

If the reduced network  $G'$  is a gunbarrel or a tree system (both of them non-cyclic systems), then the flow rates through the arcs in  $\mathcal{E}$  of  $G'$ , which correspond to the flow rates of each compressor station in  $G$ , are uniquely determined. If the reduced network  $G'$  has a loop in it, then optimal  $\mathbf{v}$  values have to be determined in the context of the underlying minimization problem. In our solution procedure, even though  $G'$  has cycles, we will start with a feasible value for  $\mathbf{v}$ , and once  $\mathbf{v}$  is fixed, we have  $r$  independent nonlinear square systems (9), each of which corresponds to a subnetwork  $G_i$ ,  $i = 1, \dots, r$  of  $G$ . By solving system (9) for all subgraphs  $G_i$ ,  $i = 1, \dots, r$ , we have a full profile of flow variables for the whole network with respect to the fixed flows of the compressor stations. The remaining problem is now to determine the pressure variables. In the following section, we consider this problem and discuss

a solution procedure.

#### 4.4 Handling the Subproblem

In the previous section, we see that once the flow values of the compressor stations are fixed, we can determine all other flow variables by solving system (9) for each subgraph independently. This in effect eliminates the flow variables from the whole formulation, leaving only pressure variables as unknowns. That is, for each subnetwork  $G_i, i = 1, \dots, r$ , we have

$$\mathbf{A}_i^T \mathbf{p}_i^2 = \phi(\mathbf{u}_i), \quad (10)$$

where  $\mathbf{u}_i$  is the vector of known pipe flows  $\dot{u}_{jk}, (j, k) \in \mathcal{L}_i$ . Using the original notation, we have the following optimization problem with only pressures involved as unknown variables:

$$\text{minimize} \quad \sum_{(i,j) \in \mathcal{M}} \bar{g}_{ij}(\dot{v}_{ij}, p_i, p_j) \quad (11a)$$

$$\text{subject to} \quad p_j^2 - p_k^2 = t_{jk} \dot{u}_{jk}^2 \quad (j, k) \in \mathcal{L} \quad (11b)$$

$$p_j \in [p_j^L, p_j^U] \quad j \in \mathcal{N} \quad (11c)$$

$$(\dot{v}_{ij}, p_i, p_j) \in D_{(i,j)} \subset R^3 \quad (i, j) \in \mathcal{M} \quad (11d)$$

where  $\dot{v}_{ij}, \dot{u}_{jk}$  are fixed flow variables through compressor station  $(i, j) \in \mathcal{M}$  and through pipe  $(j, k) \in \mathcal{L}$ , respectively.

Consider equation (10) above. Since each  $G_i$  is connected, if any pressure variable, say  $p_j, j \in \mathcal{N}_i$ , is known, then all other pressure variables defined at the other nodes in  $G_i$  are just calculated using (10). Moreover, between any two nodes  $j, k$  in  $G_i$ , if there exists a direct path  $P_{<j,k>}$  from node  $j$  to node  $k$ , then  $p_j$  ( $p_k$ ) can be written as a function of  $p_k$  ( $p_j$ ). Using these two observations, we can further reduce the problem at each subgraph. That is, we include the nonlinear relationships between the reference nodes, such as suction, discharge, source and delivery nodes of  $G_i$ , and eliminate other nonlinear equations from the problem model. Hence we simplify the problem so that the reduced problem only contains the pressure variables of the reference nodes of each subgraph  $G_i, i = 1, \dots, r$ .

At each subgraph  $G_i$ , gas flow enters  $G_i$  through either discharge nodes or supply nodes, i.e., gas sources. Similarly, some gas flow comes out of  $G_i$  through either suction nodes or delivery nodes. Let  $\mathcal{N}_i^{dc}, \mathcal{N}_i^{sc}, \mathcal{N}_i^{su}$ , and  $\mathcal{N}_i^{dv}$  be the set of discharge, suction, source, and delivery nodes in  $G_i$ , respectively, and let  $\mathcal{N}_i^{in} = \mathcal{N}_i^{dc} \cup \mathcal{N}_i^{su}$  and  $\mathcal{N}_i^{out} = \mathcal{N}_i^{sc} \cup \mathcal{N}_i^{dv}$  be the set of *input nodes* of  $G_i$  and the set of *output nodes* of  $G_i$ , respectively.

Let  $h$  and  $j$  be a discharge and a suction node of  $G_i$ , respectively, i.e.,  $h \in \mathcal{N}_i^{dc}, j \in \mathcal{N}_i^{sc}$ . Let  $P_{<h,j>}$  be a direct path from  $h$  to  $j$ . Note that nodes  $h, j$  are from two different compressor stations connected with subgraph  $G_i$ . A path,  $P_{<h,j>}$ , is represented by a sequence of pipes, i.e.,  $\{(h, i_0), (i_0, i_1), \dots, (i_k, j)\}$ , where  $i_l \in \mathcal{N}_i, l = 0, \dots, k$ . For each pipe in the path  $P_{<h,j>}$ , a pressure

drop equation (11b) is defined. Hence, we have

$$\begin{cases} p_h^2 - p_{i_0}^2 = t_{h i_0} \dot{u}_{h i_0}^2, \\ p_{i_0}^2 - p_{i_1}^2 = t_{i_0 i_1} \dot{u}_{i_0 i_1}^2, \\ \vdots \\ p_{i_k}^2 - p_j^2 = t_{i_k j} \dot{u}_{i_k j}^2. \end{cases} \quad (12)$$

Adding the above equations yields

$$p_h^2 - p_j^2 = \sum_{(l,m) \in P_{<h,j>}} t_{lm} \dot{u}_{lm}^2. \quad (13)$$

Thus equation (13) replaces equations (12) in the model. Likewise, between any two nodes  $j$  and  $k$ , where  $j \in \mathcal{N}_i^{in}$  and  $k \in \mathcal{N}_i^{out}$ , if there exist a path  $P_{<j,k>}$ , we can obtain a nonlinear relationship such as equation (13), and include it in the reduced formulation.

Based on the same logic, we can also convert the pressure bounds of node  $k \in \mathcal{N}_i$  into the bounds of the other pressure variable  $p_j$ ,  $j \in \mathcal{N}_i^{dc}, \mathcal{N}_i^{sc}$ . For example, consider the minimum pressure requirement constraint  $p_k \geq p_k^L$ ,  $k \in \mathcal{N}_i^{dv}$  for the delivery node  $k$  in  $G_i$ , and let  $P_{<j,k>}$  be a path from the discharge node  $j \in \mathcal{N}_i^{dc}$  to node  $k \in \mathcal{N}_i^{dv}$ . We can eliminate  $p_k$  from the problem formulation, and include a new pressure bound for  $p_j$ . That is, we have

$$p_j \geq \sqrt{(p_k^L)^2 + \sum_{(lm) \in P_{<j,k>}} t_{lm} \dot{u}_{lm}^2}.$$

In general, the bounding inequality  $p_k^L \leq p_k \leq p_k^U$  of any node  $k \in \mathcal{N}_i$  of  $G_i$  is replaced by the pressure bounds of the variable  $p_j$ ,  $j \in \mathcal{N}_i^{sc}$  or  $j \in \mathcal{N}_i^{dc}$  with

$$\alpha_{j,k} = \sqrt{(p_k^L)^2 + \sum_{(lm) \in P_{<j,k>}} t_{lm} \dot{u}_{lm}^2} \leq p_j \leq \sqrt{(p_k^U)^2 + \sum_{(lm) \in P_{<j,k>}} t_{lm} \dot{u}_{lm}^2} = \beta_{j,k}, \quad j \in \mathcal{N}_i^{dc}, \quad (14)$$

$$\alpha_{k,j} = \sqrt{(p_k^L)^2 - \sum_{(lm) \in P_{<k,j>}} t_{lm} \dot{u}_{lm}^2} \leq p_j \leq \sqrt{(p_k^U)^2 - \sum_{(lm) \in P_{<k,j>}} t_{lm} \dot{u}_{lm}^2} = \beta_{j,k} \quad j \in \mathcal{N}_i^{sc}. \quad (15)$$

The above procedure is motivated by the following. First, the reduced problem contains only pressure variables at node  $j \in \mathcal{N}_i^{in} \cup \mathcal{N}_i^{out}$ ,  $i = 1, \dots, r$ , and hence the problem size is reduced. Second, the reduced problem can be converted into the sequential decision structure, which enables us to efficiently apply dynamic programming. If we let  $\mathcal{B}_i = \{(j, k) : j \in \mathcal{N}_i^{in}, k \in \mathcal{N}_i^{out}\}$ , then the reduced problem, denoted as  $Q_{\hat{v}}(\mathbf{p})$ , can be formulated as follows.

$$\text{Minimize} \quad \sum_{(i,j) \in \mathcal{M}} \bar{g}_{ij}(\dot{v}_{ij}, p_i, p_j) \quad (16a)$$

$$\text{subject to} \quad p_j^2 - p_k^2 = \sum_{(l,m) \in P_{<j,k>}} t_{lm} u_{lm}^2 \quad (j, k) \in \mathcal{B}_i, i = 1, \dots, r \quad (16b)$$

$$\max\{\underline{p}_j, \alpha_{j,k}\} \leq p_j \leq \min\{\bar{p}_j, \beta_{j,k}\} \quad j \in \mathcal{N}_i^{dc}, i = 1, \dots, r \quad (16c)$$

$$\max\{\underline{p}_j, \alpha_{k,j}\} \leq p_j \leq \min\{\bar{p}_j, \beta_{k,j}\} \quad j \in \mathcal{N}_i^{sc}, i = 1, \dots, r \quad (16d)$$

$$(\dot{v}_{ij}, p_i, p_j) \in D_{(i,j)} \subset R^3 \quad (i, j) \in \mathcal{M} \quad (16e)$$

where  $\alpha_{j,k}$ ,  $\alpha_{k,j}$ ,  $\beta_{j,k}$  and  $\beta_{k,j}$  are defined in equations (14)-(15).

The degree of the sequential decision structure of this problem varies depending on the nature of the subgraph. That is, if every  $G_i, i = 1, \dots, r$  has a single input node  $j \in \mathcal{N}_i^{in}$  and a single output node  $k \in \mathcal{N}_i^{out}$ , then the reduced problem of  $G$  has an appropriate sequential decision structure. Any gunbarrel system has such structure for which DP is well defined. However, if the subgraph has multiple suction nodes or has multiple discharge nodes, then the reduced problem  $Q_b(\mathbf{p})$  has a nonsequential structure representing diverging or converging branches, respectively.

There are some limitations in applying DP for some complex configurations of the subgraph, and we assume the following.

**Assumption 6** *Each subgraph can have either multiple discharge nodes with one suction node or have multiple suction nodes with one discharge node, but cannot have multiple discharge nodes and multiple suction nodes at the same time.*

The above restriction is made for purpose of modeling via DP. Under the above assumption, we have the following cases related to the configuration of  $G_i, i = 1, \dots, r$ . That is,

**Case 1**  $G_i$  has a single node  $j \in \mathcal{N}_i^{in}$  and a single node  $k \in \mathcal{N}_i^{out}$ .

**Case 2**  $G_i$  has a single node  $j \in \mathcal{N}_i^{in}$  and multiple nodes  $k \in \mathcal{N}_i^{out}$ .

**Case 3**  $G_i$  has multiple nodes  $j \in \mathcal{N}_i^{in}$  and a single node  $k \in \mathcal{N}_i^{out}$ .

In a gunbarrel network, every subgraph  $G_i$  has the configuration stated in Case 1, and hence the sequential decision structure is maintained throughout the network. However, in the tree and the cyclic structures, we have a combination of  $G_i$ 's having the various configurations from case 1 to case 3.

## 5 Description of Heuristic

Our solution procedure is based on a two-stage iterative procedure. In a particular iteration, at the first stage, gas flow variables are fixed and optimal pressure variables are found via DP. At the second

stage, the pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. Figure 5 shows an overview of the proposed procedure.

### 5.1 Dynamic Programming for the Reduced Problem

In the previous section, we reduced the problem into a form which had only pressure variables defined at the input or output nodes of  $G_i$ . Depending on the configuration of  $G_i$ , the problem has a sequential decision structure or a nonsequential structure. DP can be applied directly to the sequential decision structure, such as a gunbarrel transmission system. For the nonsequential structure, nonsequential DP is applied.

So given DP is a fairly well studied technique and has been described widely our focus here is in elaborating in how we modify the flows. The details of the DP implementation can be found in the work of Carter [5] and in Chapter 5 of Kim's dissertation [9]. The main point here is that we do know how to solve effectively for pressures when flows are known or fixed.

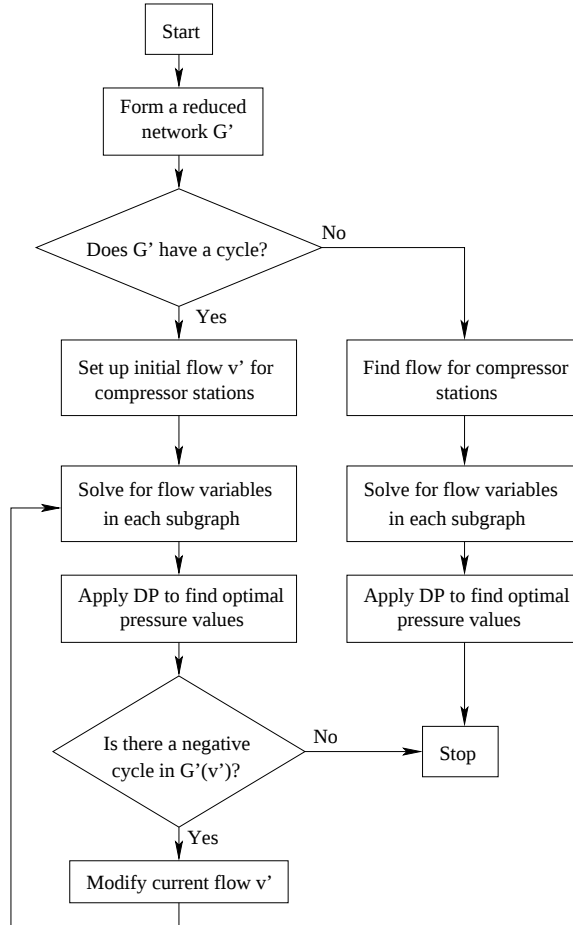


Figure 5: Overview of the solution procedure.

## 5.2 The Flow Modification Heuristic

Let  $\mathbf{x}^0 = (\mathbf{v}^0, \mathbf{p}^0)$  be an initial feasible solution to (1). As we have seen in Section 4, once the flow rates through the compressor station  $\mathbf{v}^0$  are given, the other flow variables defined at each pipe  $\mathbf{u}$  are calculated for each subgraph. Now with given flow variables  $\mathbf{v}^0$  and  $\mathbf{u}^0$ , the problem can be simplified into the reduced problem  $Q_{\mathbf{v}^0}(\mathbf{p})$ . Let  $\mathbf{p}^0$  be the solution of the reduced problem  $Q_{\mathbf{v}^0}(\mathbf{p})$ .

For a tree structured gas transmission network, since the flow variables  $\mathbf{v}^0$  are uniquely determined, no flow modification step is needed. However, for a cyclic structure, one may attempt to obtain a better objective function value by modifying the current flow setting  $\mathbf{v}^0$ . For this purpose, we make use of the residual network and negative cycle concepts ([1]), which are used in network theory for finding augmenting paths or proving optimality. Let  $G(\mathbf{v}^0)$  represent the residual network corresponding to the flow  $\mathbf{v}^0$  of the original network  $G$ .

The following two fundamental theorems, which we state here without proof (can be found in [1]), relate residual networks to negative cycles.

**Theorem 1** (*Augmenting Cycle Theorem*): Let  $\mathbf{v}$  and  $\mathbf{v}^0$  be any two feasible solutions of a network flow problem. Then  $\mathbf{v}$  equals  $\mathbf{v}^0$  plus the flow on a cycle in  $G'(\mathbf{v}^0)$ . Furthermore, the cost of  $\mathbf{v}$  equals the cost of  $\mathbf{v}^0$  plus the cost of flow on the augmenting cycle.

**Theorem 2** (*Negative Cycle Optimality Condition*): A feasible solution  $\mathbf{v}^0$  of the minimum cost flow problem is an optimal solution if and only if the residual network  $G(\mathbf{v}^0)$  contains no negative cost directed cycles.

Both theorems assume that the incremental cost of the flow is constant. From Theorem 1, we know that augmenting some amount of flow through a cycle does not violate the flow conservation. Thus the new flow is also feasible with respect to network flow constraints. Moreover, by the second part of Theorem 1, if we have a negative cycle, then augmenting some amount of flow through the negative cycle will yield a reduced total cost. By Theorem 2, if there exists a negative cycle with regard to the current feasible flow, then the current flow value is not optimal.

Using these two theorems, we want to develop a scheme to modify the current flow rates  $\mathbf{v}^0$  to the new flow rates  $\mathbf{v}'$  which will yield a better objective value. To do this, we first create the residual network  $G'(\mathbf{v}^0)$  of the reduced network  $G'$  with respect to the current flow variable vector  $\mathbf{v}^0$ . Figure 6 shows an example of a reduced network and its residual network.

Note that a self-cycle which includes both a forward arc  $(i, j)$  and a backward arc  $(j, i)$  of the compressor station  $(i, j)$ , is not considered a cycle. That is, either the forward or the backward arc of the compressor station  $(i, j)$  should be included in the cycle, but not both. For example, in Figure 6, self-cycles  $i - k - i$  and  $i - k - j - k - i$ , are not considered as cycles. Note also that the starting node of the cycle is not important. In Figure 6, there exist two cycles, the (clockwise) cycle  $i - k - j - i$ ,

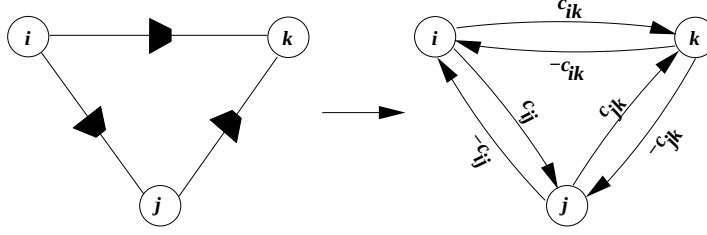


Figure 6: A reduced network and its residual network.

and the (counterclockwise) cycle  $i - j - k - i$ . We denote the cycle in the residual network by  $C$ , and denote the set of the compressor stations in a cycle  $C$  by  $\mathcal{M}_C$ .

In our heuristic flow modification step, the costs of the residual network are approximated by the derivatives of the objective function  $\bar{g}$  with respect to the flow on each compressor station. Then,

$$c_{ij} \approx a_{ij} \left\{ \left( \frac{p_j}{p_i} \right)^m - 1 \right\},$$

where  $p_i$  and  $p_j$  are the current solution values delivered by DP with fixed flow variables, and  $a_{ij} = \frac{a' Z R T_s}{m \eta'}$  as defined in Chapter 2. This cost  $c_{ij}$  is assigned to each forward arc of the residual network and  $-c_{ij}$  is assigned to each backward arc. The cycle cost  $\tau_C$ , the total cost of the cycle  $C$  in a residual network, is defined by

$$\tau_C = \sum_{(i,j) \in \mathcal{M}_C} \delta_{ij}(C) \cdot c_{ij},$$

where  $\delta_{ij}(C)$  equals 1 if arc  $(i, j)$  is contained in cycle  $C$  and is a forward arc of  $G(\mathbf{v}^0)$ ,  $-1$  if arc  $(j, i) \in C$  and is a backward arc of  $G(\mathbf{v}^0)$ , and 0 otherwise. If  $\tau_C$  is negative, then we call  $C$  a negative (cost) cycle and denote it as  $C^-$ .

Modification of the flow is done by augmenting flow through a negative cycle  $C^-$ . That is, if there exists a negative cost cycle  $C^-$ , then we augment positive flow through  $C^-$ , and hence update the current flow setting. This flow modification step can be represented by

$$\mathbf{v}^{new} = \mathbf{v}^0 + \lambda \cdot \delta(C^-), \quad (18)$$

where  $\lambda > 0$  is the positive amount of flow which will be added through the cycle, and  $\delta(C^-)$  is the vector of  $\delta_{ij}(C^-)$ 's, a vector representing the negative cycle  $C^-$ . The flow modification step can be viewed as a descent nonlinear programming algorithm in which we try to find a direction (a vector of flow modification) such that by moving  $\lambda$  units in this direction, the objective function decreases. In our heuristic procedure, a negative cycle vector  $\delta(C^-)$  corresponds to the search direction. The value  $\lambda$  is bounded below by zero and above by  $\lambda_{\max}$ , which can be obtained by considering the complex inequality constraint set  $D_{ij}$ ,  $(i, j) \in C^-$ . If  $\lambda_{\max} = 0$ , then the algorithm stops. Otherwise, we set  $\lambda = \mu \lambda_{\max} > 0$ , where  $0 < \mu < 1$ . The purpose of multiplying by a small parameter  $\mu$  is explained in the following subsection.

For the newly obtained flow setting  $\mathbf{v}^{new}$ , we need to find pressure variables, which requires us make use of DP again with the updated flow  $\mathbf{v}^{new}$ . If DP with  $\mathbf{v}^{new}$  has no feasible solution, or no improvement to the objective value has been achieved, we reduce the size of  $\lambda$  by setting  $\lambda = \gamma\lambda$ , where  $0 < \gamma < 1$ , and apply DP until we get a better result. The algorithm is summarized below.

Step 1: Find an initial feasible solution  $\mathbf{x}^0 = (\mathbf{v}^0, \mathbf{p}^0)$ , set  $t$ (iteration counter) = 0,  $\mu(0 < \mu < 1)$ , and  $\gamma(0 < \gamma < 1)$ .

Step 2: Construct the residual network  $G^r(\mathbf{v}^t)$ , and find a negative cycle  $C^-$  with negative cost  $\tau_{C^-}$ .

Step 3: If  $|\tau_{C^-}| < \varepsilon$ , where  $\varepsilon$  is a small number, stop. Otherwise, go to Step 4.

Step 4: Set  $\lambda = \mu\lambda_{\max}$ . If  $\lambda = 0$ , stop. Otherwise,

(a) Modify the current flow  $\mathbf{v}^t$  by  $\mathbf{v}^{new} = \mathbf{v}^t + \lambda \cdot \delta(C^-)$ .

(b) Calculate pressure values using DP with modified flow  $\mathbf{v}^{new}$ .

If DP yields an infeasible solution, or  $\bar{g}^{new} - \bar{g}^t > 0$ , then set  $\lambda = \gamma\lambda$ , with  $0 < \gamma < 1$ , and go to 4(a). Otherwise,  $\mathbf{v}^{t+1} = \mathbf{v}^{new}$ ,  $t = t + 1$  and go to Step 2.

To prevent cycling, Step 4(b) is executed a maximum of L times, where L is a user-specified parameter.

### 5.3 Choice of Step Size

To determine the value of the step size  $\lambda$  defined in 18, we need to consider the feasible domain  $D_{ij}^r$  for each compressor station  $(i, j) \in \mathcal{M}_C$  within a loop. As mentioned in the previous chapter, the inequality constraints set  $D_{ij}^r$  is not given in algebraic form, but is defined as a result of the curve fitting methods based on empirical data for compressor units. Consider for instance a compressor station  $(i, j) \in \mathcal{M}_C$  with four parallel compressor units. Figure 7 shows the profiles of the feasible domain with a fixed suction pressure value.

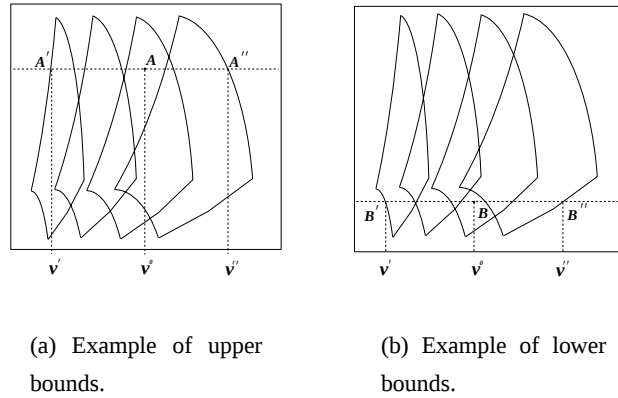


Figure 7: Profiles of the feasible region with suction pressure  $p_s$  fixed.

Assume that the triple values  $(v_{ij}^0, p_i^0, p_j^0)$  of the current solution  $\mathbf{x}^0$  are located at points  $A$  or  $B$  in Figure 7. For either cases, with the assumption that  $p_i^0$  and  $p_j^0$  are fixed at the current feasible point,  $v$  can vary from  $v'$  to  $v''$ . The values  $v'$  and  $v''$  can be obtained in a straightforward manner from the equations defining the compressor stations. The technical details can be found in [9].

For each compressor station in the loop, i.e.,  $(i, j) \in \mathcal{M}_C$ , we can get  $v_{ij}'$  and  $v_{ij}''$ , and the step size  $\lambda$  should be decided within the bounds

$$0 \leq \lambda \leq \lambda_{\max} = \min \begin{cases} v_{ij}'' - v_{ij}^0, & \delta_{ij}(C^-) = 1, \quad (i, j) \in \mathcal{M}_C, \\ v_{ij}^0 - v_{ij}', & \delta_{ij}(C^-) = -1, \quad (i, j) \in \mathcal{M}_C, \end{cases}$$

where  $\lambda_{\max}$  is the upper bound of the step size  $\lambda$ , and  $v_{ij}^0$  is the current flow setting at the compressor station  $(i, j) \in \mathcal{C}$ .

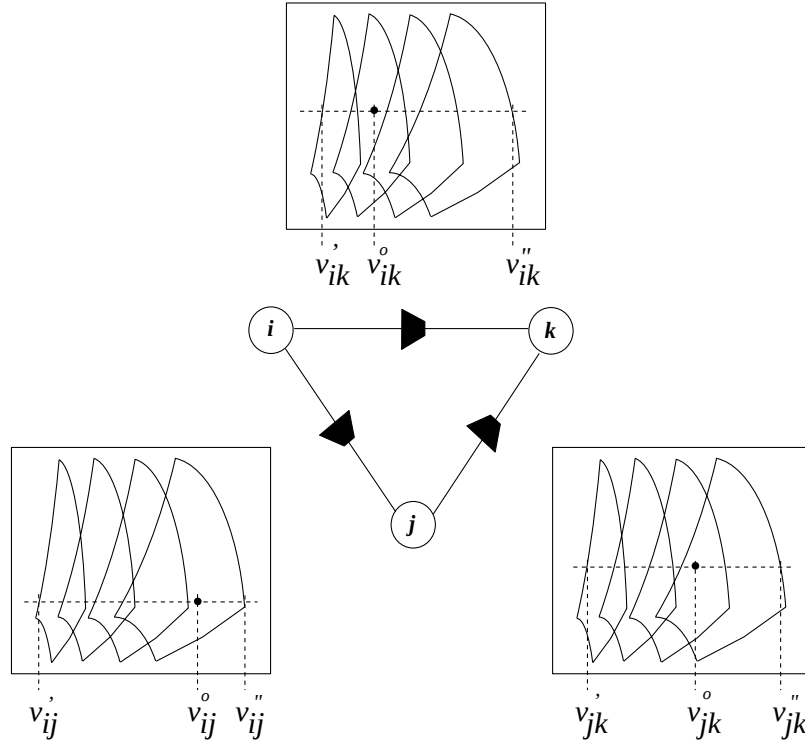


Figure 8: Choice of step size  $\lambda$ .

For instance, consider the example shown in Figure 8. Here, we assume that there is a negative cycle, whose direction is clockwise. Then the maximum step size  $\lambda_{\max}$  is determined by

$$\lambda_{\max} = \min\{v_{ik}'' - v_{ik}^0, v_{jk}^0 - v_{jk}', v_{ij}^0 - v_{ij}'\}.$$

Similarly, for the counter-clockwise negative cycle,  $\lambda_{\max}$  will be

$$\lambda_{\max} = \min\{v_{ik}^0 - v_{ik}', v_{jk}'' - v_{jk}^0, v_{ij}'' - v_{ij}^0\}.$$

Now, the step size is set to  $\lambda = \mu \lambda_{\max}$ , where  $0 < \mu < 1$ . This choice attempts to keeping feasibility and stems from the fact that taking  $\lambda = \lambda_{\max}$  causes the new solution to tend to fall outside the feasible region as the corresponding pressure values may have changed accordingly. Large values of  $\mu$  leads to larger improvements of the objective function, but increases the possibility of infeasibility. In contrast, small values of  $\mu$  may keep the updated flow within the bounds, but the value of improvement in the objective may be rather small.

## 6 Empirical Evaluation

### 6.1 Numerical Examples

In this section we provide an extensive computational evaluation of the proposed solution procedure. For that purpose we have generated problem instances based on network topology examples we have discussed previously, but using real world data for compressor stations. These data was gathered from SSI (Scientific Software Intercomp, Inc.), a Houston-based vendor specializing in gas operations. For the network topologies, we have built structures similar to the ones found in industry. The values of all data and parameters used in this experimental part can be found in [9].

*Example 1: A Tree Structured System.* Consider the following instance (depicted in Figure 9) with 64 nodes, 56 pipes, and 16 compressor stations.

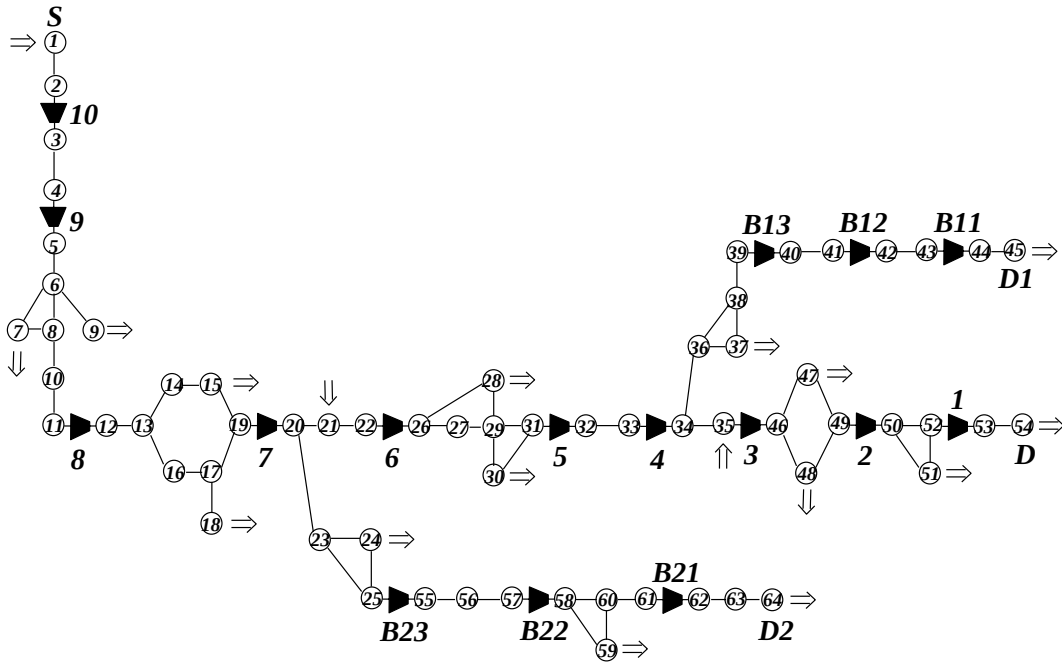


Figure 9: Network of Example 1.

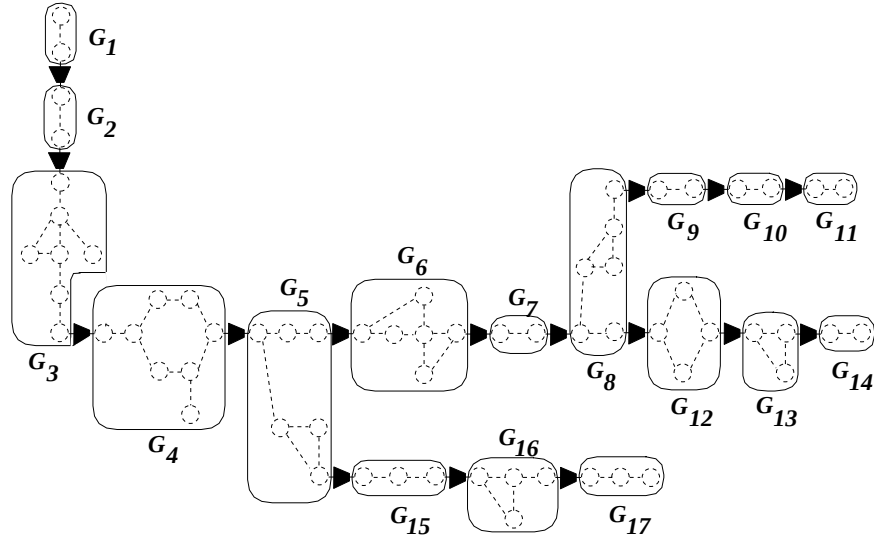


Figure 10: Reduced network  $G'$  of Example 1.

Table 1: DP solution.

| Compressor Station | Suction pressure | Discharge pressure | $v^0$ | Objective   |
|--------------------|------------------|--------------------|-------|-------------|
| 1                  | 508.16470        | 508.16470          | 50.0  | 0.00000     |
| 2                  | 509.72955        | 509.72955          | 90.0  | 79.86951    |
| 3                  | 514.96478        | 514.96478          | 180.0 | 262.60706   |
| 4                  | 523.63025        | 523.63025          | 200.0 | 1070.30823  |
| 5                  | 553.16473        | 553.16473          | 200.0 | 1139.41699  |
| 6                  | 565.62067        | 565.76611          | 250.0 | 1225.92798  |
| 7                  | 503.36340        | 585.91382          | 150.0 | 13622.09473 |
| 8                  | 540.74146        | 540.74146          | 220.0 | 13625.22949 |
| 9                  | 599.35669        | 599.35669          | 300.0 | 13694.63379 |
| 10                 | 629.71375        | 663.49652          | 300.0 | 21685.33984 |
| B11                | 504.73297        | 505.73398          | 70.0  | 269.78931   |
| B12                | 510.74597        | 510.74597          | 70.0  | 336.68485   |
| B13                | 515.37427        | 515.37427          | 70.0  | 446.73724   |
| B21                | 573.41547        | 573.41547          | 70.0  | 0.00000     |
| B22                | 574.86243        | 574.86243          | 70.0  | 0.00000     |
| B23                | 580.44147        | 580.44147          | 40.0  | 0.00000     |

The associated reduced network  $G'$  (shown in Figure 10) is a tree with 17 nodes, each of which corresponds to a subgraph, and 16 arcs, each of which corresponding to a compressor station. Using the decomposition technique explained in Section 4, at each subgraph we can calculate the flow variables beforehand by solving system (9).

We now apply our nonserial implementation of DP to solve the problem and obtain the profiles of pressure values at the compressor stations. Table 1 shows the results when the system input pressure at the super source node 1 is 700 psia and using 10 grid points at each stage of the DP.

Using different number of grid points yields different quality of solutions. Theoretically, we can obtain a finer solution by increasing the number of grid points used, but at a cost of higher computational time. For this example, we have explored using a different number of grid points in our dynamic programming solution process. Table 2 summarizes the quality of the solutions using four different numbers of grid points. We can observe that the computational time doubles when the number of grid points doubles, and the solution converges as the number of grid points increases.

Table 2: Results of DP for different grid sizes.

| Number of<br>grid points | Objective value<br>(Total fuel cost) | CPU time<br>(second) |
|--------------------------|--------------------------------------|----------------------|
| 5                        | 22916.9961                           | 2.730                |
| 10                       | 21685.3398                           | 5.030                |
| 20                       | 20508.3281                           | 10.010               |
| 40                       | 20404.4238                           | 21.830               |

In this example, since the reduced network is a tree structure, flow rates at the compressor station are uniquely determined, and thus the flow modification step is not required. This example shows how to handle insignificant loops in subgraphs and demonstrates the network decomposition scheme of the solution methodology. In the next two examples, we consider cyclic structures.

*Example 2: A Single-Cycle System.* The second example is a simple cycle network with 6 compressor stations and 9 pipeline arcs (see Figure 11). This example can be considered as one of the simplest forms of the loop structure. The associated reduced network is shown in Figure 12(a).

As discussed before, by initializing the compressor flow rates the pipeline flow rates can be easily determined (see Table 3). Table 4 shows the solution when DP is applied to solve for the pressure values. We have assumed that the system input of pressure  $p_S$  at the super source node  $S$  is 700 psia.

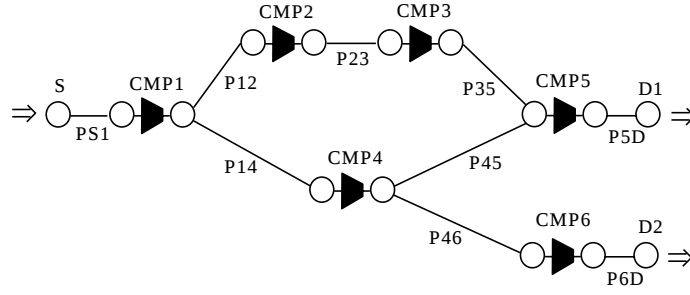


Figure 11: Network of Example 2.

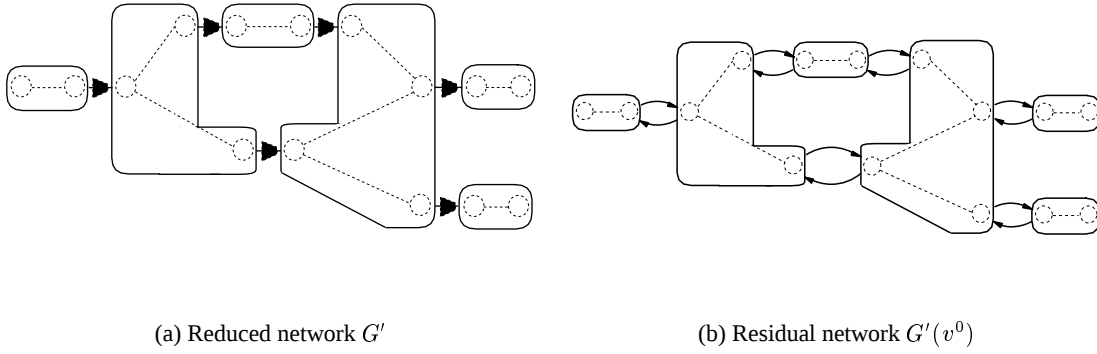


Figure 12: Reduced and corresponding residual network for Example 2.

Table 3: Flow rates on each pipe.

| Pipe | $u_{ij}$ | Pipe | $u_{ij}$ |
|------|----------|------|----------|
| PS1  | 350.0    | P45  | 150.0    |
| P12  | 100.0    | P46  | 100.0    |
| P14  | 250.0    | P5D  | 250.0    |
| P23  | 100.0    | P6D  | 100.0    |
| P35  | 100.0    |      |          |

Table 4: DP iteration 1.

| Station    | Suction pressure | Discharge pressure | $v_{ij}^0$ | Fuel cost  |
|------------|------------------|--------------------|------------|------------|
| CMP1       | 566.0268         | 689.3578           | 350.0      | 34728.6914 |
| CMP2       | 674.1254         | 674.1254           | 100.0      | 0.0000     |
| CMP3       | 655.9074         | 655.9074           | 100.0      | 0.0000     |
| CMP4       | 561.3552         | 631.0216           | 250.0      | 15418.7940 |
| CMP5       | 586.2333         | 626.7272           | 250.0      | 8275.7842  |
| CMP6       | 608.6840         | 608.6840           | 100.0      | 0.0000     |
| Total Cost |                  |                    |            | 58423.2695 |

Table 5: DP iteration 2.

| Station    | Suction pressure | Discharge pressure | $v_{ij}^0$ | Fuel cost  |
|------------|------------------|--------------------|------------|------------|
| 1          | 566.0268         | 676.2758           | 350.0000   | 31668.7832 |
| 2          | 655.5665         | 655.5665           | 115.2395   | 0.0000     |
| 3          | 630.5466         | 630.5466           | 115.2395   | 0.0000     |
| 4          | 562.2967         | 621.6626           | 234.7605   | 12376.5967 |
| 5          | 585.2015         | 626.7272           | 250.0000   | 8480.5098  |
| 6          | 598.9761         | 598.9761           | 100.0000   | 0.0000     |
| Total Cost |                  |                    |            | 52525.8906 |

Table 6: DP iteration 3.

| Station    | Suction pressure | Discharge pressure | $v_{ij}^0$ | Fuel cost  |
|------------|------------------|--------------------|------------|------------|
| 1          | 566.0268         | 700.0000           | 350.0000   | 37288.7109 |
| 2          | 674.7831         | 674.7831           | 129.1978   | 0.0000     |
| 3          | 644.1168         | 644.1168           | 129.1978   | 0.0000     |
| 4          | 604.2491         | 611.1767           | 220.8022   | 4362.0278  |
| 5          | 581.5298         | 626.7272           | 250.0000   | 9227.1436  |
| 6          | 588.0858         | 588.0858           | 100.0000   | 0.0000     |
| Total Cost |                  |                    |            | 50877.8828 |

Based on the solution given in Table 4, we can construct the residual network on  $G$ . Figure 12 show the reduced network  $G'$  of the problem and its residual network  $G'(v^0)$  with respect to the current flow setting  $\mathbf{v}^0$ . A (clockwise) negative cycle with negative cost  $\tau_C = -61.67518$  is found in Figure 12(b), and the step size  $\lambda$  was calculated as 15.2395. Now we augment flow  $\lambda$  through the negative cycle, with parameters  $\mu$  and  $\gamma$  set to 0.9 and 0.5, respectively. At iteration 4, the algorithm stops because a negative cycle is not found, i.e.,  $\tau_C = 0.0$  for both clockwise and counter-clockwise cycle  $C$ . Tables 5 through 7 show the solution of each iteration.

Table 7: Iteration 4.

| Station    | Suction pressure | Discharge pressure | $v_{ij}^0$ | Fuel cost  |
|------------|------------------|--------------------|------------|------------|
| 1          | 566.0268         | 700.0000           | 350.0000   | 37288.7109 |
| 2          | 669.7152         | 669.7152           | 141.3252   | 0.0000     |
| 3          | 632.5530         | 632.5530           | 141.3252   | 0.0000     |
| 4          | 615.1898         | 615.1898           | 208.6749   | 0.0000     |
| 5          | 583.6664         | 626.7272           | 250.0000   | 8789.2715  |
| 6          | 592.2554         | 592.2554           | 100.0000   | 0.0000     |
| Total Cost |                  |                    |            | 46077.9844 |

*Example 3: A Multiple-Cycle System.* Now let us consider the example shown in Figure 13. This instance contains multiple cycles and branches. Moreover, some of the cycles are dependent on each other.

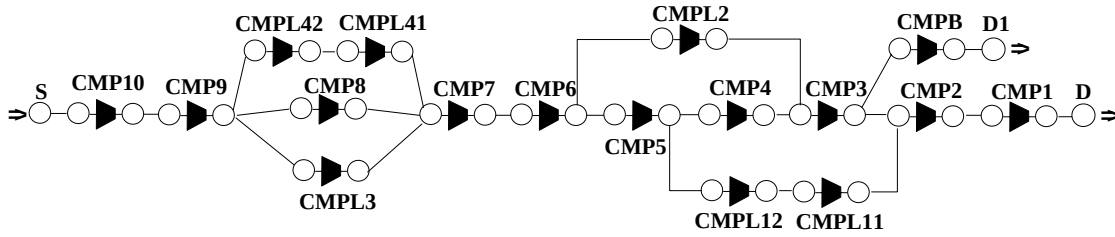


Figure 13: A complex system with multiple loops and branches.

Table 8 shows the DP solution of the given problem with different initial flow settings. We assumed that the input pressure value at the super source node  $S$  is 600 psia. The solution shown in Table 8 is checked for feasibility. Based on the current solution, we found 4 negative cycles  $C_1^-$ ,  $C_2^-$ ,  $C_3^-$ , and  $C_4^-$ , each of which corresponds to a counter-clockwise cycle with  $\mathcal{M}_{C_1^-} = \{CMP4, CMP3, CMPL22, CMPL21\}$ , a clockwise cycle with  $\mathcal{M}_{C_2^-} = \{CMPL2, CMP4, CMP5\}$ , a counter-clockwise cycle with  $\mathcal{M}_{C_3^-} = \{CMP8, CMPL3\}$ , and a counter-clockwise cycle with  $\mathcal{M}_{C_4^-} = \{CMPL41, CMPL42, CMP8\}$ , respec-

Table 8: DP solution of Example 3 at iteration 1.

| Station    | Suction pressure | Discharge pressure | $v_{ij}^0$ | Fuel cost   |
|------------|------------------|--------------------|------------|-------------|
| CMP1       | 592.14349        | 592.14349          | 250.000000 | 0.00000     |
| CMP2       | 598.64215        | 598.64215          | 250.000000 | 0.00000     |
| CMP3       | 616.95190        | 616.95190          | 300.000000 | 0.00000     |
| CMP4       | 599.94464        | 617.86743          | 100.000000 | 3587.70825  |
| CMP5       | 600.79651        | 600.79651          | 200.000000 | 0.00000     |
| CMP6       | 576.27094        | 607.03632          | 400.000000 | 11016.26074 |
| CMP7       | 599.70898        | 599.70898          | 400.000000 | 0.00000     |
| CMP8       | 558.29919        | 603.43390          | 150.000000 | 6217.34277  |
| CMP9       | 561.43622        | 561.43622          | 400.000000 | 0.00000     |
| CMP10      | 581.77887        | 581.77887          | 400.000000 | 0.00000     |
| CMPL11     | 599.27130        | 599.27130          | 100.000000 | 0.00000     |
| CMPL12     | 600.12408        | 600.12408          | 100.000000 | 0.00000     |
| CMPL2      | 602.76453        | 621.47272          | 200.000000 | 7178.70117  |
| CMPL3      | 560.14026        | 601.18829          | 100.000000 | 4086.03784  |
| CMPL41     | 579.52564        | 601.62470          | 150.000000 | 6597.18750  |
| CMPL42     | 559.81580        | 581.82019          | 150.000000 | 3329.72974  |
| CMPB       | 614.98547        | 614.98547          | 150.000000 | 0.00000     |
| Total Cost |                  |                    |            | 42012.96875 |

tively. The negative cycle costs are  $\tau_{C_1^-} = -15.37$ ,  $\tau_{C_2^-} = -0.03$ ,  $\tau_{C_3^-} = -0.59$ , and  $\tau_{C_4^-} = -0.88$ . The most negative cycle is  $C_1^-$ , and the corresponding step size  $\lambda$  is calculated to be 4.3029. Next we augment flow by 4.3029 units through negative cycle  $C_1^-$ . Table 9 presents the solutions in the successive iterations for the given problem.

Table 9: Successive solutions for example 3.

| Iteration | Total Fuel Cost | Most Negative cycle cost | Step size |
|-----------|-----------------|--------------------------|-----------|
| 1         | 42012.9698      | -15.3703                 | 4.3029    |
| 2         | 41110.9727      | -14.1674                 | 4.0354    |
| 3         | 40816.2773      | -13.9453                 | 3.7908    |
| 4         | 38750.9844      | -4.0386                  | 0.3330    |

After 4 iterations, we stop with the solution given in Table 10, because the step size is considered too small to continue. The percentile improvement of fuel cost in this example is almost 8%, and the computational time is 42.75 seconds, when 5 grid points are used at each stage of the DP.

Table 10: Final DP solution.

| Station    | Suction pressure | Discharge pressure | $v_{ij}^0$ | Fuel cost    |
|------------|------------------|--------------------|------------|--------------|
| CMP1       | 565.336609       | 565.336609         | 250.000000 | 0.000000     |
| CMP2       | 572.139832       | 572.139832         | 250.000000 | 0.000000     |
| CMP3       | 614.253479       | 614.253479         | 287.870941 | 0.000000     |
| CMP4       | 568.800598       | 614.963623         | 87.870964  | 8402.583984  |
| CMP5       | 569.494385       | 569.494385         | 200.000000 | 0.000000     |
| CMP6       | 576.073364       | 576.073364         | 400.000000 | 0.000000     |
| CMP7       | 599.519104       | 599.519104         | 400.000000 | 0.000000     |
| CMP8       | 558.299194       | 603.245239         | 150.000000 | 6191.366699  |
| CMP9       | 561.436218       | 561.436218         | 400.000000 | 0.000000     |
| CMP10      | 581.778870       | 581.778870         | 400.000000 | 0.000000     |
| CMPL11     | 567.470337       | 572.967346         | 112.129036 | 1494.207642  |
| CMPL12     | 568.602295       | 568.602295         | 112.129036 | 0.000000     |
| CMPL2      | 571.570251       | 618.794006         | 200.000000 | 8723.612305  |
| CMPL3      | 560.140259       | 600.998962         | 100.000000 | 4068.000732  |
| CMPL41     | 579.525635       | 601.435425         | 150.000000 | 6541.483887  |
| CMPL42     | 559.815796       | 581.820190         | 150.000000 | 3329.729736  |
| CMPB       | 612.278381       | 612.278381         | 150.000000 | 0.000000     |
| Total Cost |                  |                    |            | 38750.984375 |

## 6.2 Benchmark Results

Because of the lack of test problems in gas pipeline literature, another goal of this work is to provide some benchmark test results which may be used for benchmarking other methods.

The algorithm, as described previously, consists of about 15,000 lines of C code. Numerical experiments on 12 instances based on three different cyclic topologies were run on a SGI Power Challenge L running IRIX 6.2. Even though our solution methodology can handle gunbarrel and tree structures, our computational experiments are done for the cyclic structure, which is the main focus of our study. The three topologies used for our computational experiments are (A) a single cycle instance with six compressor stations (Figure 11), (B) a multi-cyclic structure with 3 cycles, 3 branches, and 21 compressor stations (Figure 14), and (C) a multi-cyclic structure with 4 cycles, 1 branch, and 17 compressor stations (Figure 13).

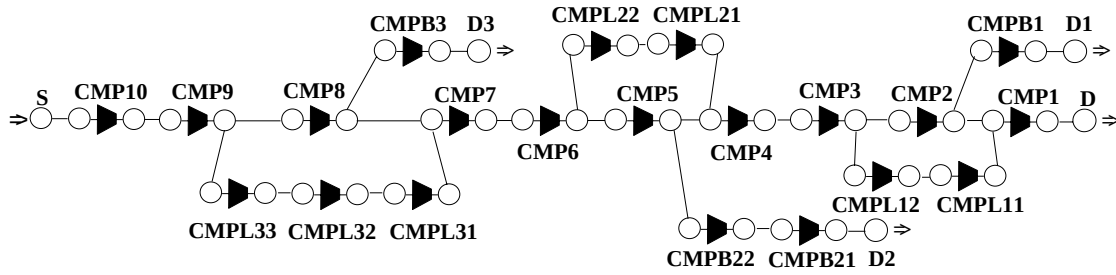


Figure 14: An example with multiple loops and multiple branches.

We have applied our solution methodology with different initial flow settings to each of these three network topologies. Table 11 shows the results of the experiments for these problems. An 0.0% implies the algorithm terminates at the very first iteration. The cost improvement obtained by applying our solution methodology ranges from 0.00% to 41.77%. According to [26], even a 1% savings on gas transshipment cost may be worth in the order of 48.6 million dollars.

Table 11: Solution benchmarks.

| Problem instance | Initial flow settings | CPU time (seconds) | % Improvement from Initial Solution |
|------------------|-----------------------|--------------------|-------------------------------------|
| A                | flow setting 1        | 2.64               | 24.88                               |
|                  | flow setting 2        | 4.57               | 21.13                               |
|                  | flow setting 3        | 20.24              | 41.77                               |
| B                | flow setting 1        | 6.07               | 0.00                                |
|                  | flow setting 2        | 6.20               | 0.00                                |
|                  | flow setting 3        | 23.84              | 17.32                               |
| C                | flow setting 1        | 41.17              | 4.62                                |
|                  | flow setting 2        | 42.75              | 3.34                                |
|                  | flow setting 3        | 74.46              | 8.20                                |

Next, we discuss finding an initial feasible solution. Like most optimization algorithms, our algorithm starts from a feasible point. Getting a feasible solution for the complex network topology is quite a difficult task. Using our solution methodology, we can obtain constructive information that can be used to find a feasible solution of the given problem. The current flow setting of the network satisfies the flow balance equations. We don't know yet whether this setting is feasible or not. The maximum capacity of the flow through the compressor station depends on the number of compressor units connected in parallel. Our implementation can be used for detecting infeasibility, that is, if the current flow setting yields an infeasible solution. The algorithm also gives us information on how much the current flow can

be augmented or reduced to meet the capacity of the specific compressor station.

It is noted that for instance A, in the neighborhood of a possible local optimal point, the solution of the proposed algorithm seems to converge. Table 12 shows the results using different starting points. Here,  $u_{23}^0$  and  $u_{23}^*$  represent the initial starting point and the final solution of the flow rate at pipe (2, 3), respectively.

Table 12: Solution from different starting points for instance A.

| Flow setting | Final Fuel Cost | $u_{23}^0$ | $u_{23}^*$ |
|--------------|-----------------|------------|------------|
| 1            | 47054.2813      | 50.0       | 132.3961   |
| 2            | 47014.1445      | 100.0      | 132.5264   |
| 3            | 47007.2152      | 120.0      | 132.9225   |

As a final remark, we have found that the two algorithmic parameters  $\mu$  and  $\gamma$  play a role in the heuristic’s performance. As explained in Section 5, the greater the value of  $\mu$ , the higher probability the iteration will move out of the feasible region. On the other hand, a larger value of  $\mu$  may yield a faster convergence. Parameter  $\gamma$  is needed if the new solution is not feasible, or if we have not obtained an better solution. During our preliminary experiments, we found that  $\mu$  ranging between 0.8 and 0.85, and  $\gamma$  around 0.5 yields the best results.

## 7 Closing Remarks

In this study we have addressed the problem of minimizing the fuel consumption of compressor stations in a steady-state gas transmission network. We have modeled this problem as a non-linear non-convex network flow problem, and derived a heuristic solution methodology. We have classified the network topologies as non-cyclic and cyclic structures and have highlighted how this type is related to the solution techniques’ success. In particular, a cyclic topology is considerable harder to solve because flow rate variables must be handled explicitly, thus making traditional DP approaches not suitable.

This motivated what constitutes the main scientific contriobution of this work, which is the derivation and implementation of a network-based heuristic that aims at providing good-quality solutions for cyclic topologies. To the best of our knowledge, there is no previous work that addresses handling both flow and pressure variables simultaneously. The solution procedure is an iterative process. First flow variables are fixed by flow modification, then DP is used to find an associated set of pressure variables. The flow modification step exploits the underlying network properties.

Our computational experimentation showed the effectiveness of the proposed approach, even in non-cyclic structures. The average cost reduction obtained from our solution methodology ranges from

3.34% to 41.77% (in the instances tested), which amounts to a great savings to the gas industry.

During our experiments, the distribution of the running time among the various types of operations in the algorithm was studied. It showed that most of time (about 95%) is spent on solving DP. These results highlight the importance of having an efficient procedure for solving DP. Our dynamic programming implementation can be improved by using time efficient interpolation techniques and perhaps parallel programming.

In this study, we consider minimizing fuel consumption at compressor stations for the steady-state model of gas transmission networks. In the operating perspective, there could be other objective functions of interest such as finding the maximal throughput. Currently the feasible domain of the compressor station is not represented algebraically. If we could represent the domain algebraically, it may be possible to develop a mathematical model in which a global optimum can be found. Finally models that consider transient gas networks are a new research area and an important future research direction.

Another line of work is to consider a decision variable that would tell us how many compressor units to operate within each compressor station. This leads to a mixed-integer nonlinear program. In fact, this line is now being pursued, and some preliminary results can be found in Cobos-Zaleta and Ríos-Mercado [7].

*Acknowledgments:* We also would like to thank Dr. Richard Carter, from Stoner Associates, for providing very valuable input while developing this work. This research has been supported by grants from the National Science Foundation (grant DMI-9622106), the Texas Higher Education Coordinating Board (grant ARP/ATP 999903-122), the Mexican National Council for Science and Technology (grant CONACYT J33187-A), and Universidad Autónoma de Nuevo León (grants PAICYT CA555-01 and CA763-02).

## References

- [1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice-Hall, Englewood Cliffs, 1993.
- [2] P. Anglard and P. David. Hierarchical steady state optimization of very large gas pipelines. In *Proceedings of the 20th PSIG Annual Meeting*, Toronto, October 1988.
- [3] R. Aris, G. L. Nemhauser, and D. J. Wilde. Optimization of multistage cyclic and branching systems by serial procedures. *A.I.Ch.E. Journal*, 10(6):913-919, 1964.
- [4] U. Bertele and F. Brioschi. *Nonserial Dynamic Programming*. Academic Press, New York, 1972.
- [5] R. G. Carter. Pipeline optimization: Dynamic programming after 30 years. In *Proceedings of the 30th PSIG Annual Meeting*, Denver, October 1998.

- [6] R. G. Carter, D. W. Schroeder, and T. D. Harbick. Some causes and effects of discontinuities in modeling and optimizing gas transmission networks. In *Proceedings of the 25th PSIG Annual Meeting*, Pittsburgh, October 1993.
- [7] D. Cobos-Zaleta and R. Z. Ríos-Mercado. A MINLP model for minimizing fuel consumption on natural gas pipeline networks. In *Proceedings of the XI Latin-Ibero-American Conference on Operations Research*, Concepción, Chile, October 2002.
- [8] S. E. Dreyfus and A. M. Law. *The Art and Theory of Dynamic Programming*. Academic Press, Orlando, 1977.
- [9] S. Kim. *Minimum-Cost Fuel Consumption on Natural Gas Transmission Network Problem*. Doctoral dissertation, Texas A&M University, College Station, December 1999.
- [10] H. S. Lall and P. B. Percell. A dynamic programming based gas pipeline optimizer. In A. Bensoussan and J. L. Lions, editors, *Analysis and Optimization of Systems*, volume 144 of *Lecture Notes in Control and Information Sciences*, pages 123–132, Berlin, 1990. Springer-Verlag.
- [11] R. E. Larson and D. A. Wismer. Hierarchical control of transient flow in natural gas pipeline networks. In *Proceedings of the IFAC Symposium on Distributed Parameter Systems*, Banff, Alberta, Canada, 1971.
- [12] C. A. Luongo, B. J. Gilmour, and D. W. Schroeder. Optimization in natural gas transmission networks: A tool to improve operational efficiency. Presented at the 3rd SIAM Conference on Optimization, Boston, April 1989.
- [13] A. J. Osiadacz. Nonlinear programming applied to the optimum control of a gas compressor station. *International Journal for Numerical Methods in Engineering*, 15(9):1287–1301, 1980.
- [14] A. J. Osiadacz. Dynamic optimization of high pressure gas networks using hierarchical systems theory. In *Proceedings of the 26th PSIG Annual Meeting*, San Diego, October 1994.
- [15] A. J. Osiadacz and D. J. Bell. A simplified algorithm for optimization of large-scale gas networks. *Optimal Control Applications & Methods*, 7:95–104, 1986.
- [16] A. J. Osiadacz and S. Swierczewski. Optimal control of gas transportation systems. In *Proceedings of the 3rd IEEE Conference on Control Applications*, pages 795–796, August 1994.
- [17] P. B. Percell and J. D. Van Reet. A compressor station optimizer for planning gas pipeline operation. In *Proceedings of the 21st PSIG Annual Meeting*, El Paso, October 1989.
- [18] P. B. Percell and M. J. Ryan. Steady-state optimization of gas pipeline network operation. In *Proceedings of the 19th PSIG Annual Meeting*, Tulsa, October 1987.

- [19] K. F. Pratt and J. G. Wilson. Optimisation of the operation of gas transmission systems. *Transactions of the Institute of Measurement and Control*, 6(5):261–269, 1984.
- [20] R. Z. Ríos-Mercado. Natural gas pipeline optimization. In P. M. Pardalos and M. G. C. Resende, editors, *Handbook of Applied Optimization*, chapter 18.8.3, pages 813–825. Oxford University Press, New York, 2002.
- [21] R. Z. Ríos-Mercado, S. Wu, L. R. Scott, and E. A. Boyd. A reduction technique for natural gas transmission network optimization problems. *Annals of Operations Research*, 117(1–4):217–234, 2002.
- [22] M. G. Singh. *Dynamical Hierarchical Control*. North-Holland, Amsterdam, 1980.
- [23] D. J. Wilde. Strategies for Optimizing Macrosystems. *Chemical Engineering Progress*, 61(3):86–93, 1965.
- [24] P. J. Wong and R. E. Larson. Optimization of natural-gas pipeline systems via dynamic programming. *IEEE Transactions on Automatic Control*, AC-13(5):475–481, 1968.
- [25] P. J. Wong and R. E. Larson. Optimization of tree-structured natural-gas transmission networks. *Journal of Mathematical Analysis and Applications*, 24(3):613–626, 1968.
- [26] S. Wu. *Steady-State Simulation and Fuel Cost Minimization of Gas Pipeline Networks*. Doctoral dissertation, University of Houston, Houston, August 1998.
- [27] S. Wu, E. A. Boyd, and L. R. Scott. Minimizing fuel consumption at gas compressor stations. In J. J.-W. Chen and A. Mital, editors, *Advances in Industrial Engineering Applications and Practice I*, pages 972–977, Cincinnati, Ohio, 1996. International Journal of Industrial Engineering.
- [28] S. Wu, R. Z. Ríos-Mercado, E. A. Boyd, and L. R. Scott. Model relaxations for the fuel cost minimization of steady-state gas pipeline networks. *Mathematical and Computer Modelling*, 31(2–3):197–220, 2000.
- [29] H. I. Zimmer. Calculating optimum pipeline operations. Presented at the AGA Transmission Conference, 1975.