Minimizing Fuel Usage on Gas Pipelines:
A Network Based Algorithm for Looped Topologies

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Abstract: In this paper we present a heuristic for the problem of minimizing fuel cost on steady-state gas transmission problems on looped networks. The procedures is based on a two-stage iterative procedure, where, in a given iteration, gas flow variables are fixed and optimal pressure variables are found via dynamic programming in the first stage. In the second stage, the pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. This work focuses on looped network topologies, that is, networks with at least one cycle containing two or more compressor stations. This kind of topologies posses the highest degree of difficulty in real-world problems.

Introduction: A gas transmission network for delivering natural gas involves a broad variety of physical components such as pipes, regulators, and compressor stations to name a few. As the gas travels through the pipe, gas pressure is lost due to friction with the pipe wall. Some of this pressure is added back at compressor stations, which raise the pressure of the gas passing through them. In a gas transmission network, the overall operating cost of the system is highly dependent upon the operating cost of the compressor stations in a network. Operating cost, however, is generally measured by the fuel consumed at the compressor station. Hence, the goal is to minimize the total fuel consumption used by the stations while satisfying specified delivery requirements throughout the system.

Gas transmission network problems differ from traditional network flow problems in some fundamental aspects. First, in gas networks, a pressure variable is defined at every node in addition to the flow variables representing mass flow rates through each pipe. Second, in addition to the network flow conservation constraint set, there exist two other type of constraints: (1) a nonlinear equality constraint on each pipe, which represent the relationships between the pressure drop and the flow; and (2) a nonlinear non-convex set for each compressor station, which represents the feasible operating limits for pressure and flow within the station.

The problem is very difficult to solve due to the presence of non-convexities in both the objective function and the set of feasible solutions. Optimization algorithms (most of them based on dynamic programming) for non-looped gas network topologies are in a relatively well-developed stage. However, effective algorithms for looped topologies are practically non-existent.

In this paper we propose a heuristics for the fuel cost minimization on gas transmission systems with a looped network topology, that is, networks with at least one cycle containing two or more compressor station arcs. The network based heuristic (NBH) is based on a two-stage iterative procedure. In a particular iteration, at a first stage, gas flow variables are fixed and optimal pressure variables are found via dynamic programming (DP). At the second stage, the pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure.

Problem Formulation: Let $G = (N, L, M)$ be a directed network defined by a set N of n nodes, a set L of l pipes, and a set M of m compressor stations. The mass flow rate on a pipe $(i, j) \in L$ is represented by u_{ij}, and the mass flow rate through a compressor station $(i, j) \in M$ is represented by v_{ij}. Note that each compressor station is represented by a special pipe which connects a pair of nodes $(i, j) \in M$, where i and j are the corresponding suction and discharge nodes, respectively. Let u and v be the vectors of u_{ij}'s and v_{ij}'s, i.e., $u = \{u_{ij} : (i, j) \in L\}$,

$v = \{v_{ij} : (i, j) \in M\}$, and let w be the vector defined by $w = (u, v)^T$. Let $p = (p_1, \ldots, p_n)^T$ be the pressure vector with p_i the pressure at node *i*. Let $s = (s_1, \ldots, s_n)^T$ be the source vector with s_i the source at node *i*. If s_i is positive (negative), this corresponds to the gas supply limit (demand requirement) at node *i*. For the steady-state model, the sum of the sources is assumed to be zero, i.e., $\sum_{i=1}^{n} s_i = 0$.

The flow balance equation at a node has the following meaning: the sum of flows coming out of the node is equal to the sum of the flow entering the node. It can be represented as

$$\sum_{j:(i,j)\in L\cup M} w_{ij} - \sum_{j:(j,i)\in L\cup M} w_{ji} = s_i \qquad i \in N \tag{1}$$

where w_{ij} represents either u_{ij} if $(i, j) \in L$ or v_{ij} if $(i, j) \in M$.

The physical law that relates the flow in the pipe to the difference of pressure at its two ends for high-pressure networks is given, as discussed in Osiadacz [4], by the Weymouth's formula:

$$p_i^2 - p_j^2 = k_{ij} u_{ij}^2 \qquad (i, j) \in L \tag{2}$$

where k_{ij} is a constant whose value depends on the pipe physical properties.

The physical operational limits at each compressor station are another set of constraints, which includes the maximum/minimum compressor speed ratio, the maximum/minimum allowable volumetric flow rate. A compressor station is typically of many compressor units (which in turn can be of many types) arranged in different configurations settings. Let us assume that each compressor station *(i,j)* has *k* centrifugal compressor units hooked up in parallel.

Let D_{ij}^k denote the feasible compressor domain for variables (v_{ij}, p_i, p_j), and let $g_{ij}^k(v_{ij}, p_i, p_j)$ denote its corresponding fuel cost function. Recent work by Wu et al. [7] contains a detailed explanation about the structure of the domain D_{ij}^k, and the behaviour of the fuel consumption function g_{ij}^k.

The fuel cost function g_{ij}^1 in a single compressor unit is computed by

$$g_{ij}^1(v_{ij}, p_i, p_j) = a_{ij} v_{ij} \left[\left(\frac{p_j}{p_i} \right)^{a} - 1 \right], \quad (v_{ij}, p_i, p_j) \in D_{ij}^1 \tag{3}$$

where a_{ij} and α are constants which are determined by the specific type of compressors involved.

The mathematical formulation of the problem is given by

$$\text{Minimize} \quad \sum_{(i,j)\in M} g_{ij}^1(v_{ij}, p_i, p_j) \tag{4a}$$

Subject to

$$\sum_{j:(i,j)\in L\cup M} w_{ij} - \sum_{j:(j,i)\in L\cup M} w_{ji} = s_i \qquad i \in N \tag{4b}$$

$$p_i^2 - p_j^2 = k_{ij} u_{ij}^2 \qquad (i, j) \in L \tag{4c}$$

$$(v_{ij}, p_i, p_j) \in D_{ij}^k \qquad (i, j) \in M \tag{4d}$$

The difficulty in solving this problem arises from the presence of non-convexity in both the set of feasible solutions and the objective function. In addition, the type of underlying network topology becomes a crucial issue. For non-looped network topologies, dynamic programming (DP) approaches have been applied with relative success. See Carter [2] and Ríos-Mercado [5] for details of the DP algorithms.

These procedures rely heavily on theoretical results establishing that, for this type of (non-looped) systems, the involved flow variables can be uniquely determined in advance, and thus, eliminated from the problem. For network topologies with loops, the problem becomes more difficult because the flow variables cannot be uniquely determined, so they indeed have to be explicitly treated in the model. Addressing looped networks becomes the main focus of this work.

The Network Based Heuristic: Let $x^0 = (v^0, p^0)$ be an initial feasible solution to problem (4). For a tree structured gas transmission network, flow variables *v* are uniquely determined. However, for looped networks, one may obtain better a objective function by modifying the current flow setting v^0. For this purpose, we introduce the residual network concept (e.g., see Ahuja et al. [1]). The residual network was originally introduced to find the optimal flow (or to prove its optimality) in minimum cost network flow problems. We define the residual network $G'(v^0)$ with respect to the current flow vector v^0 as follows. We replace each arc (*i*, *j*) in the original network by two arcs, a forward arc (*i*, *j*) and a backward arc (*j*, *i*). The arc (*i*, *j*) has cost c_{ij} and the arc (*j*, *i*) has cost $-c_{ji}$.

In our heuristic flow modification step, the costs of the residual network are approximated by the derivatives of the objective function with respect to the flow on each compressor station, that is,

$$c_{ij} \approx a_{ij}\left[\left(\frac{p_j}{p_i}\right)^{a} - 1\right] \qquad (5)$$

where p_i, p_j are the current solution values delivered by dynamic programming with fixed flow variables. This cost c_{ij} is assigned at each forward edge of the residual network, while $-c_{ji}$ is assigned at each backward edge.

The cycle cost t_C, total cost of the cycle *C* in a residual network, is defined by

$$t_C = \sum_{(i,j)\in M_C} d_{ij}(C)\cdot c_{ij}\ , \qquad (6)$$

where $d_{ij}(C)$ equals 1 if (*i*, *j*) is both contained in the cycle *C* and a forward arc of $G'(v^0)$, -1 if $(j,i)\in C$ and (*j*, *i*) is a backward arc of $G'(v^0)$, and 0 otherwise, and M_C is the set of compressor stations located in the cycle *C*. If t_C is negative, then we call it a negative cycle and denote it as C^-.

Flow modification is done by augmenting flow through a negative cycle C^-. That is, if there exists a negative cost cycle C^-, then we augment positive flow through C^-, and hence update the current flow setting. This flow modification step can be represented as

$$v^{new} = v^0 + l\cdot d(C^-)\ , \qquad (7)$$

where $l > 0$ is the positive amount of flow which will be added through the cycle, and $d(C^-)$ is the vector of $d_{ij}(C^-)$, a vector representing the negative cycle C^-. The flow modification step of NBH can be viewed as a nonlinear programming algorithm in which we try to find a direction (a vector of flow modification) such that by moving l units in this direction, the objective function decreases. In our heuristic procedure, a negative cycle vector $d(C^-)$ corresponds to the search direction.

The value l is bounded below by zero and above by $\bar{l}$, which can be obtained by considering the complex inequality constraint set D_{ij}^k, $(i,j)\in C^-$. If $\bar{l} = 0$, then the algorithm stops. Otherwise, we set $l = \bar{l} > 0$.

For the newly obtained flow setting v^{new}, we need to find pressure variables, which requires rerunning DP with fixed flow setting v^{new}. If DP with v^{new} has no feasible solution or no improvement has been achieved, we reduce the size of l by setting $l = gl$, where $0 < g < 1$, and apply DP until we get a desirable result. The algorithm is summarized below.

Step 1: Find an initial feasible solution $x^0 = (v^0, p^0)$

Step 2: Construct the residual network $G'(v^0)$, and find a negative cycle C^- with (negative) cost t_{C^-}.

Step 3: If $\left|t_{C^-}\right| < e$, where e is a small number, stop. Otherwise, go to Step 4.

Step 4: Set $l = \bar{l}$. If $l = 0$, stop. Otherwise

(a) Modify the current flow by $v^{new} = v^0 + l\cdot d(C^-)$.

(b) Calculate pressure values using dynamic programming with modified flow v^{new}. If DP yields infeasible solution, or $g^{new} - g^0 > 0$, then set $l = gl$, with $0 < g < 1$, and go to Step 4a. Otherwise, update $v^0 \leftarrow v^{new}$, and go to Step 2.

Finding an Initial Feasible Solution: Like most optimization algorithms, our algorithm starts from a feasible point. According to gas specialists in the industry, getting a feasible solution for the complex network topology is quite a difficult task. Using our implementation of the solution methodology, we can obtain constructive information that can be used to find a feasible solution of the given problem. The current flow setting of the network satisfies the flow balance equations. We do not know yet whether this setting is feasible or not. The maximum capacity of the flow through the compressor station depends on the number of compressor units connected in parallel. Our implementation can be used for detecting infeasibility, that is, if the current flow setting yields an infeasible solution. The algorithm also gives us information on how much the current flow can be augmented or reduced to meet the capacity of the specific compressor station. Starting from an initial feasible set flow rates, we arrive at an improved solution using our proposed iterative solution methodology. However, solutions vary based on the different starting points used.

Computational work: Because of the lack of test problems in gas pipeline literature, we designed our own data sets. These were carefully constructed so as to represent real-world instances. In fact, all of the compressor-related data were kindly provided by Scientific Software Intercomp, Inc, a consulting firm in the pipeline industry. The algorithm, as described previously, consists of about 15,000 lines of C code. Numerical experiments on 12 instances based on three different looped topologies were run on a SGI Power Challenge L workstation running IRIX 6.2. Even though our solution methodology can handle non-looped topologies, our computational experiments targeted the looped structure case. The problem sizes used in our work range from a single-loop six-compressor instance to a four-loop 21-compressor instance. A more complete presentation of the numerical experimentation can be found in Kim [3]. Here we limit our exposition to assess the algorithmic performance of the algorithm.

The cost improvement obtained by applying our solution methodology was found to range from 3.3% to 41.8 %, with respect to an initial feasible solution found by conventional methods. According to Wu [6], 1 % savings on gas transhipment cost is worth 48.6 million dollars. Thus even a small percentage of improvement in solution by our algorithm is a great savings for the company.

Another issue investigated was the choice of the parameters l and g . These two parameters play a significant role in terms of computational time. As explained earlier, the greater the value of parameter l , the higher probability the iteration will move out of the feasible region. On the other hand, a larger l value yields a faster algorithm convergence. Parameter g is needed if the new solution is not feasible, or if we have no improved solution obtained. During our experiments, we found that l ranging between 0.8 and 0.85, and g around 0.5 yields adequately fast convergence.

The distribution of the running time among the various types of operations in the algorithm was studied as well. It was found, that the most of time (about 95 %) is spent on solving DP. This result highlights the importance of having an efficient procedure for solving DP.

Conclusions: In this paper we have presented a heuristic for the fuel cost minimization on natural gas transmission networks in steady state. The algorithm focuses on addressing looped network topologies. The mathematical model, which has an underlying network topology, has a non-convex objective function and non-convex feasible domain, which makes it difficult to solve. The main contribution of this work is precisely on providing a method for handling looped topologies, which had not been done to the best of our knowledge. Further, our work incorporates the most accurate model of compressor stations to date. In general, the problem we have presented is too computationally complex to be solved efficiently using standard mathematical programming techniques. A dynamic programming approach has proved to be an excellent choice for simple network structures (such as gun-barrel and the tree networks). For a system with loops containing compressor stations, dynamic programming has been applied only after the flow variables have been fixed. The flow modification step of our solution methodology exploits the underlying network configuration, and is very simple yet robust.

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References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network Flows*. Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[2] R. G. Carter. Pipeline optimization: Dynamic programming after 30 years. In *Proceedings of the 30^{t}th PSIG Annual Meeting*, Denver, October 1998.

[3] S. Kim. *Minimum-Cost Fuel Consumption on Natural Gas Transmission Network Problem*.

PhD thesis, Texas A&M University, College Station, 1999.

[4] A. J. Osiadacz. *Simulation and Analysis of Gas Networks*. Gulf Publishing Company, Houston, 1987.

[5] R. Z. Ríos-Mercado. Natural gas. In P. Pardalos and M. G. C. Resende, editors, *Handbook of Applied Optimization*, Oxford University Press. Forthcoming.

[6] S. Wu. *Steady-State Simulation and Fuel Cost Minimization of Gas Pipeline Networks*. PhD thesis, University of Houston, Houston, 1998.

[7] S. Wu, R. Z. Ríos-Mercado, E. A. Boyd, and L. R. Scott. Model relaxations for the fuel cost minimization of steady-state gas pipeline networks. *Mathematical and Computer Modelling*. Forthcoming.