Towards the Simplification of Natural Gas Transmission Networks

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Abstract: We address the problem of minimizing the fuel consumption incurred by compressor stations in steady-state natural gas transmission networks. In the real world, these type of instances are very large both in terms of the number of decision variables and the number of constraints, and very complex due to the presence of non-linearity and non-convexity in both the set of feasible solutions and the objective function. In this paper we develop a technique that can be used to significantly reduce the size of the instances by exploiting the special and unique structure and properties of gas pipeline networks.

Model: In this paper we consider the problem of minimizing the fuel cost consumption incurred by compressor stations through natural gas transmission networks.

This problem is represented by a network, where arcs correspond to pipelines and compressor stations, and nodes correspond to their physical interconnection points. The decision variables are the mass flow rates through every arc, and the gas pressure level at every node. At each compressor station, there is a a cost function that depends on the inlet (suction) pressure, the outlet (discharge) pressure and the mass flow rate through the compressor. This function g is typically non-convex and nonlinear.

The objective function of the problem is the sum of the fuel costs over all the compressor stations in the network. This problem involves the following constraints:

(i) mass flow balance equation at each node;

(ii) gas flow equation through each pipe;

(iii) pressure limit constraints at each node;

(iv) operation limits in each compressor station.

The first two are also called steady-state network flow equations. We emphasize that while the mass flow balance equations are linear, the pipe flow equations are nonlinear; this has been well documented in [3]. For the medium and high pressure flows, when taking into account the fact that a change of the flow direction of the gas stream may take place in the network, the pipe flow equation takes the following form:

$$p_1^2 - p_2^2 = cu|u|^\alpha, \tag{1}$$

where p_1 and p_2 are pressures at the end nodes of the pipe, u is mass flow rate through the pipe, α is a constant ($\alpha \approx 1$), and the pipe resistance c is a positive quantity depending on the pipe physical attributes.

The steady-state network flow equations can be stated in a very concise form by using incidence matrices. Let us consider a network with n nodes, l pipes, and m compressor stations. Each pipe is assigned a direction which may or may not coincide with gas flow through the pipe. Let A_l be the $n \times l$ matrix whose elements are

$$a_{ij}^l = \begin{cases} 1, & \text{if } j^{th} \text{ pipe comes out from } i^{th} \text{ node;} \\ -1, & \text{if } j^{th} \text{ pipe goes into } i^{th} \text{ node;} \\ 0, & \text{otherwise.} \end{cases}$$

A_l is called the node-pipe incidence matrix. Let A_m be the $n \times m$ matrix whose elements are

$$a_{ik}^m = \begin{cases} 1, & \text{if } i^{th} \text{ node is the discharge node of } k^{th} \text{ station;} \\ -1, & \text{if } i^{th} \text{ node is the suction node of } k^{th} \text{ station;} \\ 0, & \text{otherwise.} \end{cases}$$

A_m is thus called a node-station incidence matrix. The matrix formed by annexing A_m to the right hand side of A_l will be denoted as A, i.e., $A = (A_l \; A_m)$, which is an $n \times (l + m)$ matrix.

A simple network example is shown in Figure 1 which has $n = 10$ nodes, $l = 6$ pipes, and $m = 3$ stations. Directions assigned to the pipes have been indicated. Note that all nodes, pipes, and stations have been labeled separately. The matrices A_l and A_m for this network are

$$A_l = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad A_m = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

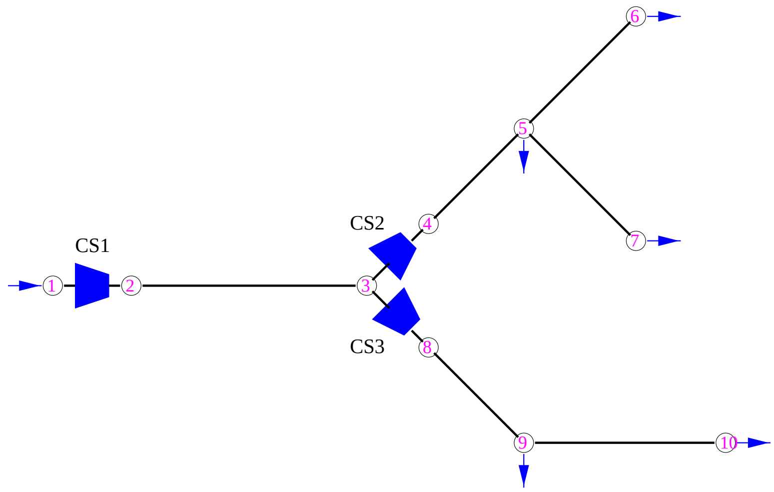

Figure 1: An example of a simple network

These matrices have a few special characteristics. To name a few, each row in matrix A_l, for example, corresponds to a node, and each column corresponds to a pipe in the network. In addition, each column contains exactly two nonzero elements, one is 1 and the other -1, which correspond to the two end nodes of the pipe.

Let $\mathbf{u} = (u_1, \ldots, u_l)^T$, and $\mathbf{v} = (v_1, \ldots, v_m)^T$ be the mass flow rate through the pipes and stations, respectively. Let $\mathbf{w} = (\mathbf{u}^T, \mathbf{v}^T)^T$. A component u_j or v_k is positive if the flow direction coincides with the assigned pipe or station direction, negative, otherwise. Let $\mathbf{p} = (p_1, \ldots, p_n)^T$ be the pressure vector (p_i the pressure at the i^{th} node), and $\mathbf{s} = (s_1, \ldots, s_n)^T$ be the source vector (s_i the source at the i^{th} node). The component s_i is positive if the node is a supply node; negative, if it is a delivery node; and zero, otherwise. We assume, without loss of generality, the sum of the sources to be zero:

$$\sum_{i=1}^{n} s_i = 0. \tag{2}$$

The network flow equations can now be stated as the following:

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ A_l^T \mathbf{p}^2 = \phi(\mathbf{u}) \end{cases}$$

where

$$\mathbf{p}^2 = (p_1^2, \ldots, p_n^2)^T, \qquad \phi(\mathbf{u}) = (\phi_1(u_1), \ldots, \phi_l(u_l))^T,$$

in which

$$\phi_j(u_j) = c_j u_j |u_j|^\alpha$$

Now suppose the source vector $\mathbf{s}$ is given satisfying the zero sum condition, and the bounds $\mathbf{p}^L$, $\mathbf{p}^U$ of pressures at every node have been specified. The problem is to determine the pressure vector $\mathbf{p}$ and the flow vector $\mathbf{w}$ so that the total fuel consumption is minimized, i.e.,

$$\begin{array}{llcll} \text{Minimize} & F(\mathbf{w},\mathbf{p}) & = & \sum_{k=1}^{m} g_k(v_k, p_{ks}, p_{kd}) & (3) \\ \text{subject to} & A\mathbf{w} & = & \mathbf{s} & (4) \\ & A_l^T \mathbf{p}^2 & = & \phi(\mathbf{u}) & (5) \\ & \mathbf{p} & \in & [\mathbf{p}^L, \mathbf{p}^U] & (6) \\ & (v_k, p_{ks}, p_{kd}) & \in & D_k \qquad k = 1, 2, \ldots, m & (7) \end{array}$$

where v_k, p_{ks}, and p_{kd} are the mass flow rate, suction pressure, and discharge pressure at the k-th station, g_k is its corresponding cost function, and D_k is the feasible domain in which the triple variables (v_k, p_{ks}, p_{kd}) may vary, See [3] for an in-depth study of the structure and properties of D_k and g_k. Note that

1. The feasible domains D_k are non-convex.
2. The fuel minimization functions g_k are nonlinear, non-convex and even discontinuous.
3. The pipe flow equations (1) define a non-convex set.

In general, a problem with these characteristics is very difficult to solve. What we do in this paper is to propose a technique that significantly reduces the size of any instance to this problem, making it more tractable. This technique uses concepts from graph theory applied to natural gas pipeline networks.

Graph Theory Concepts: A *directed graph* (or *digraph*) G consists of a set of vertices $V = \{v_1, v_2, \ldots\}$, a set of edges $E = \{e_1, e_2, \ldots\}$, and a mapping Ψ that maps every edge onto some ordered pair of vertices (v_i, v_j). If $e \in E$, $v_i, v_j \in V$, and $\Psi(e) = (v_i, v_j)$, the edge e is called *incident* with vertices v_i and v_j, or more precisely, out of vertex v_i, and into vertex v_j. If $v_i = v_j$, e is called a *self-loop*. If both V and E are finite, the digraph G is called *finite digraph*.

A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. A walk in which no vertex appears more than once is called a *path*. A walk beginning and ending with the same vertex in which no other vertex appears more than once is called a *circuit*.

A digraph G is said to be *connected* if there is at least one path between every pair of vertices in G. The digraph considered in this paper will be assumed to be finite and connected. Also we assume the digraph has no self-loops. A *tree* is a digraph without a circuit. A *spanning tree* T of a digraph G is a tree consisting of all the vertices in G. For a giving spanning tree T of a digraph G, the edge which is not in the tree T is called a *chord*. Adding a chord c to the spanning tree T forms a circuit which is called a *fundamental circuit*.

Here are some basic results regarding spanning trees and fundamental circuits of a digraph. Proofs of Theorems 1 and 2 can be found in [1].

Theorem 1 *Let n and e be the numbers of the vertices and edges, respectively, in a digraph G. Let T be a spanning tree of G. Then*

1. *The number of edges in the spanning tree T is $n-1$, the number of chords corresponding to the spanning tree T is $e-n+1$;*

2. *The number of the fundamental circuits corresponding to a spanning tree T is $e-n+1$. Every other circuit in the digraph G is a linear combination of the fundamental circuits.*

Suppose G is a self-loop-free digraph with n vertices and e edges. An *incident matrix* A of the digraph G is an n by e matrix, defined by

$$a_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge is incident out of } i^{th} \text{ vertex;} \\ -1, & \text{if } j^{th} \text{ edge is incident into } i^{th} \text{ vertex;} \\ 0, & \text{if } j^{th} \text{ edge is not incident with } i^{th} \text{ vertex.} \end{cases}$$

Each row of the incident matrix A represents how the edges are incident with a specific vertex. Each column of the incident matrix A represents how the vertices are incident with a specific edge. Since we have assumed that the digraph G has no self-loops, each column consists of one 1, one -1, and zeros. It can be proved that the rank of A is $n-1$. By deleting one row from the matrix A, the remainder matrix is called *reduced incident matrix*, denoted by A_f, which is $n-1$ by e, whose $n-1$ row vectors are linear independent. The vertex corresponding to the deleted row is called *reference vertex*. The incident matrix of a digraph completely determines the digraph.

Each circuit in the digraph G, after being arbitrarily assigned an orientation, can be represented by a vector whose components are $1, -1, 0$ according to whether and how the edge is included in the circuit. A *circuit matrix* B is a matrix each row of which corresponds a circuit vector, which is defined by

$$b_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ circuit contains } j^{th} \text{ edge} \\ & \text{and their orientations coincide,} \\ -1, & \text{if } i^{th} \text{ circuit contains } j^{th} \text{ edge} \\ & \text{but their orientations are opposite,} \\ 0, & \text{if } i^{th} \text{ circuit does not contain } j^{th} \text{ edge.} \end{cases}$$

As Theorem 1 stated, only $e-n+1$ fundamental circuit vectors with respect to a spanning tree are independent. The matrix consisting of the $e-n+1$ fundamental circuit vectors is called *reduced circuit matrix*, denoted by B_f, which is $e-n+1$ by e matrix.

Theorem 2 *Let G be a self–loop–free digraph, A_f and B_f be the reduced incident and circuit matrices using the same order of edges. Then*

$$A_f B_f^T = B_f A_f^T = 0.$$

To explain the above concepts and results, we give an example of a simple digraph in Figure 2 which consists of $n = 6$ vertices and $e = 8$ edges.

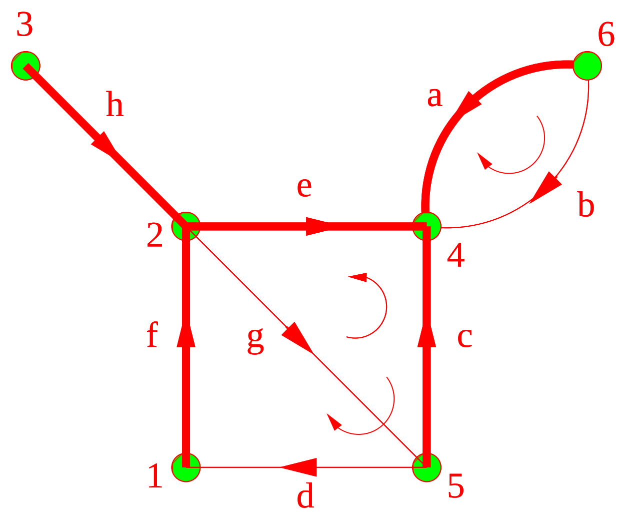

Figure 2: The fundamental circuits in a digraph

The incident matrix of the digraph, in which the vertices are arranged as 123456, and the edges $abcdefgh$, is

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced incident matrix, if taking vertex 3 as its reference vertex, is

$$A_f = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Edges $\{a, c, e, f, h\}$ form a spanning tree, indicated by the thick lines in Figure 5.1. Edges $\{b, d, g\}$ are the chords corresponding to the spanning tree. The number of fundamental circuits is $e - n + 1 = 3$. Each fundamental circuit corresponds to a chord. The reduced fundamental circuit matrix, whose orientations are shown in Figure 5.1, is

$$B_f = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

The Pipeline Network Flow Equations: Now let us consider a gas pipeline network system which consists of only nodes and pipes. We arbitrarily assign a direction for every pipe and view it as a digraph. Let G be such a digraph with n vertices and e edges. Let $\mathbf{w} = (w_1, \ldots, w_e)^T$ be the flow vector with w_j the mass flow rate through the j^{th} edge. w_j is positive if the directions of the flow and the edge coincide, negative otherwise. Let $\mathbf{s} = (s_1, \ldots, s_n)^T$ be the source vector satisfying (2). Let $\mathbf{p} = (p_1, \ldots, p_n)^T$ be the pressure vector with p_i the pressure at the i^{th} vertex.

The network flow equations can now be stated as the following:

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ A^T\mathbf{p}^2 = \phi(\mathbf{w}) \end{cases} \tag{8}$$

where $\mathbf{p}^2 = (p_1^2, \ldots, p_n^2)^T$, and $\phi(\mathbf{w}) = (\phi_1(w_1), \ldots, \phi_e(w_e))^T$, in which $\phi_j(w_j)$ is a function of w_j. In most network flow problems functions $\{\phi_j\}_1^d$ are nonlinear. In the gas pipeline network area, the most commonly used ϕ_j's are

$$\phi_j(w_j) = c_j w_j |w_j|, \quad 1 \le j \le d,$$

with $c_j > 0$. In some cases, ϕ_j's could also be

$$\phi_j(w_j) = c_j w_j |w_j|^\alpha, \quad 1 \le j \le d,$$

where $\alpha \ge 0$.

Now suppose the source vector $\mathbf{s}$ is given satisfying the zero sum condition (2) and a reference vertex has been selected whose pressure is also given (which is a necessary condition to solve the flow equations (8)). The number of the unknowns is $e + n - 1$ (since we assumed one pressure is given, the unknowns are $n - 1$ pressure variables and e flow variables), and the number of the flow equations is $e + n$. These are n node flow balance equations and e pipe flow equations. Since rank(A)$= n - 1$, only $(n-1)$ node flow balance equations are linear independent. Let A_f be the reduced incident matrix with respect to the selected vertex; let B_f be the reduced circuit matrix with respect to some spanning tree. Since $B_f A^T = 0$, system (8) is equivalent to

$$\begin{cases} A_f\mathbf{w} = \mathbf{s_f} \\ B_f\phi(\mathbf{w}) = \mathbf{0} \\ A^T\mathbf{p}^2 = \phi(\mathbf{w}) \end{cases} \tag{9}$$

where $\mathbf{s_f}$ is the $(n-1)$ by 1 vector formed by removing the source term corresponding to the selected reference vertex from $\mathbf{s}$. The advantage of system (9) is that the first two equations, i.e.,

$$\begin{cases} A_f \mathbf{w} = \mathbf{s_f} \\ B_f \phi(\mathbf{w}) = \mathbf{0} \end{cases} \tag{10}$$

contain only the flow vector $\mathbf{w}$. Notice that system (10) consists of e equations and e unknowns ($(n-1)$ equations in the first equation and $(e-n+1)$ equations in the second. Unknowns are e components in the flow vector $\mathbf{w}$). If it has a unique solution, the flow vector $\mathbf{w}$ can be solved separately from the pressure vector $\mathbf{p}$, and the pressure vector $\mathbf{p}$ can be directly computed from the third equation of the system (9) as the pressure at reference vertex is given. We now address the question on whether system (10) has a unique solution.

Uniqueness and Existence of the Solution: In this section, we show that system (10) has a unique solution. A direct corollary of this result is that system (8) has a unique solution if the source vector $\mathbf{s_f}$ and the reference pressure are given. We begin with some definitions.

Let H be a Hilbert space with a scalar product $(\cdot\, ,\, \cdot)$.

Definition 1 A mapping ϕ: $H \to H$ is said *strongly monotonic* if there exists a constant $a > 0$, such that, for every x, $y \in H$ we have

$$(\phi(x) - \phi(y), x - y) \geq a\,(x - y, x - y).$$

Definition 2 A mapping ϕ: $H \to H$ is said *strictly monotonic* if for every x, $y \in H$ we have

$$(\phi(x) - \phi(y), x - y) \geq 0,$$

and equality holds if and only if $x = y$.

Definition 3 A mapping ϕ: $H \to H$ is said to be a *basin* if for every $x_0 \in H$, the set

$$X_{x_0} = \{x \in H : (\phi(x), x - x_0) \leq 0\}$$

is bounded.

Now we prove some basic results related to the above concepts.

Lemma 1 *If ϕ: $H \to H$ is strongly monotonic, ϕ is a strictly monotonic basin.*

Proof: Obviously ϕ is strictly monotonic. To show that it is also a basin, we notice that, for every $x_0 \in H$, $x \in X_{x_0}$,

$$\begin{aligned} a\,(x - x_0, x - x_0) &\leq (\phi(x) - \phi(x_0), x - x_0) \quad \text{(since } \phi \text{ is strongly monotonic)} \\ &= (\phi(x), x - x_0) - (\phi(x_0), x - x_0) \\ &\leq -(\phi(x_0), x - x_0) \quad \text{(since } x \in X_{x_0}) \\ &\leq \|\phi(x_0)\| \|x - x_0\|. \end{aligned}$$

So

$$\|x - x_0\| \leq \frac{1}{a}\|\phi(x_0)\|.$$

Hence, X_{x_0} is bounded. ■

However, a mapping ϕ that is a strictly monotonic basin is not necessarily strongly monotonic. Here is an example.

Lemma 2 *Let $H = \mathbf{R}^d$ with the Euclid scalar product, d is a positive integer. Let ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ be a mapping as the following: for every $\mathbf{x} = (x_1, x_2, \dots, x_d)^T \in \mathbf{R}^d$,*

$$\phi(\mathbf{x}) = (\phi_1(x_1), \phi_2(x_2), \dots, \phi_d(x_d))^T$$

where

$$\phi_j(x_j) = c_j x_j |x_j|^\alpha, \quad \text{for } 1 \leq j \leq d,$$

with $c_j > 0$ and $\alpha \geq 0$. Then ϕ is strictly monotonic basin.

Proof: For every $\mathbf{x} = (x_1, x_2, \dots, x_d)^T$, $\mathbf{y} = (y_1, y_2, \dots, y_d)^T \in \mathbf{R}^d$,

$$(\phi(\mathbf{x}) - \phi(\mathbf{y}), \mathbf{x} - \mathbf{y}) = \sum_{j=1}^{d} c_j (x_j|x_j|^\alpha - y_j|y_j|^\alpha)(x_j - y_j).$$

Since $\alpha \geq 0$, function $h(s) = s|s|^\alpha$ is a strictly increasing function for all s. Hence, each term at the right-hand side (RHS) is nonnegative. Thus,

$$(\phi(\mathbf{x}) - \phi(\mathbf{y}), \mathbf{x} - \mathbf{y}) \geq 0.$$

Equality holds if and only if every term at the RHS is zero, so $x_j = y_j$, for every j. Therefore ϕ is strictly monotonic.

To show that ϕ is a basin, let $\mathbf{x}^0 \in \mathbf{R}^d$. Then $\mathbf{x} \in X_{\mathbf{x}^0}$ means

$$(\mathbf{x}, \phi(\mathbf{x})) \leq (\mathbf{x}^0, \phi(\mathbf{x})),$$

i.e.,

$$\sum_{j=1}^{d} c_j x_j^2 |x_j|^\alpha \leq \sum_{j=1}^{d} c_j x_j^0 x_j |x_j|^\alpha.$$

Let $\rho = \|\mathbf{x}\|$ and $\mathbf{y} = \mathbf{x}/\rho$, so, $\|\mathbf{y}\| = 1$. Replace x_j by ρy_j in the above inequality, we have

$$\begin{aligned} \rho &\leq \frac{\sum_{j=1}^{d} c_j x_j^0 \rho^{1+\alpha} y_j |y_j|^\alpha}{\sum_{j=1}^{d} c_j \rho^{1+\alpha} |y_j|^{2+\alpha}} \\ &\leq \|x^0\| \frac{\sum_{j=1}^{d} c_j |y_j|^{1+\alpha}}{\sum_{j=1}^{d} c_j |y_j|^{2+\alpha}}. \end{aligned}$$

Since $c_j > 0$, function

$$g(\mathbf{y}) = \frac{\sum_{j=1}^{d} c_j |y_j|^{1+\alpha}}{\sum_{j=1}^{d} c_j |y_j|^{2+\alpha}}$$

is continuous on the unit sphere $\|\mathbf{y}\| = 1$. Let

$$G = \max_{\|\mathbf{y}\|=1} g(\mathbf{y}),$$

then, for every $\mathbf{x} \in X_{\mathbf{x}^0}$, we have

$$\rho = \|\mathbf{x}\| \leq \|\mathbf{x}^0\| G.$$

Hence, $X_{\mathbf{x}^0}$ is bounded for every $\mathbf{x}^0 \in \mathbf{R}^d$; i.e., ϕ is a basin. ■

Remark: ϕ is not strongly monotonic.

Corollary 1 *The identity function ϕ: $\mathbf{R}^d \to \mathbf{R}^d$, $\phi(\mathbf{x}) = \mathbf{x}$ is a strictly monotonic basin.*

The following lemma can be found in [2].

Lemma 3 *Let H be a Hilbert space. If ϕ: $H \to H$ is continuous and strongly monotonic, then ϕ maps H onto H.*

Let $r > 0$, $t \geq 0$, be two integers, and $d = r + t$. We say a $r \times d$ matrix A and a $t \times d$ matrix B are *perpendicular to each other* if they satisfy
Hypothesis P

1. $\text{rank}(A) = r$, $\text{rank}(B) = t$;
2. $AB^T = BA^T = 0$.

Let $M = \{\mathbf{x} \in \mathbf{R}^d : A\mathbf{x} = 0\}$, and $N = \{\mathbf{y} \in \mathbf{R}^d : B\mathbf{y} = 0\}$. By Hypothesis P, we have $M = N^{\perp}$ and

$$\mathbf{R}^d = M \oplus N.$$

Let p_M: $\mathbf{R}^d \to M$ be the projection from $\mathbf{R}^d$ to M, p_N: $\mathbf{R}^d \to N$ be the projection from $\mathbf{R}^d$ to N. Then, for every $\mathbf{w} \in \mathbf{R}^d$,

$$\mathbf{w} = p_M(\mathbf{w}) + p_N(\mathbf{w}).$$

Theorem 3 (Uniqueness) *Let matrices A and B be perpendicular to each other. Suppose ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ is strictly monotonic, then, for every $\mathbf{s} \in \mathbf{R}^r$, the solution to the system of equations*

$$\left\{ \begin{array}{r} A\mathbf{w} = \mathbf{s} \\ B\phi(\mathbf{w}) = \mathbf{0} \end{array} \right. \tag{11}$$

is unique.

Proof: Suppose both $\mathbf{u}$ and $\mathbf{v}$ are solutions to the system (11), then

$$\begin{cases} A(\mathbf{u}-\mathbf{v}) = \mathbf{0} \\ B(\phi(\mathbf{u}) - \phi(\mathbf{v})) = \mathbf{0} \end{cases}$$

Hence, $\mathbf{u}-\mathbf{v} \in M$, and $\phi(\mathbf{u}) - \phi(\mathbf{v}) \in N$. Thus, $(\mathbf{u}-\mathbf{v}, \phi(\mathbf{u}) - \phi(\mathbf{v})) = 0$. Since ϕ is strictly monotonic, the above equation implies that $\mathbf{u} = \mathbf{v}$. Hence, the solution is unique. ■

The proof of the existence of the system (11) is not so straightforward. We first prove, in Lemma 4, that it is true for the continuous and strongly monotonic mapping ϕ, and then, in Theorem 4, we prove that it is also true if the mapping ϕ is a continuous and strictly monotonic basin.

Lemma 4 *Suppose ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ is continuous and strongly monotonic. Then, for every $\mathbf{s} \in \mathbf{R}^r$, system (11) has a solution.*

Proof: Let $\mathbf{w}_0$ be the unique solution of the linear system:

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ B\mathbf{w} = \mathbf{0} \end{cases} \tag{12}$$

then $\mathbf{w}_0 \in N$. We define ψ: $M \to M$ as following: for every $\mathbf{x} \in M$,

$$\psi(\mathbf{x}) = p_M(\phi(\mathbf{x} + \mathbf{w}_0)). \tag{13}$$

Then, for every $\mathbf{x}^1$, $\mathbf{x}^2 \in M$, we have

$$\begin{aligned} \left(\psi(\mathbf{x}^1) - \psi(\mathbf{x}^2), \, \mathbf{x}^1 - \mathbf{x}^2\right) &= \left(p_M(\phi(\mathbf{x}^1 + \mathbf{w}_0)) - p_M(\phi(\mathbf{x}^2 + \mathbf{w}_0)), \, \mathbf{x}^1 - \mathbf{x}^2\right) \\ &= \left(p_M(\phi(\mathbf{x}^1 + \mathbf{w}_0)) - p_M(\phi(\mathbf{x}^2 + \mathbf{w}_0)), \, \mathbf{x}^1 - \mathbf{x}^2\right) \\ &+ \left(p_N(\phi(\mathbf{x}^1 + \mathbf{w}_0)) - p_N(\phi(\mathbf{x}^2 + \mathbf{w}_0)), \, \mathbf{x}^1 - \mathbf{x}^2\right) \\ &= \left(\phi(\mathbf{x}^1 + \mathbf{w}_0) - \phi(\mathbf{x}^2 + \mathbf{w}_0), \, \mathbf{x}^1 - \mathbf{x}^2\right) \\ &= \left(\phi(\mathbf{x}^1 + \mathbf{w}_0) - \phi(\mathbf{x}^2 + \mathbf{w}_0), \, (\mathbf{x}^1 + \mathbf{w}_0) - (\mathbf{x}^2 + \mathbf{w}_0)\right) \\ &\geq a, \left((\mathbf{x}^1 + \mathbf{w}_0) - (\mathbf{x}^2 + \mathbf{w}_0), \, (\mathbf{x}^1 + \mathbf{w}_0) - (\mathbf{x}^2 + \mathbf{w}_0)\right) \\ &= a \left(\mathbf{x}^1 - \mathbf{x}^2, \, \mathbf{x}^1 - \mathbf{x}^2\right) \\ &= a \, \|\mathbf{x}^1 - \mathbf{x}^2\|. \end{aligned}$$

Hence, ψ is strongly monotonic on M. Moreover, ψ is continuous because ϕ is continuous. By Lemma 3, there is a $\mathbf{x} \in M$ such that $\psi(\mathbf{x}) = \mathbf{0}$. Thus, by (13), $p_M(\phi(\mathbf{x} + \mathbf{w}_0)) = \mathbf{0}$. Let $\mathbf{w} = \mathbf{x} + \mathbf{w}_0$. Since $\mathbf{x} \in M$, $A\mathbf{w}_0 = \mathbf{s}$, we have $A\mathbf{w} = A(\mathbf{x} + \mathbf{w}_0) = \mathbf{s}$. Moreover, since $p_M(\phi(\mathbf{w})) = 0$, we have $\phi(\mathbf{w}) = p_M(\phi(\mathbf{w})) + p_N(\phi(\mathbf{w})) = p_N(\phi(\mathbf{w})$. Hence, $\phi(\mathbf{w}) \in N$, and so $B\phi(\mathbf{w}) = \mathbf{0}$. Hence, system (11) has a solution $\mathbf{w}$. ■

Theorem 4 *Let matrices A and B be perpendicular to each other. Let ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ be continuous. Suppose*

(i) ϕ is strictly monotonic;

(ii) ϕ is a basin.

Then system (11) has a solution for every $\mathbf{s} \in \mathbf{R}^r$.

Proof: Since ϕ is strictly monotonic, for every $\epsilon > 0$, $\phi_\epsilon(\mathbf{w}) = \phi(\mathbf{w}) + \epsilon\mathbf{w}$ is strongly monotonic because

$$\begin{aligned} & (\phi_\epsilon(\mathbf{w}^1) - \phi_\epsilon(\mathbf{w}^2), \mathbf{w}^1 - \mathbf{w}^2) \\ = \ & (\phi(\mathbf{w}^1) - \phi(\mathbf{w}^2), \mathbf{w}^1 - \mathbf{w}^2) + (\epsilon\mathbf{w}^1 - \epsilon\mathbf{w}^2, \mathbf{w}^1 - \mathbf{w}^2) \\ \geq \ & \epsilon(\mathbf{w}^1 - \mathbf{w}^2, \mathbf{w}^1 - \mathbf{w}^2). \end{aligned}$$

Hence, by Lemma 4, there is a $\mathbf{w}_\epsilon \in \mathbf{R}^d$, such that

$$\begin{cases} A\mathbf{w}_\epsilon = \mathbf{s} \\ B\phi_\epsilon(\mathbf{w}_\epsilon) = \mathbf{0} \end{cases} \tag{14}$$

Let $\mathbf{w}_0$ be the solution of the linear system (12) as in Lemma 4. Then

$$A(\mathbf{w}_\epsilon - \mathbf{w}_0) = \mathbf{0}.$$

Hence, $\mathbf{w}_\epsilon - \mathbf{w}_0 \in M$. The second equation of (14) implies that $\phi_\epsilon(\mathbf{w}_\epsilon) \in N$. So,

$$(\mathbf{w}_\epsilon - \mathbf{w}_0, \phi_\epsilon(\mathbf{w}_\epsilon)) = 0, \tag{15}$$

i.e.,

$$(\mathbf{w}_\epsilon - \mathbf{w}_0, \phi(\mathbf{w}_\epsilon)) + (\mathbf{w}_\epsilon - \mathbf{w}_0, \epsilon\mathbf{w}_\epsilon) = 0.$$

Since ϕ is a basin, there is a $G_1 > 0$, such that for every $\mathbf{w} \in \mathbf{R}^d$, the inequality

$$(\mathbf{w} - \mathbf{w}_0, \phi(\mathbf{w})) \leq 0$$

implies $\|\mathbf{w}\| \leq G_1$. By Corollary 1, $\gamma(\mathbf{w}) = \mathbf{w}$ is also a basin. Hence, there is a $G_2 > 0$, such that the inequality

$$(\mathbf{w} - \mathbf{w}_0, \mathbf{w}) \leq 0$$

implies $\|\mathbf{w}\| \leq G_2$. Thus, by (15), we must have

$$\|\mathbf{w}_\epsilon\| \leq \max(G_1, G_2).$$

By Weierstrass theorem, there is a $\mathbf{w} \in \mathbf{R}^d$, such that

$$\|\mathbf{w}_{\epsilon_n} - \mathbf{w}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

for some sequence ϵ_n. Since ϕ is continuous, by (14), we have

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ B\phi(\mathbf{w}) = \mathbf{0} \end{cases}$$

Hence, system (11) has a solution. ■

Corollary 2 *System (10) has a unique solution for every* $\mathbf{s_f} \in \mathbf{R}^{n-1}$.

Proof: Apply Theorem 3 and 4, taking $A = A_f$ and $B = B_f$, then use Lemma 2. ■

Systems of nonlinear equations could have very strange behaviors. Even a single nonlinear equation could have no solution or more than one solution. Interestingly, some systems of nonlinear equations which arise from industrial and engineering problems practically should have a unique solution. We have proposed one of them in this paper.

For gas pipeline network flow problems, the presented result is quite interesting itself. One fact is that, since the function ϕ involved in gas pipeline network problems is monotonic, solving the system (10) by Newton's Method is very stable, fast, and accurate. These facts lead us to introduce the Network Reduction Method for networks consisting of nodes, pipes, and compressor stations. We will show in the next section that this method can greatly reduce the size of the problem.

The Network Reduction Method: The main result obtained in the previous section is that, with all the sources (that is, the mass flow rates at all the nodes of the network going into or out of the network) given, all the flows in pipes are completely determined, while the pressures at the nodes will be determined if the pressure at one (reference) node is given. It must be pointed out that this result is based on two facts:

1. Each node has a mass flow balance equation.
2. Each pipe has a pipe flow equation defining the relation between the flow rate and the pressures at the two end nodes.

As a gas pipeline network consists of not only nodes and pipes, but also compressor stations, we can see that, for each node, the mass flow balance is still satisfied; but for each edge representing a station there is no equation relating the flow rate through the station and the pressures at its suction and discharge sides. Flow rate, suction pressure, and discharge pressure of a station are actually independent of each other, and there are only certain inequalities these variables must satisfy. Hence, the result we obtained in the previous section can not be directly applied to such networks.

In this section, we will introduce the Network Reduction Method for networks consisting of nodes, pipes, and compressor stations. Our theory begins with deleting the compressor stations from the network.

By deleting all the stations from a network, which consists of nodes, pipes, and compressor stations, we should have several *disconnected* components, each of which, called as a *subnetwork*, consists of only nodes and pipes. There are no stations in subnetworks.

On the other hand, if we view each subnetwork as a single (big) node for the network, i.e., shrinking each subnetwork to a node, we shall get a new network which consists of only the (big) nodes, each representing a subnetwork, and the stations. There are no pipes in this network because all the pipes are encapsulated in the (big) nodes. This new network is called a *super-network* (where each node represents a subnetwork, and each edge represents a station). It is easy to see that there is only one (connected) super-network for a given network. The structure of the super-network could be either a tree or a digraph with loops, depending on the configuration of these compressor stations in the network.

To explain the concepts about the subnetworks and the super-network of a network, let us look at the following network example.

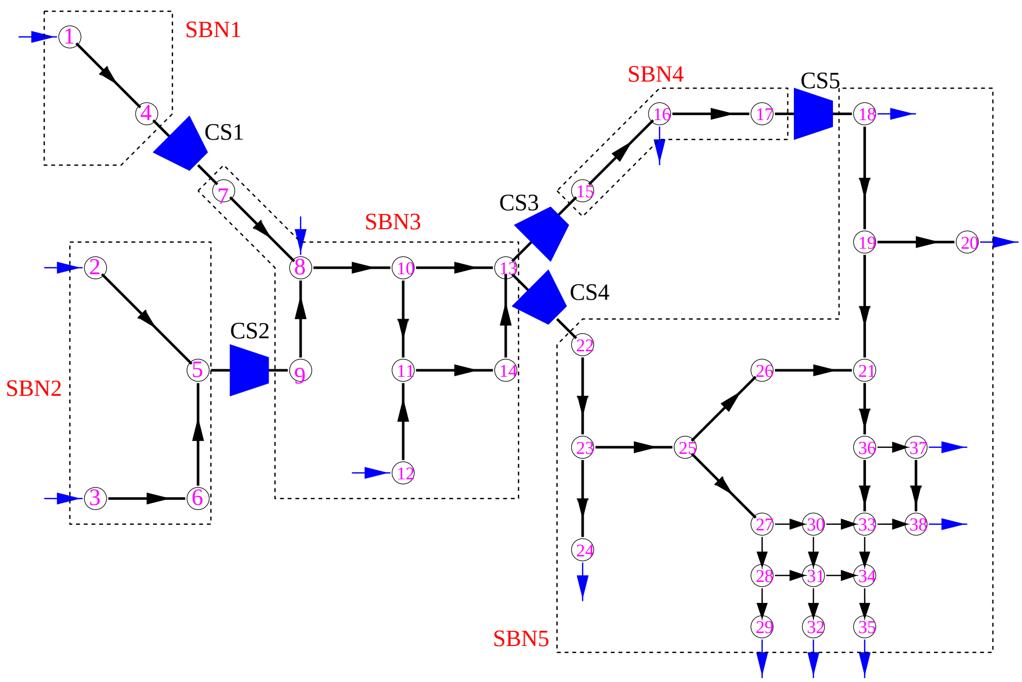

Figure 3: A network with five subnetworks

In Figure 3, a pipeline network has been drawn which consists of $n = 38$ nodes, $l = 38$ pipes, and $m = 5$ compressor stations. Stations are labeled by CS1, CS2, CS3, CS4, and CS5 in the figure. The number of edges (pipes or stations) is $e = l + m = 43$. Hence, the number of fundamental circuits is $e - n + 1 = 43 - 38 + 1 = 6$. If all the 5 stations are deleted from this digraph, we shall get 5 disconnected components, i.e., 5 subnetworks, labeled SBN1, SBN2, SBN3, SBN4, and SBN5. These subnetworks are separated by dotted lines in the figure. The super-network of this network is shown in Figure 4, which consists of 5 nodes and 5 edges, each node representing a subnetwork and each edge representing a station. In this example, the super-network has only one loop.

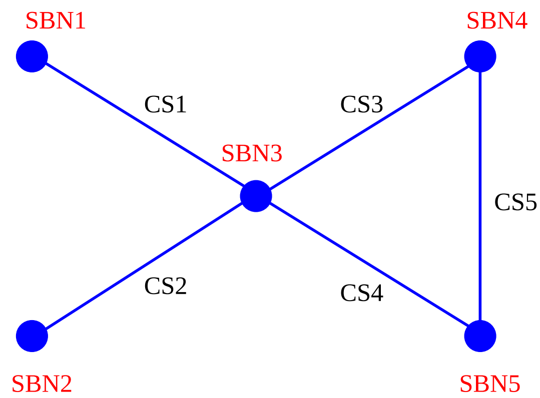

Figure 4: The super-network

For real world gas pipeline networks, we have found that the structures of the super-networks are much less complicated than those of the original networks. Although, networks themselves may have a lot of loops, especially loops in pipes, their super-networks are mostly trees. Even if their super-networks are digraphs with some loops, the numbers of the loops in the super-networks are significantly fewer than that in the original networks.

In the case that the super-network is a tree, we will show that the mass flow rates through all the stations are fixed if, as we have always assumed, the sources (supplying or delivering flow rates) at all nodes are given. This, as we should see, will greatly simplify the fuel cost minimization problem. On the other hand, if the super-network is a looped digraph, the mass flow rates through the stations are not uniquely determined but satisfy a system of linear equations. The number of the independent linear equations in the system is equal to the number of the fundamental circuits in the super-network. For an example, for the network shown in Figure 3, there is only one independent linear equation in the system.

Super-network is a Tree: In this section we assume that the super-network is a tree. In this case, since each node in the super-network represents a subnetwork, we can define the source at this node as the sum of the sources at all the nodes included in this subnetwork. In this sense, the sources at all the nodes in the super-network are fixed. Since the super-network is a tree, all the flow rates through the edges of the super-network are uniquely determined (this is Corollary 2, with $B = 0$). Since each edge in the super-network represents a station in the original network, it means that the flow rates through all the stations are known.

Now let us look at the subnetworks. We can see that, for each subnetwork, the sources at all the nodes, including the nodes connecting to stations are all known. By Theorem 4, we conclude that the flow rates through all the pipes in the subnetwork can be uniquely determined.

Moreover, the pressures at all the nodes in the subnetwork are uniquely determined by the pressure at one node, the reference node. These pressures will also be increased or decreased as the pressure at the reference node is increased or decreased, respectively.

Hence, we have the following fundamental theorem of the network reduction method

Theorem 5 *Suppose*

(i) pipeline network consists of only nodes, pipes, and stations;

(ii) sources at all the nodes are given;

(iii) super-network is a tree.

Then

1. *Flow rates through all the pipes and stations are known.*

2. *For each subnetwork, pressure p at any node is related to the pressure p_r at a reference node by*

$$p^2 - p_r^2 = c,$$

where

$$c = \sum_{j \in J} c_j \, u_j |u_j|^\alpha$$

is a constant, where J is an index set of pipes in a path connecting the node and the reference node, c_j and α are constants, u_j is the flow rate in the j^{th} pipe which is known.

Note that the constant c is independent of the selection of the path because the flow rate u_j's are solved from the equations such that summation $\sum_{j \in J} c_j \, u_j |u_j|^\alpha$ along any loop in a subnetwork is zero.

Hence, if a network is divided into b subnetworks, the total number of independent variables in the network is b, i.e., the pressure variables p_r's at the b reference nodes.

The fuel cost minimization problem (3)–(7) can now be greatly simplified by applying the network reduction method.

Firstly, since the flow rates v_k's through all the stations are known, each function g_k in (3) depends on only (p_{ks}, p_{kd}). Thus, the objective function $F(\mathbf{w}, \mathbf{p})$, depends on only the suction and discharge pressures (p_{ks}, p_{kd}), $k = 1, \ldots, m$. Let $\mathbf{z}$ be the vector of these suction and discharge pressures, i.e., $\mathbf{z} = \{p_{1s}, p_{1d}, \ldots, p_{ms}, p_{md}\}$; the objective function F can now be represented as $G(\mathbf{z})$, i.e.,

$$G(\mathbf{z}) = \sum_{k=1}^{m} g_k(v_k, p_{ks}, p_{pd}),$$

where v_k is known.

Further, suppose the network is divided into b subnetworks, the pressure variables in $\mathbf{z}$ can be partitioned into b disjoint vectors $\mathbf{z_i}$, each representing the pressures at all the suction or discharge nodes in the i^{th} subnetwork, i.e., $\mathbf{z} = (\mathbf{z_1}^T, \ldots, \mathbf{z_b}^T)^T$. Let $\mathbf{z_i} = (z_{i1}, \ldots, z_{iJ_i})^T$, where J_i is the number of the suction and discharge nodes in the i^{th} subnetwork. Let us choose z_{i1} as the reference pressure for the i^{th} subnetwork. Then, according to Theorem 5, pressure p at every node in the i^{th} subnetwork is related to z_{i1} by

$$p^2 - z_{i1}^2 = c.$$

It is easy to see that there are two constants z_i^L and z_i^U, such that the pressure limit constraints (6) for nodes in the i^{th} subnetwork are equivalent to

$$z_i^L \le z_{i1} \le z_i^U.$$

One of the reasons that we introduce the network reduction method is based on the following observations. We notice that the objective function depends only on the pressures at suction and discharge nodes. This means we need not take care of the values of pressures at the nodes other than suction or discharge nodes. However, we must keep all pressure variables within their pressure limits, or equivalently, satisfying the constraints (6), which can now be fulfilled by confining the reference pressure z_{i1} in its limits z_i^L and z_i^U. Notice that constraints (7) are irrelevant to the pressures at nodes other than suction or discharge nodes; therefore, these pressures will disappear in the minimization problem.

On the other hand, the pressures at suction or discharge nodes in the i^{th} subnetwork must be related to the reference pressure z_{i1}, i.e.,

$$z_{ij}^2 - z_{i1}^2 = c_{ij}, \quad j = 2, \ldots, J_i,$$

where c_{ij}'s are constants.

As for the compressor station constraint (7), since the v_k's are known, it becomes

$$\mathbf{z} \in Z,$$

where Z is the feasible domain of stations to the suction and discharge pressures $\mathbf{z}$.

Hence, the fuel cost minimization problem (3)–(7) can be simplified as the following

$$\text{Minimize} \qquad G(\mathbf{z}) \tag{16}$$

$$\text{subject to} \quad z_i^L \le z_{i1} \le z_i^U \quad 1 \le i \le b \tag{17}$$

$$z_{ij}^2 - z_{i1}^2 = c_{ij}, \quad 1 \le i \le b,\ j = 2, \ldots, J_i \tag{18}$$

$$\mathbf{z} \in Z \tag{19}$$

Comparing problem (3)–(7) with problem (16)–(19), the simplifications are

1. The number of variables reduces from $l + m + n$ to the size of vector $\mathbf{z}$ which is at most $2m$. Notice that, a typical pipeline network may consist of thousands of pipes and nodes, but only dozens of stations; this reduction is thus significant.

2. The nonlinear equality constraints (5) involve 3 variables, while constraints (18) involve only 2. The fact is that linearizing a nonlinear constraint involving 2 variables is much easier and more effective.

3. The number of nonlinear equality constraints reduces from l to $\sum_{i=1}^{b}(J_i - 1) \leq 2m - b$. Since nonlinear equality constraints are often the main obstacles in optimization problems, reducing the number of the nonlinear equality constraints can make the problem easier to solve.

Configuration	Before reduction				After reduction	
	l	m	n	$l+m+n$	b	$\lvert\mathbf{z}\rvert$
A	10	2	10	22	2	4
B	44	7	47	98	7	13
C	91	9	180	280	10	18
D	1462	37	1560	3059	38	73

Table 1: Size of networks before and after reduction

Table 1 displays a comparison of sizes before and after the reduction for some typical network configurations. We can see that the size of $\mathbf{z}$ is often much smaller than the number $l+m+n$, i.e., the size of the reduced problem is much smaller than that of the original problem. We must point out that the tradeoff for these simplifications is that we need to solve the network flow equations for each subnetwork. However, our numerical experiments show that a modified Newton's method is extremely fast and stable to solve these equations. Moreover, all these calculations can be done at pre-processing.

Super-network is a Looped Digraph: As the super-network is a digraph with only a few loops, the network reduction method can still be successfully used. In this case, the mass flow rate $\mathbf{v}$ through the stations satisfies a simple system of linear equations:

$$\mathcal{A}\mathbf{v} = \mathbf{S}, \tag{20}$$

where $\mathcal{A}$ is the node-edge incidence matrix for the super-network and $\mathbf{S}$ is the sources at the nodes in the super-network. The i^{th} element of $\mathbf{S}$ is the sum of the sources at all the nodes in the i^{th} subnetwork. Since the flow rate $\mathbf{v}$ must be bounded, say, $\|\mathbf{v}\| \leq v_{max}$, we can define a set V as

$$V = \{\mathbf{v} : \mathcal{A}\mathbf{v} = \mathbf{S}, \|\mathbf{v}\| \leq v_{max}\}.$$

For each $\mathbf{v} \in V$, we define a function $f(\mathbf{v})$ on V, which is

$$f(\mathbf{v}) \equiv \min\Big\{G^v(\mathbf{z}) \,:\, (z_i^L)^v \le z_{i1} \le (z_i^U)^v,\ 1 \le i \le b; \\ z_{ij}^2 - z_{i1}^2 = c_{ij}^v,\ 1 \le i \le b,\ j = 2,\ldots,J_i; \quad \mathbf{z} \in Z^v\Big\},$$

where G^v, $(z_i^L)^v$, $(z_i^U)^v$, c_{ij}^v, and Z^v depend on $\mathbf{v}$. Hence, the fuel cost minimization problem becomes minimizing $f(\mathbf{v})$ on the set V. A numerical approximation technique such as grid generation on $\mathbf{v}$ could be applied to find approximate solutions. This method might work well if the dimension of the kernel of the matrix $\mathcal{A}$ is small, or equivalently, the number of independent variables in the system (20) is small. On the other hand, the dimension of the kernel of matrix $\mathcal{A}$ is equal to the number of the fundamental circuits in the super-network; therefore, the method is especially effective to networks whose super-network has fewer loops. The extreme case is that this dimension equals to zero; i.e., the super-network is a tree. In this case, the $\mathbf{v}$ can be uniquely solved from the equation $\mathcal{A}\mathbf{v} = \mathbf{S}$.

In terms of attempting to solve the problem optimally, decomposition techniques can benefit from the network reduction method since, at a given iteration, fixing $\mathbf{v}$ implies all other flow variables in the system can also be determined due to the developments presented in the previous section.

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