Efficient Operation of Natural Gas Transmission Systems: A Network-Based Heuristic for Cyclic Structures

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Abstract

In this paper we propose a heuristic solution procedure for fuel cost minimization on gas transmission systems with a cyclic network topology, that is, networks with at least one cycle containing two or more compressor station arcs. Our heuristic solution methodology is based on a two-stage iterative procedure. In a particular iteration, at a first stage, gas flow variables are fixed and optimal pressure variables are found via dynamic programming. At a second stage, pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. Empirical evidence supports the effectiviness of the proposed procedure outperforming the best existing approach to the best of our knowledge.

Scope and Purpose

Gas transmission network problems differ from traditional network flow problems in some fundamental aspects. First, in addition to the flow variables for each arc, which in this case represent mass flow rates, a pressure variable is defined at every node. Second, besides the mass balance constraints, there exist two other types of constraints: (i) a nonlinear equality constraint on each pipe, which represents the relationship between the pressure drop and the flow; and (ii) a nonlinear non-convex set which represents the feasible operating limits for pressure and flow within each compressor station. The objective function is given by a nonlinear function of flow rates and pressures. In the real world, these type of instances are very large both in terms of the number of decision variables and the number of constraints, and very complex due to the presence of non-linearity and non-convexity in both the set of feasible solutions and the objective function.

Keywords: natural gas, pipeline optimization, transmission networks, heuristic, dynamic programming

1 Introduction

As natural gas pipeline systems have grown larger and more complex, the importance of optimum operation and planning of these facilities has increased. The investment costs and operation expenses of pipeline networks are so large that even small improvements in system utilization can involve substantial amounts of money.

The natural gas industry services include producing, moving, and selling gas. Our main interest in this study is focused on the transportation of gas through a pipeline network. Moving gas is divided into two classes: transmission and distribution. Transmission of gas means moving a large volume of gas at high pressures over long distances from a gas source to distribution centers. In contrast, gas distribution is the process of routing gas to individual customers. For both transmission and distribution networks, the gas flows through various devices including pipes, regulators, valves, and compressors. In a transmission network, gas pressure is reduced due to friction with the pipe wall as the gas travels through the pipe. Some of this pressure is added back at compressor stations, which raises the pressure of the gas passing through them.

In a gas transmission network, the overall operating cost of the system is highly dependent upon the operating cost of the compressor stations in a network. A compressor station's operating cost, however, is generally measured by the fuel consumed at the compressor station. According to Luongo, Gilmour, and Schroeder [12], the operating cost of running the compressor stations represents between 25% and 50% of the total company's operating budget. Hence, the objective for a transmission network is to minimize the total fuel consumption of the compressor stations while satisfying specified delivery flow rates and minimum pressure requirements at the delivery terminals.

Depending on how the gas flow changes with respect to time, we distinguish between systems in steady state and transient state. A system is said to be in steady state when the values characterizing the flow of gas in the system are independent of time. In this case, the system constraints, particularly the ones describing the gas flow through the pipes, can be described using algebraic non-linear equations. In contrast, transient analysis requires the use of partial differential equations to describe such relationships. This makes the problem considerably harder to solve from the optimization perspective. In fact, optimization of transient models is one of the most challenging areas of opportunity for future research. In the case of transient optimization, variables of the system, such as pressures and flows, are functions of time. In this work, we focus on steady-state gas transmission network problems with the objective of minimizing the operational costs.

Gas transmission network problems differ from traditional network flow problems in some fundamental aspects. First, in addition to the flow variables for each arc, which in this case represent mass flow rates, a pressure variable is defined at every node. Second, besides the mass balance constraints, there exist two other types of constraints: (i) a nonlinear equality constraint on each pipe, which represents the relationship between the pressure drop and the flow; and (ii) a nonlinear non-convex set which represents the feasible operating limits for pressure and flow within each compressor station. The objective function is given by a nonlinear function of flow rates and pressures.

The problem is very difficult due to the presence of a non-convex objective function and non-convex feasible region. Optimization algorithms (most of them based on dynamic programming) for non-cyclic gas network topologies are in a relatively well developed stage. However, effective algorithms for cyclic topologies are practically non-existent. While it is true that the majority of pipeline systems world-wide have non-cyclic structures, there is an important number of cyclic systems for which these results are applicable. So our work focuses on addressing gas transmission problems on cyclic topologies. A cyclic topology is a network with at least one cycle containing two or more compressor station arcs.

In this paper we propose a heuristic solution procedure for fuel cost minimization on gas transmission systems with a cyclic network topology. Our heuristic solution methodology is based on a two-stage iterative procedure. In a particular iteration, at a first stage, gas flow variables are fixed and optimal pressure variables are found via dynamic programming. At a second stage, the pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. Our procedure is compared with the best approach known to date to the best of our knowledge. Empirical evidence supports the effectiviness of this proposed procedure by providing solution with significantly better quality than those obtained by the existing approach.

The organization of this paper is as follows. In Section 2 we introduce the gas transmission network problem and present the mathematical model. Section 3 presents a survey of previous related work. In Section 4, we present the network formulation of the gas transmission system and discuss the network decomposition. In Section 5, we describe the proposed algorithm. Implementation, numerical examples and computational experiments are reported in Section 6. This is followed by our conclusions and directions for future research in Section 7.

2 Problem Definition

A gas transmission network is composed of pipelines, junction nodes, including supply and delivery nodes, and compressor stations. The existence of compressor stations in the network is one of the key characteristics of a gas transmission network. The transmission segment operates systems of pipes and compressors and attempts to move large quantities of gas over long distances. When traveling through the pipes, the gas pressure is reduced by friction with the pipe walls. Some of this pressure is added back at the compressor stations.

Compressor stations in a transmission network are complex entities typically involving a number of compressor units arranged in parallel or in serial at one location. The operating cost of the gas transmission network is usually ruled by the operating cost of the compressor stations. Therefore, representing compressor stations within a network configuration is quite an important issue and the way of representing them varies according to the solution methodology. We now present the model we use for this problem and discuss the most important assumptions.

2.1 Model for Steady-State Problem

Let $G = (\mathcal{N}, \mathcal{L}, \mathcal{M})$ be a directed network defined by a set $\mathcal{N}$ of n nodes, a set $\mathcal{L}$ of l pipes, and a set $\mathcal{M}$ of m compressor stations. Note that the set of arcs $\mathcal{A}$ in G is defined as $\mathcal{A} = \mathcal{L} \cup \mathcal{M}$, with $\mathcal{L} \cap \mathcal{M} = \emptyset$. The decision variables are v_{ij}, the mass flow rate at arc $(i,j) \in \mathcal{A}$ and p_i, the gas pressure at node $i \in \mathcal{N}$. At each node $i \in \mathcal{N}$, there is a known parameter s_i called the net flow through that node. Clearly, $s_i > 0$ ($s_i < 0$) implies node i is a source (delivery) node, whereas $s_i = 0$ means node i is just a transhipment node. In addition, pressure limits p_i^L and p_i^U are given at every node. For each pipe (i,j) the pipe resistance t_{ij}, which is computed from the pipe physical properties, is assumed to be known.

Note also that for each arc $(i,j) \in \mathcal{A}$ there are three associated variables: v_{ij}, p_i and p_j. In particular, in a compressor station arc, p_i and p_j are called suction and discharge pressures, respectively.

The goal of the problem is to minimize the total fuel consumption used by the stations while satisfying specified delivery requirements throughout the system. We assume that there is no transportation cost associated with ordinary pipes. At each compressor station (i,j), a cost $g_{ij}(v_{ij}, p_i, p_j)$ measured by the fuel consumption is incurred, so the objective function is the total amount of fuel consumed at the system.

The mathematical formulation of the gas transmission network problem (GTN) is given by

$$\text{(GTN)} \quad \text{minimize} \quad \sum_{(i,j)\in\mathcal{M}} g_{ij}(v_{ij}, p_i, p_j) \tag{1a}$$

$$\text{subject to} \quad \sum_{j:(i,j)\in\mathcal{A}} v_{ij} - \sum_{j:(j,i)\in\mathcal{A}} v_{ji} = s_i \qquad i \in \mathcal{N} \tag{1b}$$

$$p_i^2 - p_j^2 = t_{ij} v_{ij}^2 \qquad (i,j) \in \mathcal{L} \tag{1c}$$

$$p_i \in [p_i^L, p_i^U] \qquad i \in \mathcal{N} \tag{1d}$$

$$(v_{ij}, p_i, p_j) \in D_{ij} \subset R^3 \qquad (i,j) \in \mathcal{M} \tag{1e}$$

where D_{ij} is the feasible operating domain of compressor station (i,j) and $g_{ij}(v_{ij}, p_i, p_j)$ is the fuel consumed at station (i,j).

Flow balance conservation is given by Equation (1b). Constraint (1c) represents the dynamics of the pipe flow under isothermal and steady-state assumptions. The compressor station constraints (1e) are not expressed explicitly here. It suffices to know that the feasible set D_{ij} is a non-convex set. The details can be found in Wu et al. [28].

For measuring fuel consumption, we use a function g in the following form

$$g(v_{ij}, p_i, p_j) = \alpha v_{ij} \left\{ \left(\frac{p_j}{p_i} \right)^m - 1 \right\}, \quad (v_{ij}, p_i, p_j) \in D_{ij}, \tag{2}$$

where α is an assumed constant (and known) parameters which depend on the gas physical properties. A more detailed study on the nature of both the compressor station domain and the fuel consumption function is given in [28].

The problem is very difficult due to the non-convex nature of both the objective function and the feasible region. Furthermore, the feasible domain D_{ij} of the compressor station is not represented in algebraic form, but as a result of curve fitting methods based on empirical data for the compressor units used. An example of a typical 4-compressor domain in two dimensions (by fixing p_i) is given in Figure 7. Also, the type of underlying network topology becomes a crucial issue. That is, depending on the underlying network configuration, the problem can be more difficult to solve. It is well know that cyclic topologies are harder to solve than non-cyclic ones. We will discuss this issue in more detail in Section 4.

3 Literature Review

Dynamic programming (DP) has been by far the most popular technique for solving many classes of natural gas pipeline networks since the late 1960s. One of the main reasons is that, in a DP framework, it is relatively easy to satisfy the pipeline constraints and to handle the non-convexity of the feasible domain. Other techniques such as mathematical programming and hierarchical control methods have been applied as well with modest degrees of success. Mathematical programming is usually used for cyclic systems. Hierarchical control techniques can be more effective when the model of the compressor station is fairly complicated. In the following sections we present an overview of the most relevant work done on the solution methodologies used for steady state gas transmission networks.

3.1 Mathematical Programming Approaches

As we have seen in Section 2, problem GTN has a non-convex feasible region, and a non-convex objective function. In fact, it has been reported by Wu et al. [28] that this objective function typically has many local optima. These problem features make it very difficult to solve using classical techniques from mathematical programming. Several researchers have tried to apply mathematical programming techniques, but their approaches are based on inaccurate or oversimplified models of the compressor stations. Next we review the most significant works related with mathematical programming, which has been rather limited.

Pratt and Wilson [19] propose a successive mixed integer linear programming method. Their algorithm solves the nonlinear optimization problem iteratively by linearizing the pressure drop-flow equations (1c). Integer variables are included in the formulation for compressor unit selection, and the problem is solved using branch and bound. Percell and Ryan [18] propose an algorithm using a generalized reduced gradient method for minimizing the fuel consumption problem for a gas transmission network. We reemphasize the difficulties in handling the nonlinear equality constraints and the complex nature of the compressor stations in the gas transmission network as the key factors in the relatively poor success of mathematical programming approaches.

3.2 Hierarchical Control Approaches

Difficulty in solving a gas transmission network problem in an integrated way calls for other techniques, such as hierarchical structure in the solution process, which in turn demands an efficient method for decomposing of the problem. In a hierarchical control approach (Singh [22]), the overall network is decomposed into two levels: (i) the network state level and (ii) the compressor station level. The compressor station problem is the lowest level and the network state problem is the highest level. The local optimization of the compressor stations at the lowest level is the firm basis for an optimization procedure for the global minimization of total cost. Optimization of the compressor station subproblem has been studied previously by Osiadacz [13], Percell and Van Reet [17], and Wu, Boyd, and Scott [27].

Larson and Wismer [11] propose a hierarchical control approach for a transient operation of a gunbarrel pipeline system. Osiadacz and Bell [15] suggest a simplified algorithm for the optimization of the transient gas transmission network, which is based on a hierarchical control approach. The hierarchical control approach for transient models can be found in Anglard and David [2], Osiadacz [14], and Osiadacz and Swierczewski [16]. Some degree of success has been reported from these approaches as far as optimizing the compressor station subproblem. However, these approaches have limitations in globally optimizing the minimum cost. As mentioned in Carter, Schroeder, and Harbick [6], numerical simulation of the behavior of the gas transmission network is quite widespread, and with a successful compressor station optimizer, these simulations are quite accurate. However, little work has been done or even attempted in optimizing the gas transmission network. After the introduction of the hierarchical control approach, in which detailed compressor station optimization is attached as a lower level subproblem, several approaches have focused on the optimization of the higher level problem, the network state problem.

3.3 Dynamic Programming Approaches

Dynamic programming (DP) (Dreyfus and Law [8]) was first used for a steady state gas transmission system in Wong and Larson [24, 25]. The authors apply DP to the gunbarrel and diverging branch tree systems to solve the network state problem. The gunbarrel system, which is basically a single-path topology, posseses an appropriate serial structure so it can be solved via DP. For the diverging branch problem, the overall problem is decomposed into a sequence of several one-dimensional DP problems, each of which deals with a single branch.

There are, however, some limitations to the DP method described by Wong and Larson. First, its application is limited only to tree networks. Second, the method assumes that there is only a single compressor unit installed within each compressor station. In addition, the feasible domain of the compressor station is oversimplified in order to make an easier solution process. A successful commercial optimizer for tree-structured gas transmission networks using DP was developed by Zimmer [29], and Lall and Percell [10].

When applying DP, the underlying network configuration of the given problem can enhance the solution process. Note that for these non-cyclic systems, the DP formulation is one-dimensional and can be easily applied because it has been shown [21] that the flow variables can be uniquely determined beforehand and thus eliminated from the problem, so we only deal with the pressure variables. That is, in a tree network, once both supply and delivery flow rates are given, the steady state flows can be uniquely determined. This nice property does not hold for a cyclic system. The existence of cycles breaks the serial structure, so the flow variables must be explicitly handled, which makes the underlying DP multidimensional.

The DP approach for cyclic systems has been limited. There were some efforts in applying DP on the nonsequential structure for applications in chemical engineering problems. Two of the earliest works include Wilde [23] and Aris, Nemhauser, and Wilde [3]. Both groups treated cycles via "cuts", i.e., they cut one end of a cycle in the system and treat the resulting system using DP for tree structures. More general issues of the nonsequential DP can be found in Bertele and Brioschi [4].

Although several researchers over the years have addressed branched systems in the gas transmission problem, cyclic systems were not addressed until Luongo, Gilmour, and Schroeder [12]. In their study, the authors apply DP with the assumption that the flow rates through pipes and at the compressor stations are fixed. After solving the DP problem with prefixed flow rates, they use a direct search method with multiple restarts on different flow settings. Thus their approach is a hybrid of DP and either brute-force enumeration or simulated annealing, depending on problem size. Recently Carter [5] proposed a DP approach on more general structures with flow rates being fixed. A more detailed description of DP approaches to gas transmission networks can be found in Ríos-Mercado [20].

The principal obstacle to date using DP is that its application is limited in practice to simple structures. Otherwise, the computational effort becomes too large to be practical. This leads to an interesting question of how to find the optimal setting of the flow variables and how to modify the current flow setting to obtain a better objective value. This study focuses on these issues.

4 Natural Gas Network Decomposition

In our solution procedure, we first decompose the original graph into several subgraphs (not containing compressor stations), and we further contract each subgraph, yielding a reduced network. It turns out that the network configuration for each subgraph is not crucial, while the network configuration of the reduced network is critical. In this section we will see how the underlying network topology becomes a key issue, and plays a very important role in developing solution procedures for this problem.

4.1 Modeling Assumptions

In this section we consider the general network configuration of the gas transmission network and discuss several assumptions about the underlying network configuration. Gas pipeline networks are very complex entities. To make our mathematical model simple enough, several assumptions are made through our discussion.

Assumption 1 *A gas transmission network is composed of only nodes, pipes, and compressor stations.*

Under Assumption 1, the only control variables over the network are the ones in the compressor stations $((v, p_s, p_d))$. A compressor station may be in one of three states: active, bypassed, or closed. In our network model these states are not explicitly represented in the formulation, but are decided in the context of the optimization problem. If the discharge pressure value is greater than the suction pressure value at the compressor station, then the compressor station is said to be active. If the discharge pressure value equals to the suction pressure value, then the compressor station is bypassed. When the flow rate through the compressor station is zero, then the compressor station is closed. For both bypassed and closed compressor stations, the operating costs are zero.

Assumption 2 *There are no self-loops in the system.*

Assumption 3 *Bypass over any compressor station is not allowed.*

Any directed sequence of pipes connecting node i to node j, for which there is an alternate path containing at least one compressor station, is considered a bypass. In Figure 1, the dotted arcs $(3, 5)$ and $(4, 10)$ bypass compressor station $(3, 5)$ and $(8, 9)$, respectively, so they are not allowed in our model.

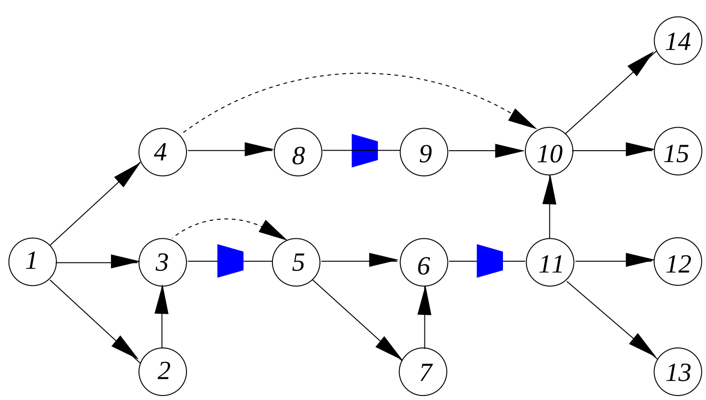


Figure 1: Example of bypasses in a gas transmission network.

Under assumptions 2 and 3, the gas transmission network problem can be viewed as a directed network whose components are nodes, pipeline arcs, and compressor station arcs.

Assumption 4 *At each node $i \in \mathcal{N}$ of G, the net inflow s_i is assumed to be known with certainty. Moreover, at each delivery node, minimum requirement of the pressure is assumed to be known.*

This assumption is not restrictive, and it stems from the fact that the amount of gas to be delivered to or taken from a given node is usually known in advance.

4.2 Decomposition

Now consider the gas transmission network $G = (\mathcal{N}, \mathcal{L}, \mathcal{M})$, with n nodes, l pipes, and m compressor stations, and its mathematical model GTN defined in (1). Let $\mathbf{A}_{\mathcal{L}}$ be the $n \times l$ node-pipe incidence matrix, $\mathbf{A}_{\mathcal{M}}$ be the $n \times m$ node-compressor station incidence matrix, and $\mathbf{A} = (\mathbf{A}_{\mathcal{L}} \mid \mathbf{A}_{\mathcal{M}})$. This partition induces a corresponding partition of the flow variables $\mathbf{v} = (\mathbf{v}^{\mathcal{L}} \mid \mathbf{v}^{\mathcal{M}})$. The equality constraint set of the problem GTN can be represented in vector form by

$$\begin{cases} \mathbf{A}\,\mathbf{v} \;=\; \mathbf{s}, \\ \mathbf{A}_{\mathcal{L}}^T\,\mathbf{p}^2 \;=\; \phi(\mathbf{v}^{\mathcal{L}}), \end{cases} \tag{3}$$

where $\mathbf{p}^2 = (p_1^2, ..., p_n^2)^T$ and $\phi(\mathbf{v}^{\mathcal{L}})$ is the vector of $\phi_{ij}(v_{ij})'s,\ (i,j) \in \mathcal{L}$, in which $\phi_{ij}(v_{ij}) = t_{ij} v_{ij}^2$.

Since we assume that there is no bypass, if we delete every compressor station from the entire system, then we have disconnected subgraphs, each one of which being composed only of pipes and nodes. Figure 2 shows an example of the subgraphs created by deleting compressor stations.

Each node or pipe of the G belongs to exactly one subgraph. It follows that the subgraphs of G determine a unique partition of its nodes and pipes. Assume that there are r subgraphs. Let $G_h = (\mathcal{N}_h, \mathcal{L}_h),\ h = 1, .., r$, be the subgraph defined by a set of nodes, $\mathcal{N}_h$, and a set of pipes, $\mathcal{L}_h$. We assume G_h has n_h nodes and l_h pipes. Note that the nodes in G can be renumbered in such a way that its node-pipe incidence matrix takes the block diagonal form

$$\mathbf{A}_{\mathcal{L}} = \begin{pmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_r \end{pmatrix}, \tag{4}$$

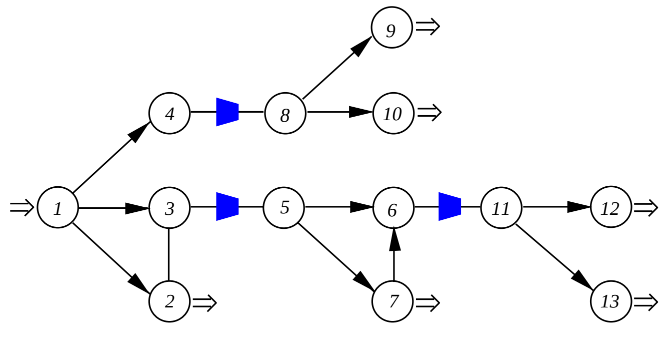
where $\mathbf{A}_h$ is the node-pipe incidence matrix for G_h, $h = 1, \ldots, r$.

Let $\mathbf{v}_h$ be the vector of mass flow rates through the pipes of G_h, and $\mathbf{p}_h$ be the pressure vector for each node of G_i, i.e.,

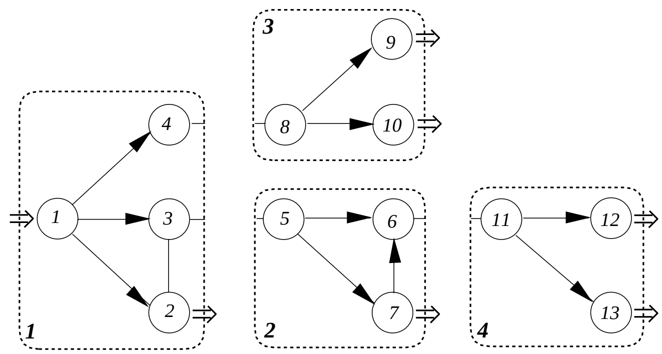
$$\mathbf{v}_h = \{v_{ij} \mid (i,j) \in \mathcal{L}_h\}, \quad \mathbf{p}_h = \{p_i \mid i \in \mathcal{N}_h\}.$$

Let $\mathbf{b}_h$ be the vector of net inflow at nodes located in subgraph G_h, that is, $\mathbf{b}_h = \{s_i \mid i \in \mathcal{N}_h\}$. Since $\mathbf{A} = (\mathbf{A}_{\mathcal{L}} \mid \mathbf{A}_{\mathcal{M}})$ and $\mathbf{v} = (\mathbf{v}^{\mathcal{L}}, \mathbf{v}^{\mathcal{M}})^T$, flow balance equations of system (3) become

$$\mathbf{A}_{\mathcal{L}}\,\mathbf{v}^{\mathcal{L}} + \mathbf{A}_{\mathcal{M}}\,\mathbf{v}^{\mathcal{M}} \;=\; \mathbf{s}. \tag{5}$$



(a) Network G



(b) Subgraphs G_1, G_2, G_3 and G_4

Figure 2: Example of the subgraphs after removal of compressor arcs.

Then, by using (4), equation (5) can be rewritten as

$$\begin{pmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_r \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_r \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{\mathcal{M}_1} \\ \mathbf{A}_{\mathcal{M}_2} \\ \vdots \\ \mathbf{A}_{\mathcal{M}_r} \end{pmatrix} \mathbf{v}^{\mathcal{M}} \;=\; \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_r \end{pmatrix},$$

where $\mathbf{A}_{\mathcal{M}_h}$ is the rearranged node-station incidence matrix corresponding to $G_h, h = 1, \ldots, r$. That is, we decompose the set of flow balance equation constraints to each subgraph G_h by

$$\mathbf{A}_h \, \mathbf{v}_h \; + \mathbf{A}_{\mathcal{M}_h} \, \mathbf{v}^{\mathcal{M}} \;=\; \mathbf{b}_h, \quad h = 1, \ldots, r.$$

Since the sets $\mathcal{L}_h$ of pipes in $G_h, h = 1, \ldots, r$ are disjoint, the nonlinear pressure drop constraint set in (3) can be naturally decomposed as

$$\mathbf{A}_h^T \mathbf{p}_h^2 = \phi(\mathbf{v}_h), \quad h = 1, ..., r, \tag{6}$$

where $\mathbf{p}_h^2$ is the vector of p_i^2's, $i \in \mathcal{N}_h$, and $\phi(\mathbf{v}_h)$ is the vector of $\phi_{ij}(v_{ij})$'s, $(i, j) \in \mathcal{L}_h$, in which

$\phi_{ij}(v_{ij}) = t_{ij}v_{ij}^2$. Therefore system (3) can be decomposed for each subgraph as

$$\begin{cases} \mathbf{A}_h \, \mathbf{v}_h \ + \mathbf{A}_{\mathcal{M}_h} \, \mathbf{v}^{\mathcal{M}} \ = \ \mathbf{b}_h, \\ \mathbf{A}_h^T \mathbf{p}_h^2 = \phi(\mathbf{v}_h), \end{cases} \qquad i = 1, \ldots, r. \tag{7}$$

Since $\mathbf{A}_h$ is the node-pipe incidence matrix for G_h, we have (by Proposition 3 in Kim [9])

$$\mathbf{A}_h \mathbf{C}_{T_h}^T = \mathbf{C}_{T_h} \mathbf{A}_h^T = \mathbf{0}, \quad h = 1, ..., r,$$

where $\mathbf{C}_{T_h}$ is the cycle matrix of G_h with respect to the spanning tree T_h of G_h. If we multiply $\mathbf{C}_{T_h}$ on both sides of (6), we have

$$\mathbf{C}_{T_h} \phi(\mathbf{v}_h) = \mathbf{0}, \quad h = 1, ..., r.$$

Hence for each $G_h, h = 1, \ldots, r$, we have the following system

$$\begin{cases} \mathbf{A}_h \, \mathbf{v}_h \ + \mathbf{A}_{\mathcal{M}_h} \, \mathbf{v}^{\mathcal{M}} \ = \ \mathbf{b}_h, \\ \mathbf{C}_{T_h} \, \phi(\mathbf{v}_h) \ = \ \mathbf{0}, \\ \mathbf{A}_h^T \mathbf{p}_h^2 = \phi(\mathbf{v}_h). \end{cases} \tag{8}$$

The advantage of system (8) over (7) is that if the mass flow rates through the compressor stations $\mathbf{v}^{\mathcal{M}}$ are known, then the flow variables, $\mathbf{v}_h$, can be solved separately from the n_h pressure variables. That is, the first two equations in (8) can be used for solving $\mathbf{v}_h$ if $\mathbf{v}^{\mathcal{M}}$ is fixed. The system of equations for solving $\mathbf{v}_h$ for the h-th subnetwork G_h becomes

$$\begin{cases} \mathbf{A}_h \, \mathbf{v}_h \ = \ \mathbf{b}'_h, \\ \mathbf{C}_{T_h} \, \phi(\mathbf{v}_h) \ = \ \mathbf{0}, \end{cases} \tag{9}$$

where $\mathbf{b}'_h = \mathbf{b}_h - \mathbf{A}_{\mathcal{M}_h} \mathbf{v}^{\mathcal{M}}$.

Formal proof of the uniqueness and existence of the solution of the above system (9) can be found in Ríos-Mercado et al. [21]. If the configuration of G_h is a tree, then no cycle matrix is defined, and we have only $\bar{\mathbf{A}}_h \, \mathbf{v}_h \ = \ \mathbf{b}'_h$, which is trivial to solve. If G_h has cycles, a method such as the modified Newton method can be used to solve the nonlinear system 9.

Decomposing the given network into subgraphs gives us insight into the structure and motivation for the development of a solution procedure. That is, once we fix the flow rates $\mathbf{v}^{\mathcal{M}}$ at the compressor stations, the rest of the flow variables are calculated at each subgraph by solving system (9). Then we are left with the pressure variables $\mathbf{p}$ as unknown variables, and with pressure drop equations $\mathbf{A}_h^T \mathbf{p}_h{}^2 = \phi(\bar{\mathbf{v}}_h)$, where $\bar{\mathbf{v}}_h$ is the known flow vector for G_h. We need to address two issues: (i) how to solve the rest of the problem which contains only pressure variables, and (ii) how to modify the flow rates through the compressor stations. The first issue will be handled by dynamic programming. To answer the second question, we further consider the network configuration of the entire system, which is the purpose of the following two subsections.

4.3 The Associated Reduced Network

As we have seen in the previous section, once the flow variables $\mathbf{v}^{\mathcal{M}}$ through the compressor stations are fixed, then at each subgraph $G_h, h = 1, \ldots, r$, regardless of its configuration, the flow variables through the pipes are uniquely determined. Hence, we focus on analyzing how the compressors participate in the entire network structure, rather than analyzing the individual subgraphs.

For this purpose, we introduce the concept of a *reduced network*. Let $G' = (\mathcal{V}, \mathcal{E})$ be the reduced network of $G = (\mathcal{N}, \mathcal{L}, \mathcal{M})$, where $\mathcal{V}$, $\mathcal{E}$ are the set of nodes and the set of arcs of G'. Each node of G' corresponds to exactly one subgraph of G. It follows that the subgraphs of a graph determine a unique partition of its nodes. Suppose we contract all those arcs, i.e., pipes, which lies in each subgraph. Then the resulting contraction network has the appearance of the network in Figure 3 which is the reduced network G' of the example network G in Figure 2(a).

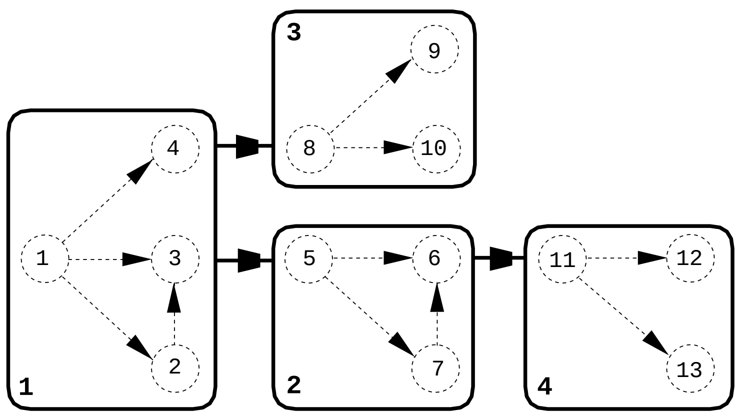
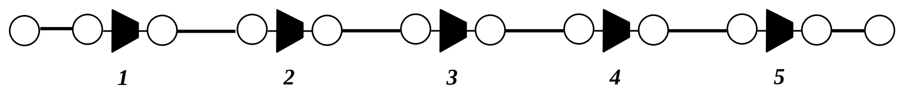


Figure 3: Example of the reduced network G'.

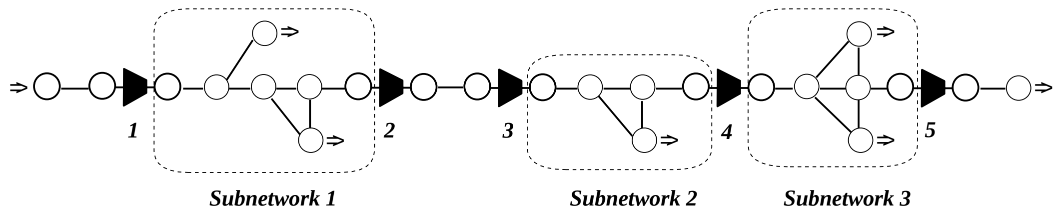
Note that each node of the reduced network G' is identified with a subgraph G_h of G. The set of arcs $\mathcal{E}$ of G' is the same as $\mathcal{M}$, the set of compressor stations of G. That is, an arc in G' corresponds to a compressor station in G. Note that G' is a connected directed network with no self-loops. Hence, it could have various configurations, which we divide into three classes: a gunbarrel system, a tree structured system, and a cyclic system. Throughout the following chapters, when we mention the network configuration of a gas transmission network, we mean the configuration of the reduced network G'.

In the reduced network G', since we contract a whole subgraph into a node, the configuration of each subgraph does not affect the configuration of a reduced network. For example, in Figure 3, even though the original network example has cycles, its reduced network is acyclic, and thus is considered a tree structure. Similarly, in Figure 4 below, the reduced networks of Figure 4(a) and 4(b) are the same, and they both are gunbarrel systems, even though the network in Figure 4 (b) has a cycle in subgraphs G_1, G_2, and G_3.

If the reduced network G' is a non-cyclic system (i.e., a gunbarrel or a tree system) then the flow



(a) Gunbarrel system with 5 compressor stations.



(b) Gunbarrel system with a subgraph between compressor stations.

Figure 4: Example of a gunbarrel system.

rates through the arcs in $\mathcal{E}$ of G', which correspond to the flow rates of each compressor station in G, are uniquely determined. If G' has a cycle, then optimal $\mathbf{v}^{\mathcal{M}}$ values have to be determined in the context of the underlying minimization problem. In our solution procedure, even though G' has cycles, we will start with a feasible value for $\mathbf{v}^{\mathcal{M}}$, and once $\mathbf{v}^{\mathcal{M}}$ is fixed, we have r independent nonlinear systems (9) (with unique solution as previously discussed), each of which corresponds to a subnetwork G_h, $h = 1, \ldots, r$ of G. By solving system (9) for all subgraphs G_h, $h = 1, \ldots, r$, we have a full profile of flow variables for the whole network with respect to the fixed flows of the compressor stations. The remaining problem is now to determine the pressure variables. In the following section, we consider this problem and discuss a solution procedure.

4.4 Handling the Subproblem

In the previous section, we see that once the flow values of the compressor stations are fixed, we can determine all other flow variables by solving system (9) for each subgraph independently. This in effect eliminates the flow variables from the whole formulation, leaving only pressure variables as unknowns. That is, for each subnetwork G_h, $h = 1, \ldots, r$, we have

$$\mathbf{A}_h^T \mathbf{p}_h{}^2 \quad = \quad \phi(\bar{\mathbf{v}}_h), \tag{10}$$

where $\bar{\mathbf{v}}_h$ is the vector of known pipe flows $\bar{v}_{ij}$, $(i,j) \in \mathcal{L}_h$. Using the original notation, by fixing the flow variables $\bar{v}_{ij}$, we have the following optimization problem with only pressures involved as unknown variables:

$$\text{minimize} \quad \sum_{(i,j)\in\mathcal{M}} \overline{g}_{ij}(\bar{v}_{ij}, p_i, p_j) \tag{11a}$$

$$
\begin{aligned}
\text{subject to} \qquad p_j^2 - p_k^2 &= t_{jk}\bar{v}_{jk}^2 \qquad && (j,k) \in \mathcal{L} && \text{(11b)}\\
p_j &\in [p_j^L, p_j^U] && j \in \mathcal{N} && \text{(11c)}\\
(\bar{v}_{ij}, p_i, p_j) &\in D_{(i,j)} \subset R^3 && (i,j) \in \mathcal{M} && \text{(11d)}
\end{aligned}
$$

Consider equation (10) above. Since each G_h is connected, if any pressure variable, say $p_j, j \in \mathcal{N}_h$, is known, then all other pressure variables defined at the other nodes in G_h are just calculated using (10). Moreover, between any two nodes j, k in G_h, if there exists a direct path $P_{<j,k>}$ from node j to node k, then p_j (p_k) can be written as a function of p_k (p_j). Using these two observations, we can further reduce the problem at each subgraph. That is, we include the nonlinear relationships between the reference nodes, such as suction, discharge, source and delivery nodes of G_h, and eliminate other nonlinear equations from the problem model. Hence we simplify the problem so that the reduced problem only contains the pressure variables of the reference nodes of each subgraph G_h, $h = 1, \ldots, r$.

At each subgraph G_h, gas flow enters G_h through either discharge nodes or supply nodes, i.e., gas sources. Similarly, some gas flow comes out of G_h through either suction nodes or delivery nodes. Let $\mathcal{N}_h^{dc}, \mathcal{N}_h^{sc}, \mathcal{N}_h^{su}$, and $\mathcal{N}_h^{dv}$ be the set of discharge, suction, source, and delivery nodes in G_h, respectively, and let $\mathcal{N}_h^{in} = \mathcal{N}_h^{dc} \cup \mathcal{N}_h^{su}$ and $\mathcal{N}_h^{out} = \mathcal{N}_h^{sc} \cup \mathcal{N}_h^{dv}$ be the set of *input nodes* of G_h and the set of *output nodes* of G_h, respectively.

Let j_d and j_s be a discharge and a suction node of G_h, respectively, i.e., $j_d \in \mathcal{N}_h^{dc}, j_s \in \mathcal{N}_h^{sc}$. Let $P_{<j_d,j_s>}$ be a direct path from j_d to j_s. Note that nodes j_d, j_s are from two different compressor stations connected with subgraph G_h. A path, $P_{<j_d,j_s>}$, is represented by a sequence of pipes, i.e., $\{(j_d, j_1), (j_1, j_2), \ldots, (j_q, j_s)\}$, where $j_l \in \mathcal{N}_h,\ l = 1, \ldots, q$. For each pipe in the path $P_{<j_d,j_s>}$, a pressure drop equation (11b) is defined. Hence, we have

$$
\begin{cases}
p_{j_d}^2 - p_{j_1}^2 = t_{j_d j_1}\bar{v}_{j_d j_1}^2, \\
p_{j_1}^2 - p_{j_2}^2 = t_{j_1 j_2}\bar{v}_{j_1 j_2}^2, \\
\qquad \vdots \\
p_{j_q}^2 - p_{j_s}^2 = t_{j_q j_s}\bar{v}_{j_q j_s}^2.
\end{cases} \tag{12}
$$

Adding the above equations yields

$$
p_{j_d}^2 - p_{j_s}^2 = \sum_{(l,m) \in P_{<j_d,j_s>}} t_{lm}\bar{v}_{lm}^2. \tag{13}
$$

Thus equation (13) replaces equations (12) in the model. Likewise, between any two nodes i and j, where $i \in \mathcal{N}_h^{in}$ and $j \in \mathcal{N}_h^{out}$, if there exist a path $P_{<i,j>}$, we can obtain a nonlinear relationship such as equation (13), and include it in the reduced formulation.

Based on the same logic, we can also convert the pressure bounds of node $i \in \mathcal{N}_h$ into the bounds of the other pressure variable $p_i, i \in \mathcal{N}_h^{dc}, \mathcal{N}_h^{sc}$. For example, consider the minimum pressure requirement constraint $p_j \geq p_j^L, j \in \mathcal{N}_h^{dv}$ for the delivery node j in G_h, and let $P_{<i,j>}$ be a path from the discharge node $i \in \mathcal{N}_h^{dc}$ to node $j \in \mathcal{N}_h^{dv}$. We can eliminate p_j from the problem formulation, and include a new pressure bound for p_i. That is, we have

$$p_i \geq \sqrt{(p_j^L)^2 + \sum_{(l,m)\in P_{<i,j>}} t_{lm}\bar{v}_{lm}^2}.$$

In general, the bounding inequality $p_j^L \leq p_j \leq p_j^U$ of any node $j \in \mathcal{N}_h$ of G_h is replaced by the pressure bounds of the variable $p_i, i \in \mathcal{N}_h^{sc}$ or $i \in \mathcal{N}_h^{dc}$ with

$$\alpha_{i,j} = \sqrt{(p_j^L)^2 + \sum_{(l,m)\in P_{<i,j>}} t_{lm}\bar{v}_{lm}^2} \leq p_i \leq \sqrt{(p_j^U)^2 + \sum_{(l,m)\in P_{<i,j>}} t_{lm}\bar{v}_{lm}^2} = \beta_{i,j}, \quad i \in \mathcal{N}_h^{dc}, \tag{14}$$

$$\alpha_{j,i} = \sqrt{(p_j^L)^2 - \sum_{(l,m)\in P_{<j,i>}} t_{lm}\bar{v}_{lm}^2} \leq p_i \leq \sqrt{(p_j^U)^2 - \sum_{(l,m)\in P_{<j,i>}} t_{lm}\bar{v}_{lm}^2}, = \beta_{i,j} \quad i \in \mathcal{N}_h^{sc}. \tag{15}$$

The above procedure is motivated by the following. First, the reduced problem contains only pressure variables at node $i \in \mathcal{N}_h^{in} \cup \mathcal{N}_h^{out}$, $h = 1, \ldots, r$, and hence the problem size is reduced. Second, the reduced problem can be converted into the sequential decision structure, which enables us to efficiently apply dynamic programming. If we let $\mathcal{B}_h = \{(i,j) \,:\, i \in \mathcal{N}_h^{in},\, j \in \mathcal{N}_h^{out}\}$, then the reduced problem, denoted as $Q_{\bar{\mathbf{v}}}(\mathbf{p})$, can be formulated as follows.

$$
\begin{aligned}
&\text{Minimize} && \sum_{(i,j)\in\mathcal{M}} \overline{g}_{ij}(\bar{v}_{ij}, p_i, p_j) && && \text{(16a)}\\
&\text{subject to} && p_j^2 - p_k^2 \;=\; \sum_{(l,m)\in P_{<j,k>}} t_{lm}\bar{v}_{lm}^2 && (j,k) \in \mathcal{B}_h,\ h = 1,\ldots,r && \text{(16b)}\\
& && \max\{p_j^L, \alpha_{j,k}\} \leq p_j \leq \min\{p_j^U, \beta_{j,k}\} && j \in \mathcal{N}_h^{dc},\ h = 1,\ldots,r && \text{(16c)}\\
& && \max\{p_j^L, \alpha_{k,j}\} \leq p_j \leq \min\{p_j^U, \beta_{k,j}\} && j \in \mathcal{N}_h^{sc},\ h = 1,\ldots,r && \text{(16d)}\\
& && (\bar{v}_{ij}, p_i, p_j) \in D_{(i,j)} \subset R^3 && (i,j) \in \mathcal{M} && \text{(16e)}
\end{aligned}
$$

where $\alpha_{j,k}, \alpha_{k,j}, \beta_{j,k}$ and $\beta_{k,j}$ are defined in equations (14)-(15).

The degree of the sequential decision structure of this problem varies depending on the nature of the subgraph. That is, if every G_h, $h = 1, \ldots, r$ has a single input node $i \in \mathcal{N}_h^{in}$ and a single output node $j \in \mathcal{N}_h^{out}$, then the reduced problem of G has an appropriate sequential decision structure. Any gunbarrel system has such structure for which DP is well defined. However, if the subgraph has multiple suction nodes or has multiple discharge nodes, then the reduced problem $Q_{\bar{v}}(\mathbf{p})$ has a nonsequential structure representing diverging or converging branches, respectively.

There are some limitations in applying DP for some complex configurations of the subgraph, and we assume the following.

Assumption 5 *Each subgraph can have either multiple discharge nodes with one suction node or have multiple suction nodes with one discharge node, but cannot have multiple discharge nodes and multiple suction nodes at the same time.*

The above restriction is made for purpose of modeling via DP. Under the above assumption, we have the following cases related to the configuration of G_h, $h = 1, \ldots, r$. That is,

Case 1 *G_h has a single node $i \in \mathcal{N}_h^{in}$ and a single node $j \in \mathcal{N}_h^{out}$.*

Case 2 *G_h has a single node $i \in \mathcal{N}_h^{in}$ and multiple nodes $j \in \mathcal{N}_h^{out}$.*

Case 3 *G_h has multiple nodes $i \in \mathcal{N}_h^{in}$ and a single node $j \in \mathcal{N}_h^{out}$.*

In a gunbarrel network, every subgraph G_h has the configuration stated in Case 1, and hence the sequential decision structure is maintained throughout the network. However, in the tree and the cyclic structures, we have a combination of G_h's having the various configurations from case 1 to case 3.

5 Heuristic Description

Our solution procedure is based on a two-stage iterative procedure. In a particular iteration, at the first stage, gas flow variables are fixed and optimal pressure variables are found via DP. At the second stage, the pressure variables are fixed and an attempt is made to find a set of flow variables that improve the objective function by exploiting the underlying network structure. Figure 5 shows an overview of the proposed procedure.

5.1 Dynamic Programming for the Reduced Problem

In the previous section, we reduced the problem into a form which had only pressure variables defined at the input or output nodes of G_i. Depending on the configuration of G_i, the problem has a sequential decision structure or a nonsequential structure. DP can be applied directly to the sequential decision structure, such as a gunbarrel transmission system. For the nonsequential structure, nonsequential DP is applied.

So given DP is a fairly well studied technique and has been described widely our focus here is in elaborating in how we modify the flows. The details of the DP implementation can be found in the work of Carter [5] and in Chapter 5 of Kim's dissertation [9]. The main point here is that we do know how to solve effectively for pressures when flows are known or fixed.

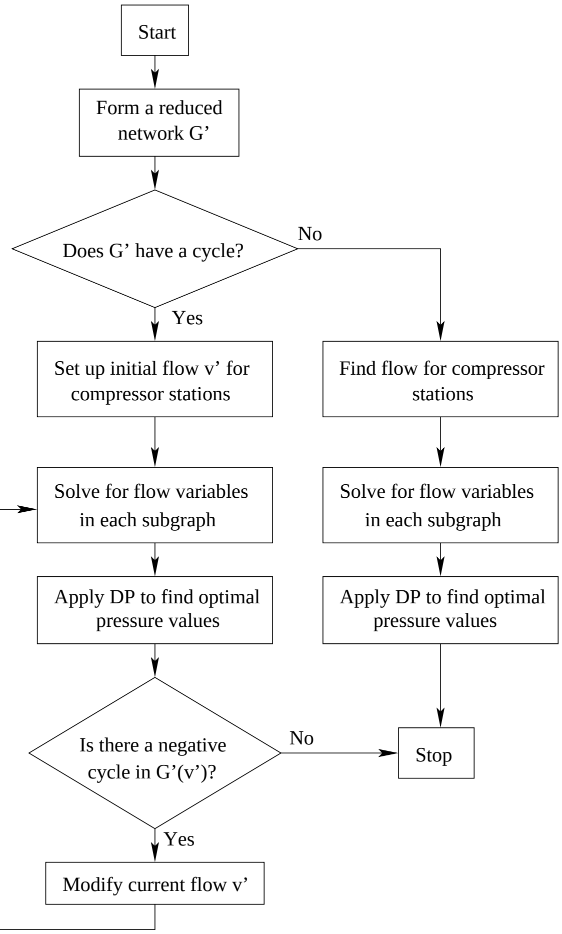


Figure 5: Overview of the solution procedure.

5.2 The Flow Modification Heuristic

Let $\mathbf{x}^0 = (\mathbf{v}^0, \mathbf{p}^0)$ be an initial feasible solution to (1). As we have seen in Section 4, once the flow rates through the compressor station $(\mathbf{v}^{\mathcal{M}})^0$ are given, the other flow variables defined at each pipe $(\mathbf{v}^{\mathcal{L}})^0$ can be computed in each subgraph. Now with given flow variables $\mathbf{v}^0$, the problem can be simplified into the reduced problem $Q_{\mathbf{v}^0}(\mathbf{p})$. Let $\mathbf{p}^0$ be the solution of the reduced problem $Q_{\mathbf{v}^0}(\mathbf{p})$.

For a tree structured gas transmission network, since the flow variables $\mathbf{v}^0$ are uniquely determined, no flow modification step is needed. However, for a cyclic structure, one may attempt to obtain a better objective function value by modifying the current flow setting $\mathbf{v}^0$. For this purpose, we make use of the residual network and negative cycle concepts ([1]), which are used in network theory for finding augmenting paths or proving optimality. Let $G(\mathbf{v}^0)$ represent the residual network corresponding to the flow $\mathbf{v}^0$ of the original network G.

The following two fundamental theorems, which we state here without proof (can be found in [1]), relate residual networks to negative cycles.

Theorem 1 *(Augmenting Cycle Theorem): Let* $\mathbf{v}$ *and* $\mathbf{v}^0$ *be any two feasible solutions of a network flow problem. Then* $\mathbf{v}$ *equals* $\mathbf{v}^0$ *plus the flow on a cycle in* $G(v^0)$*. Furthermore, the cost of* $\mathbf{v}$ *equals the cost of* $\mathbf{v}^0$ *plus the cost of flow on the augmenting cycle.*

Theorem 2 *(Negative Cycle Optimality Condition): A feasible solution* $\mathbf{v}^0$ *of the minimum cost flow problem is an optimal solution if and only if the residual network* $G(\mathbf{v}^0)$ *contains no negative cost directed cycles.*

Both theorems assume that the incremental cost of the flow is constant. From Theorem 1, we know that augmenting some amount of flow though a cycle does not violate the flow conservation. Thus the new flow is also feasible with respect to network flow constraints. Moreover, by the second part of Theorem 1, if we have a negative cycle, then augmenting some amount of flow through the negative cycle will yield a reduced total cost. By Theorem 2, if there exists a negative cycle with regard to the current feasible flow, then the current flow value is not optimal. Specifically, since our problem has a non-convex feasible region, this last result may not necessarily hold, however, it can be used in a heuristic way to guide a search for a better solution.

Using these two results, we want to develop a scheme to modify the current flow rates $\mathbf{v}^0$ to the new flow rates $\mathbf{v}'$ which will yield a better objective value. To do this, we first create the residual network $G'(\mathbf{v}^0)$ of the reduced network G' with respect to the current flow variable vector $\mathbf{v}^0$. Figure 6 shows an example of a reduced network and its residual network.

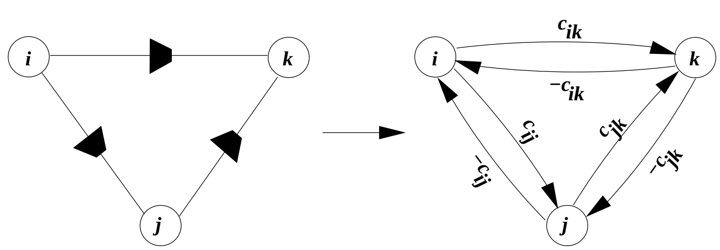


Figure 6: A reduced network and its residual network.

Note that a self-cycle which includes both a forward arc (i, j) and a backward arc (j, i) of the compressor station (i, j), is not considered a cycle. That is, either the forward or the backward arc of the compressor station (i, j) should be included in the cycle, but not both. For example, in Figure 6, self-cycles $i - k - i$ and $i - k - j - k - i$, are not considered as cycles. Note also that the starting node of the cycle is not important. In Figure 6, there exist two cycles, the (clockwise) cycle $i - k - j - i$, and the (counterclockwise) cycle $i - j - k - i$. We denote the cycle in the residual network by C, and denote the set of the compressor stations in a cycle C by $\mathcal{M}_C$.

In our heuristic flow modification step, the costs of the residual network are approximated by the derivatives of the objective function $\overline{g}$ with respect to the flow on each compressor station. Then,

$$c_{ij} \approx a_{ij} \left\{ \left(\frac{p_j}{p_i} \right)^m - 1 \right\},$$

where p_i and p_j are the current solution values delivered by DP with fixed flow variables, and α as defined in 2. This cost c_{ij} is assigned to each forward arc of the residual network and $-c_{ij}$ is assigned to each backward arc. The cycle cost τ_C, the total cost of the cycle C in a residual network, is defined by

$$\tau_C = \sum_{(i,j) \in \mathcal{M}_C} \delta_{ij}(C) \cdot c_{ij},$$

where $\delta_{ij}(C)$ equals 1 if arc (i,j) is contained in cycle C and is a forward arc of $G(\mathbf{v}^0)$, -1 if arc $(j,i) \in C$ and is a backward arc of $G'(\mathbf{v}^0)$, and 0 otherwise. If τ_C is negative, then we call C a negative (cost) cycle and denote it as C^-.

Modification of the flow is done by augmenting flow through a negative cycle C^-. That is, if there exists a negative cost cycle C^-, then we augment positive flow through C^-, and hence update the current flow setting. This flow modification step can be represented by

$$\mathbf{v}^{new} = \mathbf{v}^0 + \lambda \cdot \delta(C^-), \tag{18}$$

where $\lambda > 0$ is the positive amount of flow which will be added through the cycle, and $\delta(C^-)$ is the vector of $\delta_{ij}(C^-)$'s, a vector representing the negative cycle C^-. The flow modification step can be viewed as a descent nonlinear programming algorithm in which we try to find a direction (a vector of flow modification) such that by moving λ units in this direction, the objective function decreases. In our heuristic procedure, a negative cycle vector $\delta(C^-)$ corresponds to the search direction. The value λ is bounded below by zero and above by $\lambda_{\max}$, which can be obtained by considering the complex inequality constraint set D_{ij}, $(i,j) \in C^-$. If $\lambda_{\max} = 0$, then the algorithm stops. Otherwise, we set $\lambda = \mu\lambda_{\max} > 0$, where $0 < \mu < 1$. The purpose of multiplying by a small parameter μ is explained in the following subsection.

For the newly obtained flow setting $\mathbf{v}^{new}$, we need to find pressure variables, which requires us make use of DP again with the updated flow $\mathbf{v}^{new}$. If DP with $\mathbf{v}^{new}$ has no feasible solution, or no improvement to the objective value has been achieved, we reduce the size of λ by setting $\lambda = \gamma\lambda$, where $0 < \gamma < 1$, and apply DP until we get a better result. The algorithm is summarized below.

Step 1: Find an initial feasible solution $\mathbf{x}^0 = (\mathbf{v}^0, \mathbf{p}^0)$, set the iteration counter $t = 0$, μ $(0 < \mu < 1)$, and γ $(0 < \gamma < 1)$.

Step 2: Construct the residual network $G'(\mathbf{v}^t)$, and find a negative cycle C^- with negative cost τ_{C^-}.

Step 3: If $|\tau_{C^-}| < \varepsilon$, where ε is a small number, stop. Otherwise, go to Step 4.

Step 4: Set $\lambda = \mu\lambda_{\max}$. If $\lambda = 0$, stop. Otherwise,

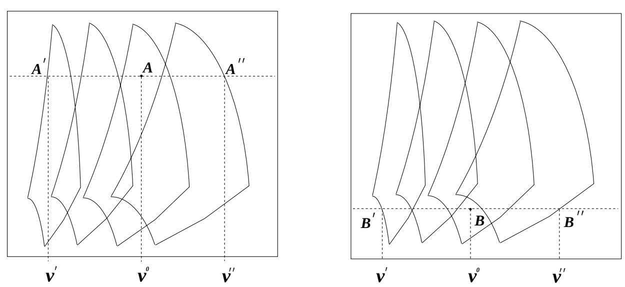
(a) Modify the current flow $\mathbf{v}^t$ by $\mathbf{v}^{new} = \mathbf{v}^t + \lambda \cdot \delta(C^-)$.

(b) Calculate pressure values using DP with modified flow $\mathbf{v}^{new}$.
If DP yields an infeasible solution, or $\overline{g}^{new} - \overline{g}^t > 0$, then set $\lambda = \gamma\lambda$, with $0 < \gamma < 1$, and go to 4(a). Otherwise, $\mathbf{v}^{t+1} = \mathbf{v}^{new}$, $t = t + 1$ and go to Step 2.

To prevent cycling, Step 4(b) is executed a maximum of L times, where L is a user-specified parameter.

5.3 Choice of Step Size

To determine the value of the step size λ defined in (18), we need to consider the feasible domain D_{ij} for each compressor station $(i, j) \in \mathcal{M}_C$ within a loop. As mentioned in the previous chapter, the inequality constraints set D_{ij} is not given in algebraic form, but is defined as a result of the curve fitting methods based on empirical data for compressor units. Consider for instance a compressor station $(i, j) \in \mathcal{M}_C$ with four parallel compressor units. Figure 7 shows the profiles of the feasible domain with a fixed suction pressure value. Discharge pressure p_j and mass flow rate v_{ij} are shown in the y- and x-axes, respectively.



(a) Example of upper bounds.

(b) Example of lower bounds.

Figure 7: Profiles of the feasible region with suction pressure p_i fixed.

Assume that the triple values (v^0_{ij}, p^0_i, p^0_j) of the current solution $\mathbf{x}^0$ are located at points A or B in Figure 7. For either cases, with the assumption that p^0_i and p^0_j are fixed at the current feasible point, v can vary from v' to v''. The values v' and v'' can be obtained in a straightforward manner from the equations defining the compressor stations. The technical details can be found in [9].

For each compressor station in the loop, i.e., $(i, j) \in \mathcal{M}_C$, we can get v'_{ij} and v''_{ij}, and the step size λ should be decided within the bounds

$$0 \leq \lambda \leq \lambda_{\max} = \min \begin{cases} v''_{ij} - v^0_{ij}, & \delta_{ij}(C^-) = 1, \quad (i, j) \in \mathcal{M}_C, \\ v^0_{ij} - v'_{ij}, & \delta_{ij}(C^-) = -1, \quad (i, j) \in \mathcal{M}_C, \end{cases}$$

where $\lambda_{\max}$ is the upper bound of the step size λ, and v_{ij}^0 is the current flow setting at the compressor station $(i, j) \in \mathcal{C}$.

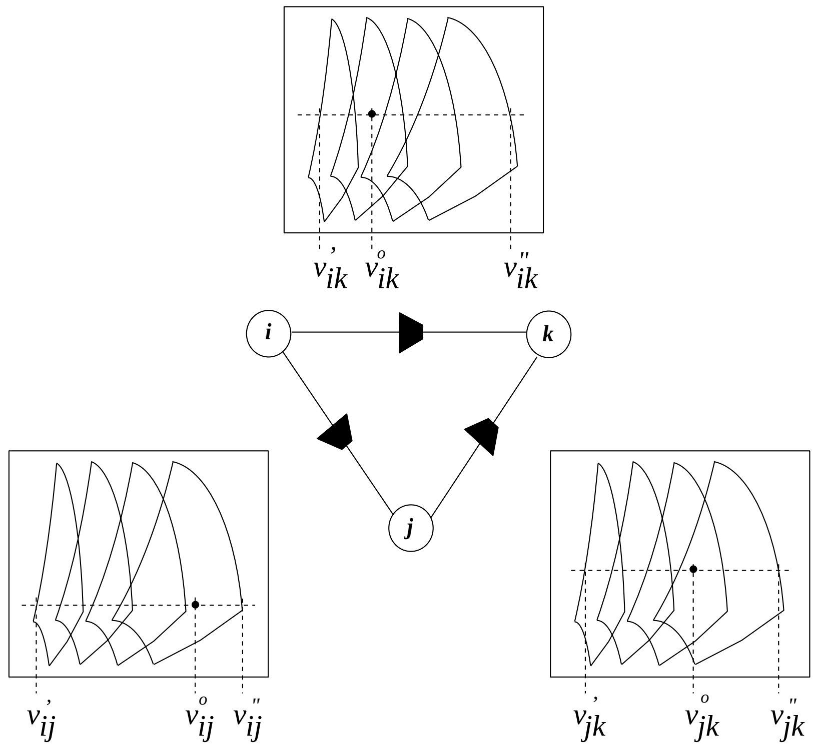


Figure 8: Choice of step size λ.

For instance, consider the example shown in Figure 8. Here, we assume that there is a negative cycle, whose direction is clockwise. Then the maximum step size $\lambda_{\max}$ is determined by

$$\lambda_{\max} = \min\{v_{ik}'' - v_{ik}^0, v_{jk}^0 - v_{jk}', v_{ij}^0 - v_{ij}'\}.$$

Similarly, for the counter-clockwise negative cycle, $\lambda_{\max}$ will be

$$\lambda_{\max} = \min\{v_{ik}^0 - v_{ik}', v_{jk}'' - v_{jk}^0, v_{ij}'' - v_{ij}^0\}.$$

Now, the step size is set to $\lambda = \mu\lambda_{\max}$, where $0 < \mu < 1$. This choice attempts to keeping feasibility and stems from the fact that taking $\lambda = \lambda_{\max}$ causes the new solution to tend to fall outside the feasible region as the corresponding pressure values may have changed accordingly. Large values of μ leads to larger improvements of the objective function, but increases the possibility of infeasibility. In contrast, small values of μ may keep the updated flow within the bounds, but the value of improvement in the objective may be rather small.

6 Empirical Evaluation

6.1 Numerical Examples

In this section we provide an extensive computational evaluation of the proposed solution procedure. For that purpose we have generated problem instances based on network topology examples we have discussed previously, but using real world data for compressor stations. These data was gathered from SSI (Scientific Software Intercomp, Inc.), a Houston-based vendor specializing in gas operations. For the network topologies, we have built structures similar to the ones found in industry. The values of all data and parameters used in this experimental part can be found in [9].

Example 1: A Tree Structured System. Consider the following instance (depicted in Figure 9) with 64 nodes, 56 pipes, and 16 compressor stations.

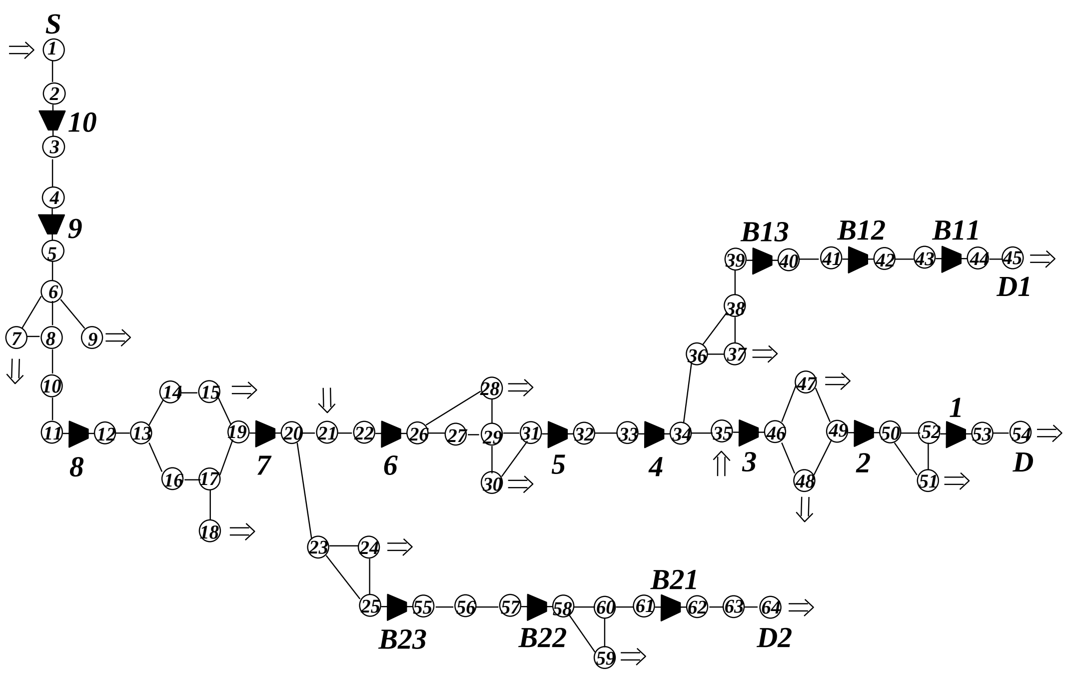


Figure 9: Network of Example 1.

The associated reduced network G' (shown in Figure 10) is a tree with 17 nodes, each of which corresponds to a subgraph, and 16 arcs, each of which corresponding to a compressor station. Using the decomposition technique explained in Section 4, at each subgraph we can calculate the flow variables beforehand by solving system (9).

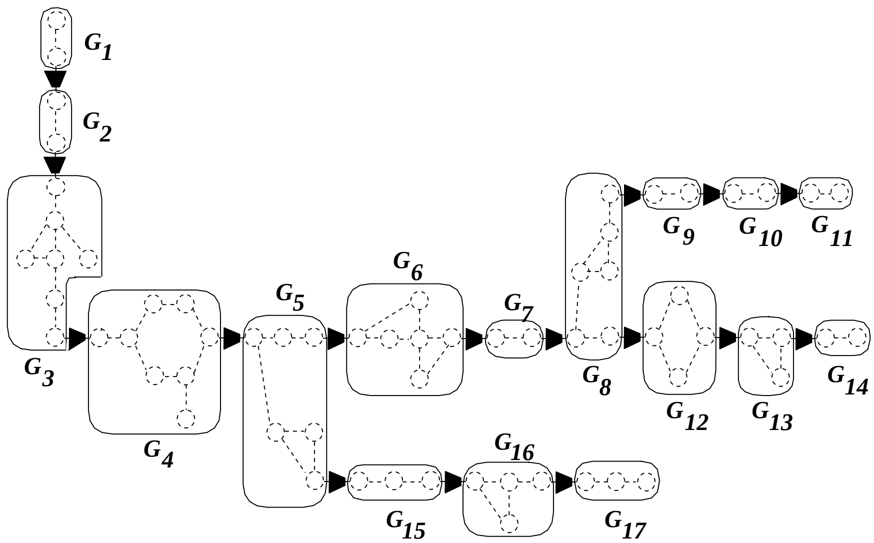


Figure 10: Reduced network G' of Example 1.

Table 1: DP solution.

Compressor Station	Suction pressure	Discharge pressure	v^0	Objective
1	508.16470	508.16470	50.0	0.00000
2	509.72955	509.72955	90.0	79.86951
3	514.96478	514.96478	180.0	262.60706
4	523.63025	523.63025	200.0	1070.30823
5	553.16473	553.16473	200.0	1139.41699
6	565.62067	565.76611	250.0	1225.92798
7	503.36340	585.91382	150.0	13622.09473
8	540.74146	540.74146	220.0	13625.22949
9	599.35669	599.35669	300.0	13694.63379
10	629.71375	663.49652	300.0	21685.33984
B11	504.73297	505.73398	70.0	269.78931
B12	510.74597	510.74597	70.0	336.68485
B13	515.37427	515.37427	70.0	446.73724
B21	573.41547	573.41547	70.0	0.00000
B22	574.86243	574.86243	70.0	0.00000
B23	580.44147	580.44147	40.0	0.00000

We now apply our nonserial implementation of DP to solve the problem and obtain the profiles of pressure values at the compressor stations. Table 1 shows the results when the system input pressure at the super source node 1 is 700 psia and using 10 grid points in each pressure range at each stage of the DP. It is clear that a finer discretization yields more precise solution values, but at a more expensive computational cost. So, the issue here is to investigate this trade-off. For this example, we have explored using a different mesh sizes for pressure ranges (meassured by the number of grid points) in our dynamic programming solution process. Table 2 summarizes the quality of the solutions using four different numbers of grid points. We can observe that the computational time doubles when the number of grid points doubles, and the solution converges as the number of grid points increases. The relative difference in solution quality between a 10-grid and a 20-grid solution is about 1%. Going from a 20-point to a 40-point grid yields an approximate relative improvement of 0.1%.

Table 2: Results of DP for different grid sizes.

Number of grid points	Objective value (Total fuel cost)	CPU time (second)
5	22916.9961	2.73
10	21685.3398	5.03
20	20508.3281	10.01
40	20404.4238	21.83

In this example, since the reduced network is a tree structure, flow rates at the compressor station are uniquely determined, and thus the flow modification step is not required. This example shows how to handle insignificant loops in subgraphs and demonstrates the network decomposition scheme of the solution methodology. In the next two examples, we consider cyclic structures.

Example 2: A Single-Cycle System. The second example is a simple cycle network with 6 compressor stations and 9 pipeline arcs (see Figure 11). This example can be considered as one of the simplest forms of the loop structure. The associated reduced network is shown in Figure 12(a).

As discussed before, by initializing the compressor flow rates the pipeline flow rates can be easily determined (see Table 3). Table 4 shows the solution when DP is applied to solve for the pressure values. We have assumed that the system input of pressure p_S at the super source node S is 700 psia.

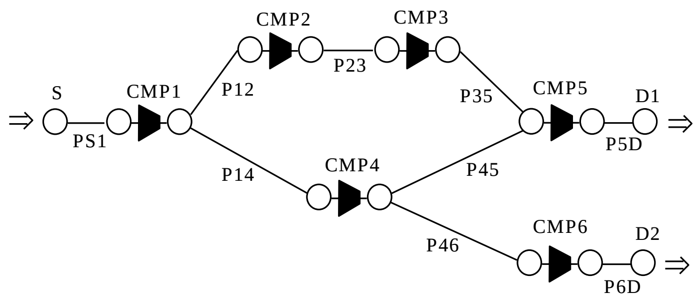
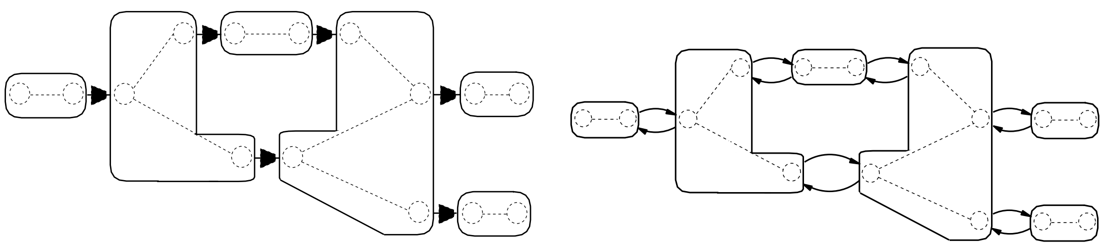


Figure 11: Network of Example 2.



(a) Reduced network G'

(b) Residual network $G'(v^0)$

Figure 12: Reduced and corresponding residual network for Example 2.

Table 3: Flow rates on each pipe.

Pipe	u_{ij}	Pipe	u_{ij}
PS1	350.0	P45	150.0
P12	100.0	P46	100.0
P14	250.0	P5D	250.0
P23	100.0	P6D	100.0
P35	100.0		

Table 4: DP iteration 1.

Station	Suction pressure	Discharge pressure	v_{ij}^0	Fuel cost
CMP1	566.0268	689.3578	350.0	34728.6914
CMP2	674.1254	674.1254	100.0	0.0000
CMP3	655.9074	655.9074	100.0	0.0000
CMP4	561.3552	631.0216	250.0	15418.7940
CMP5	586.2333	626.7272	250.0	8275.7842
CMP6	608.6840	608.6840	100.0	0.0000
Total Cost				58423.2695

Table 5: DP iteration 2.

Station	Suction pressure	Discharge pressure	v_{ij}^0	Fuel cost
1	566.0268	676.2758	350.0000	31668.7832
2	655.5665	655.5665	115.2395	0.0000
3	630.5466	630.5466	115.2395	0.0000
4	562.2967	621.6626	234.7605	12376.5967
5	585.2015	626.7272	250.0000	8480.5098
6	598.9761	598.9761	100.0000	0.0000
Total Cost				52525.8906

Table 6: DP iteration 3.

Station	Suction pressure	Discharge pressure	v_{ij}^0	Fuel cost
1	566.0268	700.0000	350.0000	37288.7109
2	674.7831	674.7831	129.1978	0.0000
3	644.1168	644.1168	129.1978	0.0000
4	604.2491	611.1767	220.8022	4362.0278
5	581.5298	626.7272	250.0000	9227.1436
6	588.0858	588.0858	100.0000	0.0000
Total Cost				50877.8828

Based on the solution given in Table 4, we can construct the residual network on G'. Figure 12 shows the reduced network G' of the problem and its residual network $G'(v^0)$ with respect to the current flow setting $\mathbf{v}^0$. A (clockwise) negative cycle with negative cost $\tau_C = -61.67518$ is found, and the step size λ was calculated as 15.2395. Now we augment flow λ through the negative cycle, with parameters μ and γ set to 0.9 and 0.5, respectively. At iteration 4, the algorithm stops because a negative cycle is not found, i.e., $\tau_C = 0.0$ for both clockwise and counter-clockwise cycle C. Tables 5 through 7 show the solution of each iteration.

Table 7: Iteration 4.

Station	Suction pressure	Discharge pressure	v_{ij}^0	Fuel cost
1	566.0268	700.0000	350.0000	37288.7109
2	669.7152	669.7152	141.3252	0.0000
3	632.5530	632.5530	141.3252	0.0000
4	615.1898	615.1898	208.6749	0.0000
5	583.6664	626.7272	250.0000	8789.2715
6	592.2554	592.2554	100.0000	0.0000
Total Cost				46077.9844

Example 3: A Multiple-Cycle System. Now let us consider the example shown in Figure 13. This instance contains multiple cycles and branches. Moreover, some of the cycles are dependent on each other.

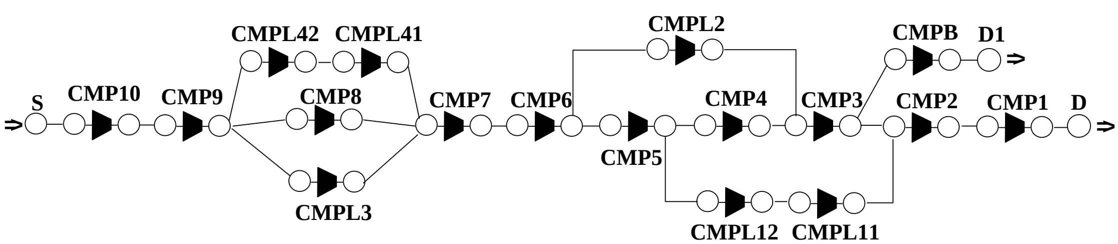


Figure 13: A complex system with multiple loops and branches.

Table 8 shows the DP solution of the given problem with different initial flow settings. We assumed that the input pressure value at the super source node S is 600 psia. The solution shown in Table 8 is checked for feasibility. Based on the current solution, we found 4 negative cycles C_1^-, C_2^-, C_3^-, and C_4^-, each of which corresponds to a counter-clockwise cycle with $\mathcal{M}_{C_1^-}$ = {CMP4,CMP3,CMPL22,CMPL21}, a clockwise cycle with $\mathcal{M}_{C_2^-}$ = {CMPL2,CMP4,CMP5}, a counter-clockwise cycle with $\mathcal{M}_{C_3^-}$ = {CMP8,CMPL3}, and a counter-clockwise cycle with $\mathcal{M}_{C_4^-}$ = {CMPL41,CMPL42,CMP8}, respec-

Table 8: DP solution of Example 3 at iteration 1.

Station	Suction pressure	Discharge pressure	v_{ij}^0	Fuel cost
CMP1	592.14349	592.14349	250.000000	0.00000
CMP2	598.64215	598.64215	250.000000	0.00000
CMP3	616.95190	616.95190	300.000000	0.00000
CMP4	599.94464	617.86743	100.000000	3587.70825
CMP5	600.79651	600.79651	200.000000	0.00000
CMP6	576.27094	607.03632	400.000000	11016.26074
CMP7	599.70898	599.70898	400.000000	0.00000
CMP8	558.29919	603.43390	150.000000	6217.34277
CMP9	561.43622	561.43622	400.000000	0.00000
CMP10	581.77887	581.77887	400.000000	0.00000
CMPL11	599.27130	599.27130	100.000000	0.00000
CMPL12	600.12408	600.12408	100.000000	0.00000
CMPL2	602.76453	621.47272	200.000000	7178.70117
CMPL3	560.14026	601.18829	100.000000	4086.03784
CMPL41	579.52564	601.62470	150.000000	6597.18750
CMPL42	559.81580	581.82019	150.000000	3329.72974
CMPB	614.98547	614.98547	150.000000	0.00000
Total Cost				42012.96875

tively. The negative cycle costs are $\tau_{C_1^-} = -15.37$, $\tau_{C_2^-} = -0.03$, $\tau_{C_3^-} = -0.59$, and $\tau_{C_4^-} = -0.88$. The most negative cycle is C_1^-, and the corresponding step size λ is calculated to be 4.3029. Next we augment flow by 4.3029 units through negative cycle C_1^-. Table 9 presents the solutions in the successive iterations for the given problem.

Table 9: Successive solutions for example 3.

Iteration	Total Fuel Cost	Most Negative cycle cost	Step size
1	42012.9698	-15.3703	4.3029
2	41110.9727	-14.1674	4.0354
3	40816.2773	-13.9453	3.7908
4	38750.9844	-4.0386	0.3330

After 4 iterations, we stop with the solution given in Table 10, because the step size is considered too small to continue. The percentile improvement of fuel cost in this example is almost 8%, and the computational time is 42.75 seconds, when 5 grid points are used at each stage of the DP.

Table 10: Final DP solution.

Station	Suction pressure	Discharge pressure	v_{ij}^0	Fuel cost
CMP1	565.336609	565.336609	250.000000	0.000000
CMP2	572.139832	572.139832	250.000000	0.000000
CMP3	614.253479	614.253479	287.870941	0.000000
CMP4	568.800598	614.963623	87.870964	8402.583984
CMP5	569.494385	569.494385	200.000000	0.000000
CMP6	576.073364	576.073364	400.000000	0.000000
CMP7	599.519104	599.519104	400.000000	0.000000
CMP8	558.299194	603.245239	150.000000	6191.366699
CMP9	561.436218	561.436218	400.000000	0.000000
CMP10	581.778870	581.778870	400.000000	0.000000
CMPL11	567.470337	572.967346	112.129036	1494.207642
CMPL12	568.602295	568.602295	112.129036	0.000000
CMPL2	571.570251	618.794006	200.000000	8723.612305
CMPL3	560.140259	600.998962	100.000000	4068.000732
CMPL41	579.525635	601.435425	150.000000	6541.483887
CMPL42	559.815796	581.820190	150.000000	3329.729736
CMPB	612.278381	612.278381	150.000000	0.000000
Total Cost				38750.984375

An important observation is that in these two cases our procedure shows a considerable improvement over the initial solution, where this initial solution is found by a single application of the non-sequential DP approach, which is the current state of the art to the best of our knowledge.

6.2 Benchmark Results

Because of the lack of test problems in gas pipeline literature, another goal of this work is to provide some benchmark test results which may be used for testing other methods.

The algorithm, as described previously, consists of about 15,000 lines of C code. Numerical experiments on 12 instances based on three different cyclic topologies were run on a SGI Power Challenge L running IRIX 6.2. Even though our solution methodology can handle gunbarrel and tree structures, our computational experiments are done for the cyclic structure, which is the main focus of our study. The three topologies used for our computational experiments are (A) a single cycle instance with six compressor stations (Figure 11), (B) a multi-cyclic structure with 3 cycles, 3 branches, and 21 compressor stations (Figure 14), and (C) a multi-cyclic structure with 4 cycles, 1 branch, and 17 compressor stations (Figure 13).

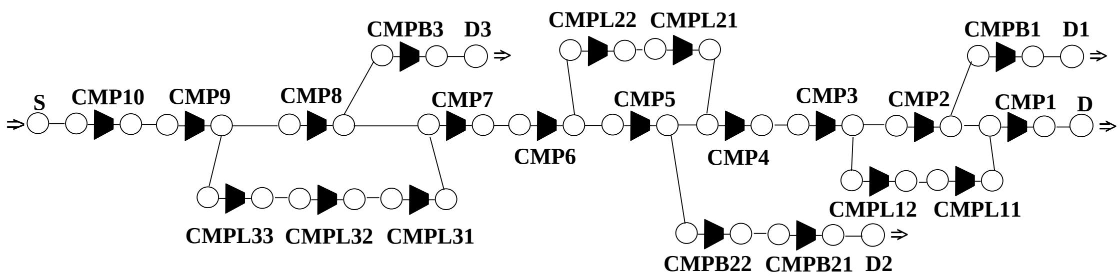


Figure 14: An example with multiple loops and multiple branches.

The two algorithmic parameters μ and γ play a role in the heuristic's performance. As explained in Section 5, the greater the value of μ, the higher probability the iteration will move out of the feasible region. On the other hand, a larger value of μ may yield a faster convergence. Parameter γ is needed if the new solution is not feasible, or if we have not obtained an better solution. During our preliminary experiments, we found that μ around 0.8, and γ around 0.5 gave the best results, so we used these values for our experiments.

We have applied our solution methodology with different initial flow settings to each of these three network topologies. In this part of our experiment we compare the solution delivered by existing work to the one delivered by our procedure. The way the current approach works is actuyally done in industrial settings is as follows. One first attempts to guess, based on experience and practical considerations, a value for the flow variables at compressor stations. Once this is done, flow values for pipelines can be computed by using the model equations. Finally, the non-sequential DP approach (which operates in fixed flows) is applied to obtain an optimal set of pressure flows. We use this actual solution as an input to our algorithm.

Table 11 shows the results of the experiments for these problems. The last column shows the relative improvement in objective function value of the proposed procedure procedure iver the existing approach. An 0.0% implies the algorithm terminates at the very first iteration because no improvement was found. The cost improvement obtained by applying our solution methodology ranges from 0.00% to 41.77%. According to [26], even a 1% savings on gas transportation cost may be worth in the order of 48.6 million dollars. Thus the economical impact is significant.

It is noted that for instance A, in the neighborhood of a possible local optimal point, the solution of the proposed algorithm seems to converge. Table 12 shows the results using different starting points. Here, u_{23}^0 and u_{23}^* represent the initial starting point and the final solution of the flow rate at pipe $(2,3)$, respectively.

Table 11: Solution benchmarks.

Problem instance	Initial flow settings	CPU time (seconds)	Relative Improvement (%)
A	flow setting 1	2.64	24.88
	flow setting 2	4.57	21.13
	flow setting 3	20.24	41.77
B	flow setting 1	6.07	0.00
	flow setting 2	6.20	0.00
	flow setting 3	23.84	17.32
C	flow setting 1	41.17	4.62
	flow setting 2	42.75	3.34
	flow setting 3	74.46	8.20

Table 12: Solution from different starting points for instance A.

Flow setting	Final Fuel Cost	u_{23}^0	u_{23}^*
1	47054.2813	50.0	132.3961
2	47014.1445	100.0	132.5264
3	47007.2152	120.0	132.9225

7 Closing Remarks

In this study we have addressed the problem of minimizing the fuel consumption of compressor stations in a steady-state gas transmission network. We have modeled this problem as a non-linear non-convex network flow problem, and derived a heuristic solution methodology. We have classified the network topologies as non-cyclic and cyclic structures and have highlighted how this type is related to the solution techniques' success. In particular, a cyclic topology is considerable harder to solve because flow rate variables must be handled explicitly, thus making traditional DP approaches not suitable.

This motivated what constitutes the main scientific contribution of this work, which is the derivation and implementation of a network-based heuristic that aims at providing good-quality solutions for cyclic topologies. To the best of our knowledge, there is no previous work that addresses handling both flow and pressure variables simultaneously. The solution procedure is an iterative process. First flow variables are fixed by flow modification, then DP is used to find an associated set of pressure variables. The flow modification step exploits the underlying network properties.

Our computational experimentation showed the effectiviness of the proposed approach, even in non-cyclic structures. Significant improvements were found in most of the tested instances outperforming the best existing approach to the best of our knowledge. The average cost reduction obtained from our solution methodology was about 27% over all instances. The maximum improvement was of about 41.77% (in the instances tested), which amounts to considerable fuel savings.

During our experiments, the distribution of the running time among the various types of operations in the algorithm was studied. It showed that most of time (about 95%) is spent on solving DP. These results highlight the importance of having an efficient procedure for solving DP. Our dynamic programming implementation can be improved by using time efficient interpolation techniques and perhaps parallel programming.

In this study, we consider minimizing fuel consumption at compressor stations for the steady-state model of gas transmission networks. In the operating perspective, there could be other objective functions of interest such as finding the maximal throughput. Currently the feasible domain of the compressor station is not represented algebraically. If we could represent the domain algebraically, it may be possible to develop a mathematical model in which a global optimum can be found. Finally models that consider transient gas networks are a new research area and an important future research direction.

Another line of work is to consider a decision variable that would tell us how many compressor units to operate within each compressor station. This leads to a mixed-integer nonlinear program. In fact, this line is now being pursued, and some preliminary results can be found in Cobos-Zaleta and Ríos-Mercado [7].

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