A Reduction Technique for Natural Gas Transmission Network Optimization Problems

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Abstract

We address the problem of minimizing the fuel consumption incurred by compressor stations in steady-state natural gas transmission networks. In the practical world, these type of instances are very large, in terms of the number of decision variables and the number of constraints, and very complex due to the presence of non-linearity and non-convexity in both the set of feasible solutions and the objective function. In this paper we present a study of the properties of gas pipeline networks, and exploit them to develop a technique that can be used to reduce significantly problem dimension, without disrupting problem structure, making it more amenable to solution.

Keywords: natural gas, pipeline optimization, transmission networks, preprocessing

1 Introduction

We consider the problem of minimizing the fuel cost consumption incurred by compressor stations through natural gas transmission networks. It is represented by a network, where arcs correspond to pipelines and compressor stations, and nodes correspond to their physical interconnection points. The decision variables are the mass flow rate through each arc, and the gas pressure level at each node. At each compressor station, there is a cost that depends on the inlet (suction) pressure, the outlet (discharge) pressure and the mass flow rate through the compressor. The function representing this cost is typically non-convex and nonlinear. In addition, the set of feasible solutions is typically non-convex as well.

In general, a problem with these characteristics is very difficult to solve. This can be clearly seen in many of the approaches that have been taken in past to deal with this problem, such as those of Wong and Larson (1968a, 1968b), Tsal et al. (1986), Percell and Ryan (1987), Lall and Percell (1990), Mallinson et al. (1993), to name a few. The main contribution of our work is to provide a way to reduce significantly problem dimension at preprocessing without disrupting problem structure. In fact, our approach has been successfully incorporated in recent work, such as that of Wu et al. (2000), Kim (1999), and Kim, Ríos-Mercado, and Boyd (2000). For a more complete review on algorithms for pipeline optimization the reader is referred to the work of Carter (1998) and Ríos-Mercado (2002).

The rest of the paper is organized as follows. In Section 2 we introduce the problem and present a mathematical formulation. In Section 3, we present the relevant results related to the pipeline network flow equations. In Section 4 we develop the main theoretical results about uniqueness and the existence of solutions using techniques from nonlinear functional analysis. We continue in Section 5 with the description of the proposed network reduction method and show how to apply it to the two basic cases of network topologies. We provide conclusions in Section 6.

2 Model Description

The objective function of the problem is the sum of the fuel costs over all the compressor stations in the network. This problem involves the following constraints:

(i) mass flow balance at each node;

(ii) gas flow relation through each pipe;

(iii) pressure limit constraints at each node; and

(iv) operation limits at each compressor station.

The first two of the just-mentioned constraints are also called *steady-state network flow equations*. We emphasize that while the mass flow balance equations (i) are linear, the pipe flow equations (ii) are nonlinear. This has been well documented in (Wu, 1998; Wu et al., 2000). For medium and high pressure flows, when taking into account the fact that a change of the flow direction of the gas stream may take place in the network, the pipe flow equation takes the form:

$$p_i^2 - p_j^2 \quad = \quad c_{ij} u |u|^{\alpha}, \tag{1}$$

where p_i and p_j are pressures at the end nodes of pipe (i, j), u is the mass flow rate through the pipe, α is a constant ($\alpha \approx 1$), and the pipe resistance c_{ij} is a positive quantity depending on the physical attributes of pipe (i, j).

The steady-state network flow equations can be stated in a very concise form by using incidence matrices. Let us consider a network with n nodes, l pipes, and m compressor stations. Each pipe is assigned a direction which, may or may not, coincide with the direction of the gas flow through the pipe. Let A_l be the $n \times l$ matrix whose elements are given by

$$a_{ij}^l \quad = \quad \begin{cases} 1, & \text{if pipe } j \text{ comes out of node } i\text{;} \\ -1, & \text{if pipe } j \text{ goes into node } i\text{; and} \\ 0, & \text{otherwise.} \end{cases}$$

A_l is called the *node-pipe incidence matrix.* Similarly, let A_m be the $n \times m$ matrix whose elements are given by

$$a_{ik}^m \quad = \quad \begin{cases} 1, & \text{if node } i \text{ is the discharge node of station } k\text{;} \\ -1, & \text{if node } i \text{ is the suction node of station } k\text{; and} \\ 0, & \text{otherwise.} \end{cases}$$

A_m is called the *node-station incidence matrix.* The matrix formed by appending A_m to the right hand side of A_l is denoted by A, i.e., $A = (A_l \; A_m)$, which is an $n \times (l + m)$ matrix.

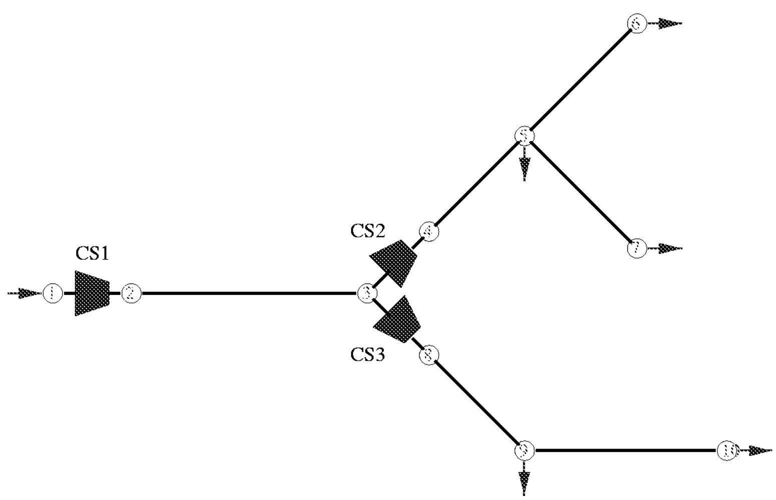

Figure 1: An example of a simple network.

Figure 1 shows a simple network example with $n = 10$ nodes, $l = 6$ pipes, and $m = 3$ stations. Directions assigned to the pipes have been indicated. Note that all nodes, pipes, and stations have been labeled separately. The matrices A_l and A_m for this network are given by,

$$A_l = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad A_m = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the i-th row in each matrix corresponds to node i, the columns in A_l correspond to pipes (2,3), (4,5), (5,6), (5,7), (8,9), and (9,10), respectively, and the k-th column in A_m to compressor station k (CSk) in the network. Note that in each matrix every column contains exactly two nonzero elements, 1 and -1, which correspond to the two end nodes of the pipe or compressor.

Let $\mathbf{u} = (u_1, \ldots, u_l)^T$ and $\mathbf{v} = (v_1, \ldots, v_m)^T$ be the mass flow rate through the pipes and stations, respectively. Let $\mathbf{w} = (\mathbf{u}^T, \mathbf{v}^T)^T$. A component u_j or v_k is positive if the flow direction coincides with the assigned pipe or station direction, negative, otherwise. Let p_i be the pressure at node i, $\mathbf{p} = (p_1, \ldots, p_n)^T$, and $\mathbf{s} = (s_1, \ldots, s_n)^T$ be the source vector, where the source s_i at node i is positive (negative) if the node is a supply (delivery) node. A node that is neither a supply or delivery node is called a *transition node* and has its s_i set equal to zero. We assume, without loss of generality, the sum of the sources to be zero:

$$\sum_{i=1}^{n} s_i = 0. \tag{2}$$

The network flow equations can now be stated as the following:

$$\begin{cases} A\mathbf{w} = \mathbf{s}, \quad \text{and} \\ A_l^T \mathbf{p}^2 = \phi(\mathbf{u}), \end{cases} \tag{3}$$

where $\mathbf{p}^2 = (p_1^2, \ldots, p_n^2)^T$, $\phi(\mathbf{u}) = (\phi_1(u_1), \ldots, \phi_l(u_l))^T$, with $\phi_j(u_j) = c_j u_j |u_j|^\alpha$ being the pipe flow equation at pipe j.

Now suppose the source vector $\mathbf{s}$ satisfying the zero sum condition (2), and the bounds $\mathbf{p}^L$, $\mathbf{p}^U$ of pressures at every node are given. The problem is to determine the pressure vector $\mathbf{p}$ and the flow vector $\mathbf{w}$ so that the total fuel consumption is minimized, that is,

$$\text{Minimize} \quad F(\mathbf{w}, \mathbf{p}) = \sum_{k=1}^{m} g_k(v_k, p_{\text{in}(k)}, p_{\text{out}(k)}), \tag{4a}$$

$$\text{subject to} \quad A\mathbf{w} = \mathbf{s}, \tag{4b}$$

$$A_l^T \mathbf{p}^2 \quad = \quad \phi(\mathbf{u}), \tag{4c}$$

$$\mathbf{p} \quad \in \quad [\mathbf{p}^L, \mathbf{p}^U], \quad \text{and} \tag{4d}$$

$$(v_k, p_{\text{in}(k)}, p_{\text{out}(k)}) \quad \in \quad D_k, \qquad k = 1, 2, \ldots, m, \tag{4e}$$

where v_k, $p_{\text{in}(k)}$, and $p_{\text{out}(k)}$ are the mass flow rate, suction pressure, and discharge pressure at station k, respectively. That is, in(k) and out(k) denote the indices associated with the nodes defining compressor station k. In Figure 1, for instance, compressor 2 is formed by arc (3,4), which means in(2)=3 and out(2)=4. Function g_k is the corresponding cost function, and D_k is the feasible domain in station k for vector $(v_k, p_{\text{in}(k)}, p_{\text{out}(k)})$. See (Wu, 1998; Wu et al., 2000) for an in-depth study of the structure and properties of D_k and g_k. Note the following:

1. The feasible domains D_k are typically non-convex.
2. The fuel minimization functions g_k are nonlinear, non-convex and sometimes discontinuous.
3. The pipe flow equations (4c) define a non-convex set.

In general, a problem with these characteristics is very difficult to solve. What we do here is propose a technique that significantly reduces problem size, making it more tractable. This technique uses concepts from graph theory that can be found in standard works, such as Deo (1974).

3 The Pipeline Network Flow Equations

Now let us consider a gas pipeline network subsystem which consists of nodes and pipes only, that is with no compressor stations. We arbitrarily assign a direction for every pipe and view it as a digraph. Let G be such a digraph with n vertices and e edges. Following notation from Section 2, $\mathbf{w} = (w_1, \ldots, w_e)^T$ denotes the flow vector with w_j the mass flow rate through the j-th edge. The flow w_j is *positive* if the directions of the flow and the edge coincide, *negative* otherwise. Let $\mathbf{s} = (s_1, \ldots, s_n)^T$ be the source vector satisfying (2).

Given that no compressor stations are considered, the compressor flows u_j can be ignored so that system (3) can now be restated as:

$$\begin{cases} A\mathbf{w} = \mathbf{s}, \quad \text{and} \\ A^T \mathbf{p}^2 = \phi(\mathbf{w}). \end{cases} \tag{5}$$

In many network flow problems functions, $\{\phi_j\}$ describing the relationship between arc flows and node variables at end points of the arc, are nonlinear. In the case of gas transmission networks, the most commonly used functions are of the following form:

$$\phi_j(w_j) = c_j w_j |w_j|, \quad 1 \le j \le e,$$

with $c_j > 0$. In some cases, ϕ_j's could also be of the form:

$$\phi_j(w_j) = c_j w_j |w_j|^\alpha, \quad 1 \leq j \leq e,$$

where $\alpha \geq 0$.

Now suppose a source vector $\mathbf{s}$ is given, satisfying the zero sum condition (2), and a reference vertex has been selected whose pressure is also given (which is a necessary condition to solve system (5)). The number of unknowns is $e+n-1$ (since we assumed one pressure value is given, the unknowns are $n-1$ pressure variables and e flow variables), and the number of flow equations is $e+n$. These are n node flow balance equations and e pipe flow equations. Since rank(A)= $n-1$, only $n-1$ node flow balance equations are linearly independent. Let A_f be the reduced incident matrix with respect to the selected vertex; let B_f be the reduced cycle matrix with respect to some spanning tree. Since $B_f A^T = 0$ (from Theorem 2 in Ríos-Mercado et al. (2000)), system (5) is equivalent to

$$\left\{ \begin{array}{r} A_f \mathbf{w} = \mathbf{s}_f, \\ B_f \phi(\mathbf{w}) = \mathbf{0}, \quad \text{and} \\ A^T \mathbf{p}^2 = \phi(\mathbf{w}), \end{array} \right. \tag{6}$$

where $\mathbf{s}_f$ is a $(n-1)$-vector formed by removing from $\mathbf{s}$ the source term corresponding to the selected reference vertex. The advantage of system (6) is that the first two sets of equations:

$$\left\{ \begin{array}{r} A_f \mathbf{w} = \mathbf{s}_f, \quad \text{and} \\ B_f \phi(\mathbf{w}) = \mathbf{0}, \end{array} \right. \tag{7}$$

contain only the flow vector $\mathbf{w}$. Notice that system (7) comprises e equations and e unknowns ($(n-1)$ equations in the first equation and $(e-n+1)$ equations in the second, and e components in $\mathbf{w}$). If it has a unique solution, the flow vector $\mathbf{w}$ can be solved separately from the pressure vector $\mathbf{p}$, and the pressure vector $\mathbf{p}$ can be directly computed from the third equation of system (6), if the pressure at a reference vertex is given. We now address the question of whether system (7) has a unique solution.

4 Uniqueness and Existence of the Solution

In this section, we show that system (7) has a unique solution. A direct implication of this result is that system (5) has a unique solution if the source vector $\mathbf{s}_f$, and the pressure value at a reference node, are given. We begin with some definitions.

Let H be a Hilbert space with a scalar product $(\cdot\,,\cdot)$, and let $\|\cdot\|$ denote the associated norm, i.e. $\|x\| = \sqrt{(x,x)}$ for any $x \in H$.

Definition 1 A mapping ϕ: $H \to H$ is said *strongly monotonic* if there exists a constant $a > 0$, such that, for every $x, y \in H$:

$$(\phi(x) - \phi(y), x - y) \geq a\,(x-y, x-y).$$

Definition 2 A mapping ϕ: $H \to H$ is said to be *strictly monotonic* if for every $x, y \in H$ we have

$$(\phi(x) - \phi(y), x - y) \geq 0,$$

and equality holds if and only if $x = y$.

Definition 3 A mapping ϕ: $H \to H$ is said to be a *basin*, if for every $x_0 \in H$, the set

$$X_{x_0} = \{x \in H : (\phi(x), x - x_0) \leq 0\}$$

is bounded.

Now we prove some basic results related to the above concepts.

Lemma 1 *If ϕ: $H \to H$ is strongly monotonic, ϕ is a strictly monotonic basin.*

Proof: Obviously ϕ is strictly monotonic. To show that it is also a basin, we notice that, for every $x_0 \in H, x \in X_{x_0}$,

$$\begin{aligned} a\,(x - x_0, x - x_0) &\leq (\phi(x) - \phi(x_0), x - x_0) && \text{(since } \phi \text{ is strongly monotonic)} \\ &= (\phi(x), x - x_0) - (\phi(x_0), x - x_0) && \\ &\leq -(\phi(x_0), x - x_0) && \text{(since } x \in X_{x_0}) \\ &\leq \|\phi(x_0)\| \|x - x_0\| && \text{(by Schwartz' inequality).} \end{aligned}$$

So

$$\|x - x_0\| \leq \frac{1}{a} \|\phi(x_0)\|.$$

Hence, X_{x_0} is bounded. ■

However, a mapping ϕ that is a strictly monotonic basin is not necessarily strongly monotonic. Here is an example.

Lemma 2 *Let $H = \mathbf{R}^d$ with the Euclidean scalar product, where d is a positive integer. Let ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ be a mapping as follows: for every $\mathbf{x} = (x_1, x_2, \ldots, x_d)^T \in \mathbf{R}^d$,*

$$\phi(\mathbf{x}) = (\phi_1(x_1), \phi_2(x_2), \ldots, \phi_d(x_d))^T,$$

where

$$\phi_j(x_j) = c_j x_j |x_j|^\alpha, \quad \textit{for } 1 \leq j \leq d,$$

with $c_j > 0$ and $\alpha \geq 0$. Then ϕ is a strictly monotonic basin.

Proof: For every $\mathbf{x} = (x_1, x_2, \ldots, x_d)^T$, $\mathbf{y} = (y_1, y_2, \ldots, y_d)^T \in \mathbf{R}^d$,

$$(\phi(\mathbf{x}) - \phi(\mathbf{y}), \mathbf{x} - \mathbf{y}) = \sum_{j=1}^{d} c_j (x_j |x_j|^\alpha - y_j |y_j|^\alpha)(x_j - y_j).$$

Since $\alpha \geq 0$, the function $h(s) = s|s|^\alpha$ is a strictly increasing function for all s. Hence, each term on the right-hand side (RHS) is nonnegative. Thus,

$$(\phi(\mathbf{x}) - \phi(\mathbf{y}), \mathbf{x} - \mathbf{y}) \geq 0.$$

Equality holds if, and only if, every term on the RHS is zero, so that $x_j = y_j$, for every j. Therefore ϕ is strictly monotonic.

To show that ϕ is a basin, let $\mathbf{x}^0 \in \mathbf{R}^d$. Then $\mathbf{x} \in X_{\mathbf{x}^0}$ means

$$(\mathbf{x}, \phi(\mathbf{x})) \leq (\mathbf{x}^0, \phi(\mathbf{x})),$$

i.e.,

$$\sum_{j=1}^{d} c_j x_j^2 |x_j|^\alpha \leq \sum_{j=1}^{d} c_j x_j^0 x_j |x_j|^\alpha.$$

Let $\rho = \|\mathbf{x}\|$ and $\mathbf{y} = \mathbf{x}/\rho$, so, $\|\mathbf{y}\| = 1$. Replace x_j by ρy_j in the above inequality, we have

$$\rho \leq \frac{\sum_{j=1}^d c_j x_j^0 \rho^{1+\alpha} y_j |y_j|^\alpha}{\sum_{j=1}^d c_j \rho^{1+\alpha} |y_j|^{2+\alpha}}$$

$$\leq \|x^0\| \frac{\sum_{j=1}^d c_j |y_j|^{1+\alpha}}{\sum_{j=1}^d c_j |y_j|^{2+\alpha}}.$$

Since $c_j > 0$, function

$$g(\mathbf{y}) = \frac{\sum_{j=1}^d c_j |y_j|^{1+\alpha}}{\sum_{j=1}^d c_j |y_j|^{2+\alpha}}$$

is continuous on the unit sphere $\|\mathbf{y}\| = 1$. Let

$$G = \max_{\|\mathbf{y}\|=1} g(\mathbf{y}).$$

Then, for every $\mathbf{x} \in X_{\mathbf{x}^0}$, we have

$$\rho = \|\mathbf{x}\| \leq \|\mathbf{x}^0\| G.$$

Hence, $X_{\mathbf{x}^0}$ is bounded for every $\mathbf{x}^0 \in \mathbf{R}^d$; i.e., ϕ is a basin. ■

Remark: ϕ is not strongly monotonic if $\alpha > 0$. If $\alpha = 0$, we obtain the following.

Corollary 1 *The identity function ϕ: $\mathbf{R}^d \to \mathbf{R}^d$, $\phi(\mathbf{x}) = \mathbf{x}$ is a strictly monotonic basin.*

The following lemma can be found in Schwartz (1969).

Lemma 3 *Let H be a Hilbert space. If ϕ: $H \to H$ is continuous and strongly monotonic, then ϕ maps H onto H.*

Let $r > 0$, $t > 0$, be two integers, and $d = r + t$. We say an $r \times d$ matrix A and a $t \times d$ matrix B are *perpendicular to each other* if they satisfy the following hypothesis.
Hypothesis P

1. $\text{rank}(A) = r$, $\text{rank}(B) = t$, and
2. $AB^T = BA^T = 0$.

Let $M = \{\mathbf{x} \in \mathbf{R}^d : A\mathbf{x} = 0\}$, and $N = \{\mathbf{y} \in \mathbf{R}^d : B\mathbf{y} = 0\}$. By Hypothesis P, we have $M = N^{\perp}$ and

$$\mathbf{R}^d = M \oplus N.$$

Let p_M: $\mathbf{R}^d \to M$ be the projection from $\mathbf{R}^d$ to M, p_N: $\mathbf{R}^d \to N$ be the projection from $\mathbf{R}^d$ to N. Then, for every $\mathbf{w} \in \mathbf{R}^d$,

$$\mathbf{w} = p_M(\mathbf{w}) + p_N(\mathbf{w}).$$

Theorem 1 (Uniqueness) *Let matrices A and B be perpendicular to each other. Suppose ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ is strictly monotonic. Then, for every $\mathbf{s} \in \mathbf{R}^r$, the solution to the system of equations*

$$\begin{cases} A\mathbf{w} = \mathbf{s}, \quad \textit{and} \\ B\phi(\mathbf{w}) = \mathbf{0} \end{cases} \tag{8}$$

is unique.

Proof: Suppose both $\mathbf{u}$ and $\mathbf{v}$ are solutions to system (8), then

$$\begin{cases} A(\mathbf{u} - \mathbf{v}) = \mathbf{0}, \quad \text{and} \\ B(\phi(\mathbf{u}) - \phi(\mathbf{v})) = \mathbf{0}. \end{cases}$$

Hence, $\mathbf{u} - \mathbf{v} \in M$, and $\phi(\mathbf{u}) - \phi(\mathbf{v}) \in N$. Thus, $(\mathbf{u} - \mathbf{v}, \phi(\mathbf{u}) - \phi(\mathbf{v})) = 0$. Since ϕ is strictly monotonic, the above equation implies that $\mathbf{u} = \mathbf{v}$. Hence, the solution is unique. ∎

The proof of the existence of a solution to system (8) is not so straightforward. We first prove, in Lemma 4, that it is true for the continuous and strongly monotonic mapping ϕ. Then, in Theorem 2, we prove that it is also true if the mapping ϕ is a continuous and strictly monotonic basin.

Lemma 4 *Suppose ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ is continuous and strongly monotonic. Then, for every $\mathbf{s} \in \mathbf{R}^r$, system (8) has a solution.*

Proof: Let $\mathbf{w}_0$ be the unique solution to the linear system:

$$\begin{cases} A\mathbf{w} = \mathbf{s}, \quad \text{and} \\ B\mathbf{w} = \mathbf{0}, \end{cases} \tag{9}$$

so that $\mathbf{w}_0 \in N$. We define ψ: $M \to M$ as follows. For every $\mathbf{x} \in M$,

$$\psi(\mathbf{x}) = p_M(\phi(\mathbf{x} + \mathbf{w}_0)). \tag{10}$$

Then, for every $\mathbf{x}^1$, $\mathbf{x}^2 \in M$, we have

$$\begin{aligned}
\left(\psi(\mathbf{x}^1) - \psi(\mathbf{x}^2),\, \mathbf{x}^1 - \mathbf{x}^2\right) &= \left(p_M(\phi(\mathbf{x}^1 + \mathbf{w}_0)) - p_M(\phi(\mathbf{x}^2 + \mathbf{w}_0)),\, \mathbf{x}^1 - \mathbf{x}^2\right) \\
&= \left(p_M(\phi(\mathbf{x}^1 + \mathbf{w}_0)) - p_M(\phi(\mathbf{x}^2 + \mathbf{w}_0)),\, \mathbf{x}^1 - \mathbf{x}^2\right) \\
&+ \left(p_N(\phi(\mathbf{x}^1 + \mathbf{w}_0)) - p_N(\phi(\mathbf{x}^2 + \mathbf{w}_0)),\, \mathbf{x}^1 - \mathbf{x}^2\right) \\
&= \left(\phi(\mathbf{x}^1 + \mathbf{w}_0) - \phi(\mathbf{x}^2 + \mathbf{w}_0),\, \mathbf{x}^1 - \mathbf{x}^2\right) \\
&= \left(\phi(\mathbf{x}^1 + \mathbf{w}_0) - \phi(\mathbf{x}^2 + \mathbf{w}_0),\, (\mathbf{x}^1 + \mathbf{w}_0) - (\mathbf{x}^2 + \mathbf{w}_0)\right) \\
&\geq a \left((\mathbf{x}^1 + \mathbf{w}_0) - (\mathbf{x}^2 + \mathbf{w}_0),\, (\mathbf{x}^1 + \mathbf{w}_0) - (\mathbf{x}^2 + \mathbf{w}_0)\right) \\
&= a \left(\mathbf{x}^1 - \mathbf{x}^2,\, \mathbf{x}^1 - \mathbf{x}^2\right) \\
&= a\, \|\mathbf{x}^1 - \mathbf{x}^2\|^2.
\end{aligned}$$

Hence, ψ is strongly monotonic on M. Moreover, ψ is continuous because ϕ is continuous. By Lemma 3, there is a $\mathbf{x} \in M$ such that $\psi(\mathbf{x}) = \mathbf{0}$. Thus, by (10), $p_M(\phi(\mathbf{x}+\mathbf{w}_0)) = \mathbf{0}$. Let $\mathbf{w} = \mathbf{x}+\mathbf{w}_0$. Since $\mathbf{x} \in M$, $A\mathbf{w}_0 = \mathbf{s}$, we have $A\mathbf{w} = A(\mathbf{x} + \mathbf{w}_0) = \mathbf{s}$. Moreover, since $p_M(\phi(\mathbf{w})) = 0$, we have $\phi(\mathbf{w}) = p_M(\phi(\mathbf{w})) + p_N(\phi(\mathbf{w})) = p_N(\phi(\mathbf{w})$. Hence, $\phi(\mathbf{w}) \in N$, and so $B\phi(\mathbf{w}) = \mathbf{0}$. Hence, system (8) has a solution $\mathbf{w}$. ■

Theorem 2 *Let matrices A and B be perpendicular to each other. Let ϕ: $\mathbf{R}^d \to \mathbf{R}^d$ be continuous. Suppose*

(i) ϕ is strictly monotonic, and

(ii) ϕ is a basin.

Then system (8) has a solution for every $\mathbf{s} \in \mathbf{R}^r$.

Proof: Since ϕ is strictly monotonic, for every $\epsilon > 0$, $\phi_\epsilon(\mathbf{w}) = \phi(\mathbf{w}) + \epsilon\mathbf{w}$ is strongly monotonic because

$$\begin{aligned}
& (\phi_\epsilon(\mathbf{w}^1) - \phi_\epsilon(\mathbf{w}^2), \mathbf{w}^1 - \mathbf{w}^2) \\
= \;& (\phi(\mathbf{w}^1) - \phi(\mathbf{w}^2), \mathbf{w}^1 - \mathbf{w}^2) + (\epsilon\mathbf{w}^1 - \epsilon\mathbf{w}^2, \mathbf{w}^1 - \mathbf{w}^2) \\
\geq \;& \epsilon(\mathbf{w}^1 - \mathbf{w}^2, \mathbf{w}^1 - \mathbf{w}^2).
\end{aligned}$$

Hence, by Lemma 4, there is a $\mathbf{w}_\epsilon \in \mathbf{R}^d$, such that

$$\begin{cases} A\mathbf{w}_\epsilon = \mathbf{s}, \quad \text{and} \\ B\phi_\epsilon(\mathbf{w}_\epsilon) = \mathbf{0}. \end{cases} \tag{11}$$

Let $\mathbf{w}_0$ be the solution to linear system (9), as in Lemma 4. Then

$$A(\mathbf{w}_\epsilon - \mathbf{w}_0) = \mathbf{0}.$$

Hence, $\mathbf{w}_\epsilon - \mathbf{w}_0 \in M$. The second equation of (11) implies that $\phi_\epsilon(\mathbf{w}_\epsilon) \in N$. So,

$$(\mathbf{w}_\epsilon - \mathbf{w}_0, \phi_\epsilon(\mathbf{w}_\epsilon)) = 0, \tag{12}$$

i.e.,

$$(\mathbf{w}_\epsilon - \mathbf{w}_0, \phi(\mathbf{w}_\epsilon)) + (\mathbf{w}_\epsilon - \mathbf{w}_0, \epsilon\mathbf{w}_\epsilon) = 0.$$

Since ϕ is a basin, there is a $G_1 > 0$, such that for every $\mathbf{w} \in \mathbf{R}^d$, the inequality

$$(\mathbf{w} - \mathbf{w}_0, \phi(\mathbf{w})) \leq 0,$$

implies $\|\mathbf{w}\| \leq G_1$. By Corollary 1, $\gamma(\mathbf{w}) = \mathbf{w}$ is also a basin. Hence, there is a $G_2 > 0$, such that the inequality

$$(\mathbf{w} - \mathbf{w}_0, \mathbf{w}) \leq 0,$$

implies $\|\mathbf{w}\| \leq G_2$. Thus, by (12), we must have

$$\|\mathbf{w}_\epsilon\| \leq \max(G_1, G_2).$$

By Weierstrass' theorem, there is a $\mathbf{w} \in \mathbf{R}^d$, such that

$$\|\mathbf{w}_{\epsilon_n} - \mathbf{w}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

for some sequence ϵ_n. Since ϕ is continuous, by (11), we have

$$\begin{cases} A\mathbf{w} = \mathbf{s}, & \text{and} \\ B\phi(\mathbf{w}) = \mathbf{0}. \end{cases}$$

Hence, system (8) has a solution. ■

Corollary 2 *System (7) has a unique solution for every* $\mathbf{s}_f \in \mathbf{R}^{n-1}$.

Proof: By applying Theorems 1 and 2, taking $A = A_f$ and $B = B_f$, then using Lemma 2. ■

Even a single nonlinear equation can have no solution or more than one solution. Interestingly, some systems of nonlinear equations which arise in industrial engineering, have a unique solution, as do (7) and (8), as proposed in this paper.

Since the function ϕ involved in gas pipeline network problems is monotonic, solving the system (7) by Newton's method is very stable, fast, and accurate. These facts lead us to introduce the Network Reduction Method for networks consisting of nodes, pipes, and compressor stations. We shall show in the next section that this method can greatly reduce the size of the problem, without modifying its mathematical structure.

5 The Network Reduction Method

The main result obtained in the previous section is that, if all the sources (that is, the mass flow rates at all the nodes of the network going into or out of the network) are given, then all the flows in the pipes are completely determined. Also the pressures at the nodes can be determined if the pressure at one (reference) node is given. It must be pointed out that this result is based on two facts:

1. Each node has mass flow balance.
2. There is a relation between the flow rate and the pressures at the two end nodes of each pipe.

This result is valid in networks consisting of pipes only. Let us take a step further and consider now a network comprising both pipes and compressor stations. The mass flow balance equations must still be satisfied at each node, and a pipe flow equation (relating flow rate through the pipe, and the pressure at the end points) must be satisfied at each edge representing a pipe. However, for each edge representing a station, there is no equation relating the flow rate through the station and the pressures at its suction and discharge sides. The flow rate, suction pressure, and discharge pressure of a station are actually independent of each other, and there are only certain inequalities that these variables must satisfy. Hence, the result obtained in the previous section cannot be directly applied to such networks.

In this section, we shall introduce the Network Reduction Method for networks consisting of nodes, pipes, and compressor stations. In the sequel, we refer to the latter simply as "stations."

Let us first start by introduing the concept of a reduced network. By removing all the stations' arcs from a network, which consists of nodes, pipes, and compressor stations, we are left with *disconnected* components, each of them called a *subnetwork*, consisting of only nodes and pipes. By construction, there are no stations in any subnetwork.

On the other hand, if we view each subnetwork as a single node for the network, i.e., shrinking each subnetwork to a node, and replacing the compressor arcs we had previously removed, we get a new network which consists only of nodes, each representing a subnetwork, and the station arcs. There are no pipes in this network because all the pipes are encapsulated in the nodes. This new network is called a *reduced network* (where each node represents a subnetwork, and each edge represents a station). It is easy to see that there is a unique (connected) reduced network associated with a given (original) network. The structure of the undirected graph associated with the reduced network is called *reduced graph* and can be either a tree or a graph with cycles, depending on the configuration of the compressor stations in the network.

In order to illustrate these concepts let us look at the following example. In Figure 2, a pipeline network has been drawn which comprises $n = 38$ nodes, $l = 38$ pipes, and $m = 5$ compressor stations. Stations are labeled: CS1, CS2, CS3, CS4, and CS5. The number of edges (pipes or stations) is $e = l + m = 43$. Hence, the number of fundamental circuits is $e - n + 1 = 43 - 38 + 1 = 6$. If all 5 stations are removed from this digraph, we get 5 disconnected components, i.e., 5 subnetworks, labeled

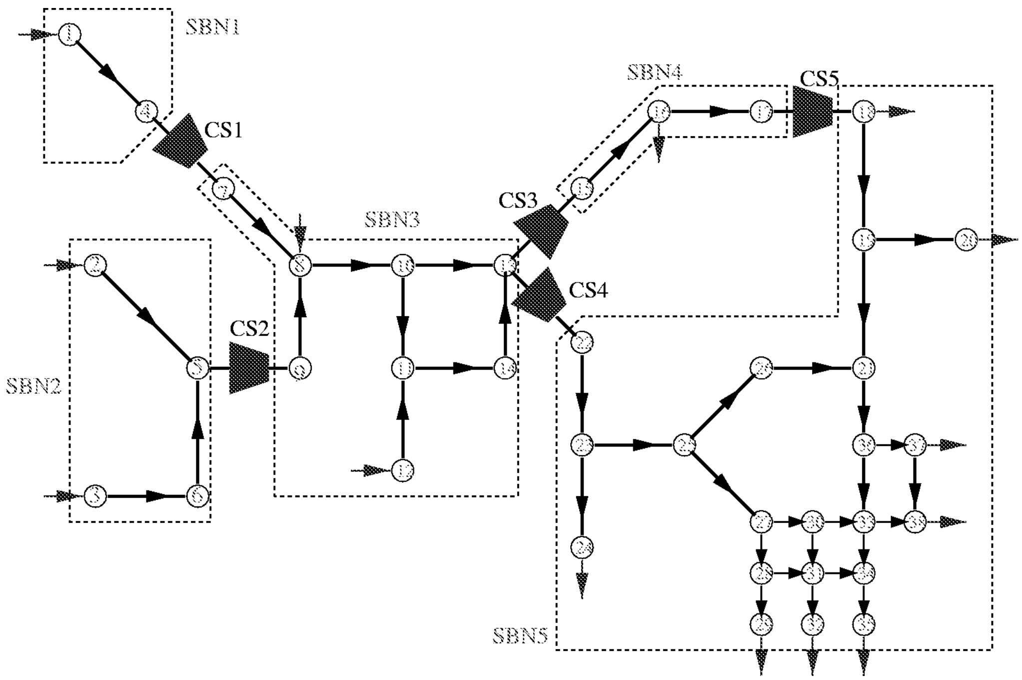

Figure 2: A network with five subnetworks.

SBN1, SBN2, SBN3, SBN4, and SBN5. Each subnetwork is indicated by dotted lines. The undirected graph of the associated reduced network is shown in Figure 3, which comprises 5 nodes and 5 edges, each node representing a subnetwork and each edge representing a station. In this example, the reduced graph has only one cycle.

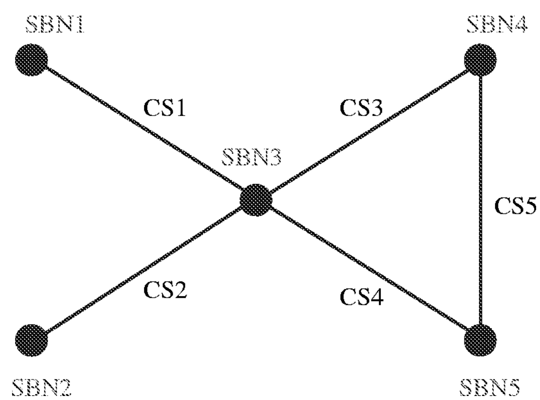

Figure 3: The reduced graph for network of Figure 2.

For practical instances of pipeline networks, we have found that the topology of a reduced network is much less complicated than that of the original network. Although a network may have a number of cycles, especially cycles in pipes, its associated reduced graph is usually a tree. Even if the associated reduced graph is not a tree, the number of cycles in the reduced graph is often significantly less than that in the original network.

We distinguish two cases in terms of the network topology. In the case where the reduced graph is a tree, we shall show that the mass flow rates through all the stations can be fixed if, as we have always assumed, the sources (supplying or delivering flow rates) at all nodes are given. This greatly simplifies the fuel cost minimization problem. On the other hand, if the reduced graph possesses at least one cycle, the mass flow rates through the stations are not uniquely determined but satisfy a system of linear equations. The number of independent linear equations in the system is equal to the number of the fundamental circuits in the super-network. For example, for the network shown in Figure 2, there is only one independent linear equation in the system.

5.1 Case 1: Reduced Graph is a Tree

In this section we assume that the reduced graph is a tree. In this case, since each node in the reduced network represents a subnetwork, we can define the source value at this node as the sum of the source values at all the nodes included in this subnetwork. In this sense, the sources at all the nodes in the reduced network are fixed. Since the reduced graph is a tree, all the flow rates through the edges of the reduced network are uniquely determined (by Corollary 2, with $B = 0$). Since each edge in the reduced network represents a station in the original network, it means that the flow rates through all the stations are known.

Now we examine the subnetworks. We can see that, for each subnetwork, all the sources at the nodes, including the nodes connected to stations are known. By Theorem 2, we conclude that the flow rates through all the pipes in the subnetwork can be uniquely determined. Moreover, the pressures at all the nodes in the subnetwork are uniquely determined by the pressure at one node, the reference node. These pressures will also be increased or decreased as the pressure at the reference node is increased or decreased, respectively.

Hence, we have the following fundamental theorem that underpins the network reduction method.

Theorem 3 *Suppose that*

(i) The pipeline network comprises only nodes, pipes, and stations;

(ii) the sources at all the nodes are given; and

(iii) the associated reduced graph is a tree.

Then

1. *Flow rates through all the pipes and stations are known.*

2. *For each subnetwork, pressure p at any node is related to the pressure p_r at a reference node by*

$$p^2 - p_r^2 = c,$$

where

$$c = \sum_{j \in J} c_j \, u_j |u_j|^{\alpha}$$

is a constant, J is an index set of pipes in a path connecting the node and the reference node, c_j and α are constants, and u_j is the flow rate in the j-th pipe, which is known.

Note that the constant c is independent of the selection of the path because the flow rate u_j's are solved from the equations such that summation $\sum_{j \in J} c_j \, u_j |u_j|^{\alpha}$ along any cycle in a subnetwork is zero. Hence, if a network is divided into b subnetworks, the total number of independent variables in the network is b, i.e., the pressure variables p_r at the b reference nodes.

The fuel cost minimization problem (4a)–(4e) can now be greatly simplified by applying the network reduction method. First, since the flow rates v_k through all the stations are known, each function g_k in (4a) depends on $(p_{\text{in}(k)}, p_{\text{out}(k)})$ only. Thus, the objective function $F(\mathbf{w}, \mathbf{p})$, depends on only the suction and discharge pressures $(p_{\text{in}(k)}, p_{\text{out}(k)})$, $k = 1, \ldots, m$. Let $\mathbf{z}$ be the vector of these suction and discharge pressures, i.e., $\mathbf{z} = \{p_{\text{in}(1)}, p_{\text{out}(1)}, \ldots, p_{\text{in}(m)}, p_{\text{out}(m)}\}$. The objective function F can now be represented by

$$G(\mathbf{z}) = \sum_{k=1}^{m} g_k(v_k, p_{\text{in}(k)}, p_{\text{out}(k)}), \tag{13}$$

where v_k is known.

Further, suppose the network is divided into b subnetworks. The pressure variables in $\mathbf{z}$ can be partitioned into b disjoint vectors $\mathbf{z}_i$, each representing the pressures at all the suction or discharge nodes in the i-th subnetwork, i.e., $\mathbf{z} = (\mathbf{z}_1^T, \ldots, \mathbf{z}_b^T)^T$. Let $\mathbf{z}_i = (z_{i1}, \ldots, z_{iJ_i})^T$, where J_i is the number of the suction and discharge nodes in the i-th subnetwork. Let us choose z_{i1} as the reference pressure for the i-th subnetwork. Then, according to Theorem 3, pressure p at every node in the i-th subnetwork is related to z_{i1} by

$$p^2 - z_{i1}^2 = c.$$

The fact that each pressure value is bounded implies that there are two constants z_i^L and z_i^U, such that the pressure limit constraints (4d) for nodes in the i-th subnetwork are equivalent to

$$z_i^L \leq z_{i1} \leq z_i^U.$$

The effectiveness of the network reduction method is based on the following observations. The objective function depends only on the pressures at suction and discharge nodes. This means that we do not need to calculate the values of pressures at the nodes other than the suction or discharge nodes. However, we must keep all pressure variables within their pressure limits, i.e., satisfying the constraints (4d). These can be fulfilled by confining the reference pressure z_{i1} within its limits z_i^L and z_i^U. Constraints (4e) are irrelevant to the pressures at nodes other than suction or discharge nodes; therefore, these pressures will disappear in the minimization problem.

On the other hand, the pressures at suction or discharge nodes in the i-th subnetwork must be related to the reference pressure z_{i1}, i.e.,

$$z_{ij}^2 - z_{i1}^2 = c_{ij}, \quad j = 2, \ldots, J_i,$$

where c_{ij}'s are constants.

Since the v_k's are known, the compressor station constraint (4e) becomes $\mathbf{z} \in Z$, where Z is the feasible domain of stations for the suction and discharge pressures $\mathbf{z}$.

Hence, the fuel cost minimization problem (4a)–(4e) can be simplified to the following:

$$\text{Minimize} \quad G(\mathbf{z}), \tag{14a}$$

$$\text{subject to} \quad z_i^L \le z_{i1} \le z_i^U, \quad 1 \le i \le b, \tag{14b}$$

$$z_{ij}^2 - z_{i1}^2 = c_{ij}, \quad 1 \le i \le b,\ j = 2, \ldots, J_i, \quad \text{and} \tag{14c}$$

$$\mathbf{z} \in Z, \tag{14d}$$

where $G(\mathbf{z})$ is defined in (13).

Comparing problem (4a)–(4e) with problem (14a)–(14d), the simplifications are:

1. The number of variables is reduced from $l+m+n$ to the size of vector $\mathbf{z}$, which is at most $2m$. A typical pipeline network may comprise thousands of pipes and nodes, but only dozens of stations. Hence this reduction is typically significant.

2. The (nonlinear) constraints (4c), involving 3 variables, are replaced by the (nonlinear) constraints (14c), involving 2 variables. Linearizing a nonlinear constraint involving 2 variables is much easier and more effective.

3. The number of nonlinear equality constraints is reduced from l to $\sum_{i=1}^{b}(J_i - 1) \le 2m - b$. Since nonlinear equality constraints are often the main obstacles in optimization problems, reducing the number of the nonlinear equality constraints can make the problem easier to solve.

	Before reduction				After reduction	
Configuration	l	m	n	$l+m+n$	b	$\|\mathbf{z}\|$
A	10	2	10	22	2	4
B	44	7	47	98	7	13
C	91	9	180	280	10	18
D	1462	37	1560	3059	38	73

Table 1: Size of networks before and after reduction.

Table 1 displays a comparison of sizes before and after the reduction for some typical network configurations. We can see that the size of $\mathbf{z}$ is often much smaller than the number $l + m + n$, i.e., the size of the reduced problem is much smaller than that of the original problem. We must point out that the tradeoff for these simplifications is that we need to solve the network flow equations for each subnetwork. However, our numerical experiments show that a modified Newton's method is an extremely fast and stable way to solve these equations. Moreover, all these calculations can be done at pre-processing.

5.2 Case 2: Reduced Graph Has Cycles

If the reduced graph is not acyclic, the flow rates cannot be uniquely determined, although the network reduction method can still be successfully used. In this case, the mass flow rate $\mathbf{v}$ through the stations satisfies a simple system of linear equations:

$$\mathcal{A}\mathbf{v} = \mathbf{S}, \tag{15}$$

where $\mathcal{A}$ is the node-edge incidence matrix for the super-network and $\mathbf{S}$ is the vector of sources at the nodes in the reduced network. The i-th element of $\mathbf{S}$ is the sum of the sources at all the nodes in the i-th subnetwork. Since the flow rate $\mathbf{v}$ must be bounded, say, $\|\mathbf{v}\| \le v_{max}$, we can define a set V as

$$V = \{\mathbf{v} : \mathcal{A}\mathbf{v} = \mathbf{S}, \|\mathbf{v}\| \le v_{max}\}.$$

Theorem 4 *The number of independent variables in system (15) is equal to the number of fundamental cycles in the associated reduced network.*

For each $\mathbf{v} \in V$, we define a function $f(\mathbf{v})$ on V, which is

$$\begin{aligned} f(\mathbf{v}) \;\; \equiv \;\; \min \Big\{ G^v(\mathbf{z}) \; : \; (z_i^L)^v \le z_{i1} \le (z_i^U)^v, \; 1 \le i \le b; \\ z_{ij}^2 - z_{i1}^2 = c_{ij}^v, \; 1 \le i \le b, \; j = 2, \ldots, J_i; \quad \mathbf{z} \in Z^v \Big\}, \end{aligned}$$

where G^v, $(z_i^L)^v$, $(z_i^U)^v$, c_{ij}^v, and Z^v all depend upon $\mathbf{v}$. Hence, the fuel cost minimization problem becomes one of minimizing $f(\mathbf{v})$ over V. A numerical approximation technique, such as grid generation on $\mathbf{v}$ could be applied to find approximate solutions. This method might work well if the dimension of the kernel of the matrix $\mathcal{A}$ is small, or equivalently, if the number of independent variables in system (15) is small. On the other hand, the dimension of the kernel of matrix $\mathcal{A}$ is equal to the number of the fundamental cycles in the reduced graph. Therefore, the method is relatively effective for networks whose reduced graph has relatively few cycles. The extreme case occurs when this dimension equals zero; i.e., the reduced graph is a tree. This is Case 5.1. In this case, $\mathbf{v}$ can be uniquely solved via equation $\mathcal{A}\mathbf{v} = \mathbf{S}$.

The efficiency of decomposition techniques can be increased via the network reduction method. This is so because, at a given iteration, fixing $\mathbf{v}$ implies that all other flow variables in the system can be determined, due to the developments presented in the previous section.

6 Conclusions

We have proposed a reduction technique for gas pipeline optimization problems. The justification of the technique was based on a novel combination of graph theory and nonlinear functional analysis. The reduction technique can decrease the problem size by more than an order of magnitude in practice, without disrupting its mathematical structure.

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