

Natural Gas

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Abstract

In this chapter we introduce the reader with the problem of minimizing the fuel cost incurred by the compressor stations driving the gas in steady-state pipeline networks. This is one of the problems that has received most of the attention from both practitioners and researchers by virtue of its tremendous economical impact.

After presenting the basic mathematical framework and discussing the most important modeling assumptions, we establish the fundamental network topologies, discuss their importance in the selection of the solution technique, and present current solution methodologies based on dynamic programming. We conclude by highlighting the research potential of this exciting application area of operations research.

1 Introduction

A typical pipeline network for delivering natural gas requires a tremendous amount of fuel per day to operate the compressor stations driving the gas. Efficient design and operation of these complex networks can substantially reduce airborne emissions, increase safety, and decrease the often multi-million dollar daily operating costs.

Models describing gas flows in networks are in a reasonably well developed descriptive stage. However, algorithms for actually finding good solutions relative to some cost criterion are not

nearly as well developed up to date. From an optimization standpoint, the domain of feasible solutions prescribed by the models is mathematically complicated by virtue of the existence of both nonlinearity and non-convexity. In addition, the related cost functions are typically nonlinear and non-convex, and, in many cases, discontinuous, depending on the modeling assumptions. While classical nonlinear programming techniques (Chapter 2.5) can handle these complications on a practical basis for small pipeline networks, they are wholly inadequate for many of the large and growing pipeline networks in operation today.

We would also like to emphasize that despite the great amount of research on network flows by discrete and nonlinear optimizers within the operations research community, little work has been devoted by this community to optimization problems for compressor driven network flows.

In this chapter we introduce the reader with the problem of minimizing the fuel cost incurred by the compressor stations driving the gas in steady-state pipeline networks. This is one of the problems that has received most of the attention from both practitioners and researchers by virtue of its tremendous economical impact. Other application areas in gas networks where optimization techniques have been used include pipeline design (Edgar et al. 1978; Mariani et al. 1997; De Wolf and Smeers 1996) and operations planning (Tomlin 1988; Creegan and Monforte 1989) to name a few.

In Section 2 we present a basic mathematical framework for the steady-state case, and discuss the most important modeling assumptions. In Section 3 we establish the fundamental network topologies and discuss their importance in the selection of the solution technique. In Section 4 we overview the current solution methodologies, and present a basic solution technique within a dynamic programming framework. We conclude in Section 5 with a direction of current research trends, putting in perspective the research potential of this exciting application area of operations research.

2 Model Formulation

2.1 Gas Transmission Elements

In modeling gas transmission networks, we consider three basic types of entities: pipelines and compressor stations, which are represented by arcs; and interconnection points, represented by nodes. Let \mathcal{A} denote the set of arcs and \mathcal{V} the set of nodes. We call \mathcal{A}_p and \mathcal{A}_c the set of pipeline and compressor (station) arcs, respectively. Note that $(\mathcal{A}_p, \mathcal{A}_c)$ is a partition of \mathcal{A} , i.e., $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$ and $\mathcal{A}_p \cap \mathcal{A}_c = \emptyset$. A node $i \in \mathcal{V}$ is used to represent the physical interconnection among the arcs. We also have different types of nodes. A receipt or supply node represents a gas production/storage facility. A delivery node represents a place where gas is to be taken out of the system. Let \mathcal{V}_s and \mathcal{V}_d be the set of supply and delivery nodes, respectively.

Likewise, each compressor arc $e = (i, j) \in \mathcal{A}_c$ defines another two types of nodes technically referred to as suction and discharge nodes (i and j , respectively). In this case, variable x_e represents the gas mass flow rate through arc $e \in \mathcal{A}$. In addition to the flow variables, there are pressure variables p_i that must be determined at every node $i \in \mathcal{V}$.

For a given compressor station $(i, j) \in \mathcal{A}_c$, there are three relevant decision variables: the mass flow rate through the station, the inlet (suction) pressure p_i at node i , and the outlet (discharge) pressure p_j at node j . In the compressor, these variables are further restricted by a set of constraints that depend on the operating attributes of the compressor. Let \mathcal{D}_{ij} denote the set of feasible operating values for x_{ij} , p_i , and p_j within compressor $(i, j) \in \mathcal{A}_c$.

At each compressor station (i, j) , the amount of fuel used g_{ij} is a function of x_{ij} , p_i , and p_j , that is, $g_{ij} : \mathcal{D}_{ij} \rightarrow R$. In general, the nature of this function g_{ij} is very complex and depends also on considerations such as the number of compressor stations running within the compressor station, how the compressor units are configured (i.e., in series, in parallel, combination of

both, etc.), physical properties of the compressor units themselves, and type of compressor unit. One of the most common configurations encountered in practice is that of compressor stations consisting of identical centrifugal compressor units operating in parallel. Even with this assumption, g_{ij} is in general nonlinear and non-convex. The overall gas consumption is obtained by adding up all the g_e for each $e \in \mathcal{A}_c$.

For pipeline $(i, j) \in \mathcal{A}_p$, there is an equation that describes the relationship between the mass flow rate through the pipe and the pressure at the end points of the pipe. Let \mathcal{P}_{ij} be the set of feasible values for mass flow rate and pressures in pipe (i, j) . As it will be illustrated later, \mathcal{P}_{ij} is a non-convex set.

The problem is to determine $(x, p) \in R^{|\mathcal{A}|+|\mathcal{V}|}$ (flow rates and pressures) that minimizes the total fuel consumption $g(x, p) = \sum_{(i,j) \in \mathcal{A}_c} g_{ij}(x_{ij}, p_i, p_j)$ subject to (a) mass balance constraints at each node, (b) compressor operating limits, (c) pipeline constraints, and (d) single lower, upper bounds for all the variables.

2.2 The Basic Model

In the development we make use of the following notation.

Indices and sets

i, j, k receipt, delivery, interconnection points; $i, j, k \in \mathcal{V} = \{1, 2, \dots, n\}$

\mathcal{A}_p set of pipeline arcs

\mathcal{A}_c set of compressor station arcs

\mathcal{A} set of all arcs; $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$

$\delta^+(j)$ set of outgoing arcs from point j ; $\delta^+(j) \subset \mathcal{A}$; $j \in \mathcal{V}$

$\delta^-(j)$ set of ingoing arcs into point j ; $\delta^-(j) \subset \mathcal{A}$; $j \in \mathcal{V}$

\mathcal{V}_S set of supply points; $\mathcal{V}_S \subset \mathcal{V}$

\mathcal{V}_D set of delivery points; $\mathcal{V}_D \subset \mathcal{V}$

Input data

S_i supply limit at point i ; $i \in \mathcal{V}_S$

D_i forecasted demand at point i ; $i \in \mathcal{V}_D$

p_i^L, p_i^U lower, upper pressure bounds at node i ; $i \in \mathcal{V}$

C_{ij} capacity of pipeline (i, j) ; $(i, j) \in \mathcal{A}$

Decision variables

x_{ij} flow rate on arc (i, j) ; $(i, j) \in \mathcal{A}$

p_i pressure at node i ; $i \in \mathcal{V}$

The problem (P_0) has the general mathematical framework:

$$\text{Minimize} \quad f(x, p) = \sum_{(i,j) \in \mathcal{A}_c} g_{ij}(x_{ij}, p_i, p_j) \quad (1a)$$

$$\text{subject to} \quad \sum_{k \in \delta^+(j)} x_{jk} - \sum_{i \in \delta^-(j)} x_{ij} = b(j) \quad j \in \mathcal{V} \quad (1b)$$

$$x_{ij} \in [0, C_{ij}] \quad (i, j) \in \mathcal{A} \quad (1c)$$

$$(x_{ij}, p_i, p_j) \in \mathcal{D}_{ij} \quad (i, j) \in \mathcal{A}_c \quad (1d)$$

$$(x_{ij}, p_i, p_j) \in \mathcal{P}_{ij} \quad (i, j) \in \mathcal{A}_p \quad (1e)$$

$$p_i \in [p_i^L, p_i^U] \quad i \in \mathcal{V} \quad (1f)$$

where

$$b(j) = \begin{cases} S_j & \text{if } j \in \mathcal{V}_S \\ -D_j & \text{if } j \in \mathcal{V}_D \\ 0 & \text{otherwise} \end{cases}$$

Constraints (1b)–(1c) are the classical network flow constraints. Note that the overall objective function is an aggregate of the fuel consumption at every compressor station. The difficulty of this problem is due to the non-convexity of the objective function (1a), and the non-convexity of both \mathcal{D}_{ij} and \mathcal{P}_{ij} (compressor station operating domain (1d) and pipeline domain (1e)). In the following sections, a more detailed discussion of these formulation components will be given.

2.3 Compressor Stations

In general, compressor stations in gas pipeline networks are be very complex entities because they may consist of several compressor units (typically 15–20) connected in a number of different configurations (e.g., in series, in parallel, combinations of both). In addition, there exists different types of compressor units, two of the most important being *centrifugal* and *reciprocating* units. Of these two, it is the centrifugal type the most commonly used in today’s industry. Thus, it is no surprise that most of the research done up to date has focused on centrifugal units. For a detailed description of these and other types of compressor units, the reader is referred to Osiadacz (1987). In this chapter we present the reader with a modeling description based on centrifugal compressor units.

We recall there are three modeling variables involved in a compressor station (i, j) , namely the mass flow rate, and the inlet and outlet pressures, x_{ij} , p_i and p_j , respectively. However, the variables that are actually manipulated in the compressor are (i) compressor speed s , (ii) volumetric flow rate q , and adiabatic head h . These variables are related by the following

equation

$$h/s^2 = \phi(q, s)$$

where ϕ is a polynomial function of the ratio q/s , typically of degree 3. There are also physical operational limits set on both on s and q , given by

$$s \in [s^L, s^U]$$

$$q \in [q^L, q^U]$$

A consequence of these is an operational limit on the ratio q/s technically referred as *surge* and *stonewall* for the lower and upper bound limit, respectively.

The (x_{ij}, p_i, p_j) and (s, q, h) spaces are related by the following transformation.

$$h = \beta[(p_j/p_i)^m - 1]$$

$$q = \beta(x_{ij}/p_i)$$

where β and m are parameters that depend on gas properties, with $m \in (0, 1)$. Hence, the feasible operating domain $\mathcal{D}^{\text{unit}}$ for a single centrifugal compressor unit in the (x_{ij}, p_i, p_j) space is given by $\mathcal{D}^{\text{unit}} = \{(x, p_i, p_j) : \text{constraints (2a)–(2c) below are satisfied}\}$.

$$(s^L)^2 \phi(s^L, \beta x_{ij}/p_i) \leq \frac{\beta}{m} \left[\left(\frac{p_j}{p_i} \right)^m - 1 \right] \leq (s^U)^2 \phi(s^U, \beta x_{ij}/p_i) \quad (2a)$$

$$\left(\frac{s^U}{q^U} \right)^2 \phi(s^U, q^U)(\beta x_{ij}/p_i)^2 \leq \frac{\beta}{m} \left[\left(\frac{p_j}{p_i} \right)^m - 1 \right] \leq \left(\frac{s^L}{q^L} \right)^2 \phi(s^L, q^L)(\beta x_{ij}/p_i)^2 \quad (2b)$$

$$p_i \in [p_i^L, p_i^U] \quad (2c)$$

$\mathcal{D}^{\text{unit}}$ is non-convex, as can be seen in Figure 1, which shows the feasible domain as a function of x_{ij} and p_j when p_i is fixed in a centrifugal compressor unit. In this figure, the function joining points A and D (and C and B) is convex, whereas the function joining points A and B (and C and D) is non-convex.

Now, let us consider a compressor station with N identical units running in parallel. In this case, the suction and discharge pressures of the station are the same as those of the individual units in the station. If the units are identical, it is fair to assume that the mass flow rate x_{ij} through the station is equally divided among the individual units. Hence, if only one unit is running, then the feasible domain, denoted as \mathcal{D}^1 , is the same as the feasible domain $\mathcal{D}^{\text{unit}}$ for a single unit described above. When r units are selected to run, $1 \leq r \leq N$, then the feasible domain, denoted as \mathcal{D}^r , is given by

$$\mathcal{D}^r = \left\{ (v, p_s, p_d) : (v/r, p_s, p_d) \in \mathcal{D}^1 \right\}. \quad (3)$$

As we can see, if r is selected before hand, \mathcal{D}^r can be seen as extending domain \mathcal{D}^1 r times in the direction of x_{ij} and is essentially the same type of domain. However, r can be another decision variable. If this is the case, and we assume that at least one unit must be running, the feasible domain \mathcal{D} is thus given by

$$\mathcal{D} = \bigcup_{r=1}^N \mathcal{D}^r. \quad (4)$$

The (x_{ij}, p_j) profile of domain \mathcal{D} when p_i is fixed in a station with 3 identical units is shown in Figure 2.

2.4 Pipeline Equations

The steady-state relationship between pressure and mass flow rate along the pipe can be described by different equations depending on the working pressure of the network. Basically, three types of equations are considered (for low-, medium-, and high-pressure networks). For isothermal models (e.g., gas temperature assumed constant throughout the pipe), the general formula takes the following form

$$\phi(\Delta p) = K_{ij} x_{ij}^2 \quad (5)$$

where ϕ is the pressure function, and K_{ij} is the pipe parameter which depends on the physical properties of the pipeline. For low-pressure networks, Lacey's equation, we have $\phi(\Delta p) = p_i - p_j$, $K_{ij} = 11.7 \times 10^3 L/D^5$. For medium-pressure networks, Polyflo equation, we have $\phi(\Delta p) = p_i^2 - p_j^2$, $K_{ij} = 27.24L/E^2 D^5$. For high-pressure networks, Panhandle equation, $\phi(\Delta p) = p_i^2 - p_j^2$, $K_{ij} = 18.43L/E^2 D^5$. Here, the physical properties L (length of pipe), D (inside pipe diameter), and E (efficiency factor) are assumed to be known with certainty.

Note that in some instances, the direction of the flow through a given pipeline may not be known in advance and could go either way (this is represented by an undirected edge (i, j)). If this is the case, the undirected edge e is replaced by two directed arcs e^+ and e^- and equation (5) becomes:

$$\phi(\Delta p) = K_{ij}(x_{ij}^+ + x_{ij}^-)(x_{ij}^+ - x_{ij}^-) \quad (6)$$

When doing this, a constraint $(x_{ij}^+)(x_{ij}^-) = 0$ must be added since the objective function is non-convex. Note that there is only one equation (6) per pipeline. For non isothermal models, it is still possible to derive a pipe-leg equation describing the relationship between mass flow rate and pressure drops. This equation, though, is more complex and includes gas temperature as a variable.

2.5 Objective Function

The fuel consumption in a single centrifugal compressor unit $(i, j) \in \mathcal{A}_c$ is governed by the following equation:

$$g_{ij}^{\text{unit}} = \gamma \frac{x_{ij}}{\eta(x_{ij}, p_i, p_j)} \frac{\alpha}{m} \left[\left(\frac{p_j}{p_i} \right)^m - 1 \right]$$

where γ is a parameter, and η is the adiabatic efficiency function. In general, η is a complex expression, not implicitly defined. To make one function evaluation of η , one must solve a linear

system of equations. In practice, though, one employs polynomial approximation functions that fit the function relatively well and are simpler to evaluate. On other cases, when the fluctuations of η are small enough, η can be assumed to be a constant, leaving us with the function:

$$g_{ij}^{\text{unit}} = \alpha' x_{ij} \left[\left(\frac{p_j}{p_i} \right)^m - 1 \right] \quad (7)$$

which is not polynomial, but easy to evaluate ($\alpha' = (\eta\alpha)/m$ is a constant). This function is non-convex.

For the case of r identical units connected in parallel, the flow entering the compressor station is equally divided among the units, and its fuel consumption can be described with the following mixed integer nonlinear function:

$$g_{ij}(x_{ij}, p_i, p_j, r) = r g_{ij}^{\text{unit}}(x_{ij}/r, p_i, p_j).$$

3 Network Topologies

There exist two fundamental classes of gas network topologies: non-looped (or tree) networks and looped networks. For this classification, rather than considering the original topology, an associated network is formed in the following form:

1. By removing temporarily all compressor arcs from the original network, we are left with a set of connected components or sub-networks consisting each of nodes and pipeline arcs.
2. Each of these sub-networks is merged into one single node.
3. The compressor arcs are restored and the result is the associated reduced network.

This is illustrated in Figure 3. The solid lines represent original arcs (pipelines and compressor stations). In the reduced network, each sub-network shown within the dotted line is viewed as a node. Here, the reduced network is a tree with 4 nodes and 3 arcs. We call a

network a *non-looped* network if and only if the undirected version of its corresponding reduced graph has no cycles. Otherwise, we say the network is a *looped* network. A special case of a non-looped network where the associated reduced network is a simple path is called a serial (or gun-barrel) system.

It is important to realize that looped networks possess a higher degree of difficulty than non-looped networks. First, for non-looped networks it can be shown that if the parameter $b(i)$ in the mass balance constraints (1b) is specified for every node i in the network (which is usually the case), then the flow variables can indeed be uniquely determined for each arc in the network, even if some of the sub-networks have cycles. To prove this result one uses the mass balance constraints in combination with the pipeline constraints to derive a system of equations with a unique solution for the flow variables. The proof can be found in Wu et al. (1999b). Note that this result is unique for this type of gas networks and in general does not apply to other network flow problems.

Hence, for non-looped networks, it is possible to fix the flow variables and remove them from the problem, reducing its size significantly. This has a direct impact on the selection of the solution procedure. For instance, for non-looped networks, if the continuous pressure variables are discretized, dynamic programming can be applied to find the optimal pressures. However, when considering looped networks, the flow variables cannot be uniquely determined, and the use of dynamic programming has now to deal with the issue of multi-dimensionality. More will be said about optimization algorithms in Section 4.

4 Optimization Algorithms

Dynamic programming (DP) has been by far the most popular technique for solving many classes of natural gas pipeline networks since the late 1960s. One of the main reasons is that, in

a DP framework, is relatively straightforward to satisfy the pipeline constraints and to handle the non-convexity of the feasible domain. Today, DP can be successfully applied to optimally solve very large non-looped networks, and some subclasses of looped systems.

DP approaches include that of Wong and Larson (1968), applied to gun-barrel networks; Lall and Percell (1990), applied to tree networks; and, more recently, Carter (1998), applied to looped networks. Other approaches based on local optima techniques (Percell and Ryan 1987) and hierarchical methods (Anglard and David 1988; Luongo et al. 1989; Osiadacz 1994) have been applied with modest success. These methods deliver local optimum solutions, which makes them unattractive for this particular type of problems given the typical non-convex behavior of the objective function and its large number of local optima. We must point out, though, that many of the methodologies in the operations research repertoire have yet to be explored. See Carter (1998) for a survey.

4.1 Dynamic Programming on Non-Looped Networks

We assume the reader is familiar with the basic concepts of DP (Chapter 2.3.4). As we have discussed, in non-looped systems it is possible to determine unique values for mass flow rate variables so they can be eliminated from further consideration, leaving us with the problem of determining the optimal set of pressures. Let \bar{x} denote this unique set of mass flow rates. Let us also assume throughout this section that we are dealing with high-pressure networks, that is pipeline constraints (5) are given by

$$p_i^2 - p_j^2 = K_{ij}x_{ij}^2 \quad (i, j) \in \mathcal{A}_p \quad (8)$$

Furthermore, we assume without loss of generality that compressor station arcs are joined by a single pipeline arc. Note that this is possible when the mass flow rates are fixed due to the presence of the pipeline constraints (8). To see this, consider compressor arcs (i_s, i_d) and

(j_s, j_d) (subindex s and d stand for suction and discharge nodes, respectively) and suppose that between these two there is a sub-network $G' = (\mathcal{V}', \mathcal{A}')$. Let P be a path (of arcs) in \mathcal{A}' from node i_d to j_s . Then, by adding up each pipeline equation corresponding to arcs in P , we obtain $p_{i_d}^2 - p_{j_s}^2 = K'_{i_d, j_s}$, where $K'_{i_d, j_s} = \sum_{(k,l) \in P} K_{kl} \bar{x}_{kl}$ is a constant.

Serial Systems: We now define the DP formulation for a serial system consisting of T compressor stations. We identify each compressor station (i, j) with a stage t , $t = 1, 2, \dots, T$. Boundary stages 0 and $T + 1$ are associated with single nodes 1 and n , respectively. For each stage (see Figure 4), we define:

y_t state variable; $y_t \equiv p_i$ (suction pressure), $t = 1, \dots, T + 1$

u_t decision (control) variable; $u_t \equiv p_j$ (discharge pressure), $t = 0, \dots, T$

$U_t(y_t)$ feasible domain; $U_t(y_t) = \{u : (\bar{x}_{ij}, y_t, u) \text{ satisfies (1d)}\}$, $t = 1, \dots, T$, and

$$U_0(y_0) = \{u : p_1^L \leq u \leq p_1^U\}$$

$f_t(u_t)$ state transition function; from eq. (8), $y_{t+1} = f_t(u_t) = (u_t^2 - K_{jk} \bar{x}_{jk}^2)^{1/2}$, $t = 0, \dots, T$, where (j, k) is the pipeline arc succeeding compressor (i, j)

$g_t(y_t, u_t)$ cost function; from eq. (7), (for a single unit) given by $g_t(y_t, u_t) = g_{ij}^{\text{unit}}(\bar{x}_{ij}, y_t, u_t)$

The backwards recurrence relationship is given by

$$J_t(y_t) = \min_{u_t \in U_t(y_t)} \{g_t(y_t, u_t) + J_{t+1}(y_{t+1})\}, \quad t = 0, \dots, T,$$

where $J_t(y_t)$ is the minimum fuel cost from station t to T when the suction pressure at stage t is y_t . The boundary condition is given by $J_{T+1}(y_{T+1}) = 0$, and the solution by $J_0(y_0)$, where $g(y_0, u_0) = 0$ for all (y_0, u_0) . Note that for given y_t , the feasible domain $U_t(y_t)$ is merely a closed interval for u , so the non-convexity issue is easily handled in DP.

The application of the recurrence relationship is straightforward. A discharge pressure grid is constructed at each stage by discretizing over its feasible range. The recurrence relationship is thus applied to a finite number of discharge pressures. The optimal discharge pressure at each stage is obtained by applying the recurrence relationship backwards, that is, by first analyzing stage T , then $T - 1$, and so on, until all stages are included.

It is also possible to derive a forward formulation for a serial system. To this end, we define the following entities at each compressor (i, j) .

- z_t state variable; $z_t \equiv p_j$ (discharge pressure), $t = 0, \dots, T$
- v_t decision (control) variable; $v_t \equiv p_i$ (suction pressure), $t = 1, \dots, T + 1$
- $V_t(z_t)$ feasible domain; $V_t(z_t) = \{v : (\bar{x}_{ij}, v, z_t) \text{ satisfies (1d)}\}$, $t = 1, \dots, T$, and
 $V_{T+1}(z_{T+1}) = \{v : p_n^L \leq v \leq p_n^U\}$
- $f_t(u_t)$ state transition function; from eq. (8), given by $z_{t-1} = f_t(v_t) = (v_t^2 + K_{ki}\bar{x}_{ki}^2)^{1/2}$,
 $t = T + 1, \dots, 1$, where (k, i) is the pipeline arc preceding compressor (i, j)
- $h_t(v_t, z_t)$ cost function; from eq. (7), (for a single unit) given by $h_t(v_t, z_t) = g_{ij}^{\text{unit}}(\bar{x}_{ij}, v_t, z_t)$

The forward recurrence relationship is thus given by

$$J_t(z_t) = \min_{v_t \in V_t(z_t)} \{h_t(v_t, z_t) + J_{t-1}(z_{t-1})\}, \quad t = 1, \dots, T + 1,$$

where $J_t(z_t)$ is the minimum fuel cost from station 1 to t when the discharge pressure at stage t is z_t . The boundary condition is given by $J_0(z_0) = 0$, and the solution by $J_{T+1}(z_{T+1})$, where $h(v_{T+1}, z_{T+1}) = 0$ for all (v_{T+1}, z_{T+1}) .

Diverging Branch Systems: Diverging branch systems (i.e., networks whose nodes have all indegree equal to one, except for a single node (root) which has indegree equal to zero) can

be solved with relatively the same amount of effort as the one required for serial systems. Let node 1 denote the root node and r the corresponding root station. The DP formulation is modified as follows:

- (i) Let $J_t(y_t)$ denote the minimum fuel cost of the sub-network rooted at station t when the suction pressure is y_t ; $t = 1, \dots, T$.
- (ii) The (backward) recurrence relationship is given by

$$J_t(y_t) = \min_{u_t \in U_t(y_t)} \left\{ g_t(y_t, u_t) + \sum_{w \in \mathcal{C}_t} J_w(y_w) \right\}, \quad t = 1, \dots, T$$

where \mathcal{C}_t is the set of compressor stations succeeding t . Note that for each $w \in \mathcal{C}_t$, the transition function $y_w = f_t(u_t)$ is accordingly defined by $f_t(u_t) = (u_t^2 - K_{jk}\bar{x}_{jk}^2)^{1/2}$, where (j, k) is the pipeline arc joining compressors t and w .

- (iii) For given y_r , the solution is given by $J_r(y_r)$.

Note that if t is a station in the boundary (i.e., a station with no succeeding stations) then we have $\mathcal{C}_t = \emptyset$. The stages must be carefully labeled so as to ensure that, a stage with diverging branches is computed after its corresponding succeeding stages. Figure 5 shows a diverging stage with three branches, where $\mathcal{C}_t = \{w_1, w_2, w_3\}$.

Converging Branch Systems: Converging branch systems (i.e., networks whose nodes have all outdegree equal to one, except for a single node (root) which has outdegree equal to zero) are handled essentially in the same way as diverging branch systems. In this case, a forward recurrence formulation is employed.

General Tree Systems: In the case of general tree structures with multiple converging and diverging branches, the DP formulation makes use of both backward and forward recursion to

deal with diverging and converging branches, respectively. To help understand the formulation, let us introduce some notation. Let a be any (source) node with zero indegree and let b be any node with zero outdegree (reachable from a). The path from a to b is called the *main path* or main stream of the network. The path merely acts as a reference and its selection is arbitrary. We then proceed to label all paths in the remaining branches as follows. Starting with a leave node (i.e, either indegree or outdegree equal to one), which does not belong to the main path, we form a path by iteratively adding adjacent nodes whose indegree and outdegree are both equal to one (i.e., right before we reach a branching node). We call this path (along with its respective nodes and stations contained in it) *diverging* if the leave node where we started has indegree equal to one; otherwise, we call it *converging*. We remove the path from the network (just for labeling purposes) and repeat all over until all nodes in the network have been labeled as either diverging or converging. As a convention, the main path is referred to as diverging.

We now introduce the supported sub-network concept for each stage t . Let (i, j) be the compressor arc associated with compressor t and let $G = (V, E)$ be the undirected version of the network. The *sub-network supported by t* , G_t , is the sub-network (containing t) that remains after the edges in E in the $a-i$ or $a-j$ (whichever is shorter) path are removed. This is illustrated in Figure 6, where each of the supported sub-networks is shown within dotted frames. Consider $t = 4$, for instance. A path from a to 4 is formed by arcs 1 and C . By removing these arcs, G_4 is the connected component containing station 4.

We now give a DP formulation for the entire network based on backward recursion for the main and diverging paths and forward recursion for converging paths. These are the fundamental components:

- (i) Let $J_t(\alpha_t)$ denote the minimum fuel cost of the network supported by t when the state variable at stage t is α_t ; $t = 1, \dots, T$. If t is a diverging stage, $\alpha_t = y_t$ refers to the

suction pressure at t . Otherwise, $\alpha_t = z_t$ refers to the discharge pressure at t .

(ii) The recurrence relationship is thus given by

$$J_t(y_t) = \min_{u_t \in U_t(y_t)} \left\{ g_t(y_t, u_t) + \sum_{w \in \mathcal{C}_t^{\text{div}}} J_w(y_w) + \sum_{w \in \mathcal{C}_t^{\text{conv}}} J_w(z_w) \right\}, \text{ if } t \text{ is diverging,}$$

and

$$J_t(z_t) = \min_{v_t \in V_t(z_t)} \left\{ h_t(v_t, z_t) + \sum_{w \in \mathcal{C}_t^{\text{div}}} J_w(y_w) + \sum_{w \in \mathcal{C}_t^{\text{conv}}} J_w(z_w) \right\}, \text{ if } t \text{ is converging,}$$

where $\mathcal{C}_t^{\text{div}}$ ($\mathcal{C}_t^{\text{conv}}$) is the set of all compressor stations in G_t succeeding (preceding) t .

Note that for $w \in \mathcal{C}_t^{\text{div}}$, the transition function $y_w = f_t(u_t)$ is defined by $f_t(u_t) = (u_t^2 - K_{jk}\bar{x}_{jk}^2)^{1/2}$ (pipeline arc (j, k) succeeding compressor (i, j)), whereas for $w \in \mathcal{C}_t^{\text{conv}}$, the transition function is $z_w = f_t(v_t) = (v_t^2 + K_{ki}\bar{x}_{ki}^2)^{1/2}$ (pipeline arc (k, i) preceding (i, j)).

(iii) For given y_a , the solution is given by $J_a(y_a)$, where a is the preselected source node.

Again, the stages are carefully labeled so as to ensure that, a stage with branches is computed after its corresponding stages in the sub-network supported by it.

4.2 DP on Looped Networks

As mentioned in Section 3, looped networks are more difficult to handle as we must deal with both pressure and flow variables. One way to simplify this task is to assume the flow variables are known. This is a very common assumption from the practical stand-point. By doing this, we avoid the multi-dimensionality issue in DP; however, DP is applied in a slightly different way. Rather than attempting to formulate DP recursively, we proceed to combine simple elements of the system into an equivalent composite element. This process is applied iteratively until there is exactly one element left, which completely characterizes the behavior of the entire

system. At a given iteration, the elements can be selected from any position in the system, so this is referred as Nonserial Dynamic Programming (NDP) (Bertelè and Brioschi 1972). This technique has been successfully applied to solve large instances with complex looped network topologies in reasonable times. The details of the implementation of NDP for looped networks can be found in Carter (1998).

5 Conclusion

The research potential in the natural gas industry from the optimization perspective is tremendous. In this chapter, we have presented only one of the problems in the field that have received a significant amount of attention over the past thirty years. We have seen how DP can be used to solve (within a given pressure discretization) non-looped network structures and how NDP is an attractive technique for addressing looped systems (for the particular modeling assumptions). However, we feel many of the important operations research techniques have not been used yet at full capacity on this type of problems. For instance, in addition to DP, there are global optimization methods (Chapter 2.6) that can be used to attempt to solve the problem optimally. In that direction, we have seen some preliminary work consisting of development of lower bounding schemes (Wu et al. 1999a) that can be used within these methodologies.

There is also an area of opportunity for discrete optimization methods, such as meta-heuristics (Chapter 2.3.7), applied to this continuous optimization problem. One can devise discrete neighborhoods in such a way that it is computationally efficient to generate feasible neighbors and then use local search techniques such as Tabu Search (Chapter 2.3.7.6), for instance, to try to overcome local optimality.

Likewise, it is an entire open area of research to investigate other variations of this problem.

- Other compressor station configurations such as reciprocating units, non-identical com-

processor units, non-serial units.

- Non-isothermal models, which would involve considering gas temperature as a system variable.
- Medium- and low-pressure networks, which would lead to different pipeline equations (5).
- Non-steady models, which implies considering analyzing the system as a function of time and undoubtedly stands as one of the most important challenges.

It is expected that within the next few years, more researchers with optimization background become involved in helping address these issues.

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