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Universidad Autónoma de Nuevo León

Facultad de Ingeniería Mecánica y Eléctrica

Computational Experience with Heuristics for the P - Center Problem

PROPOSAL

COURSE : *Selected Topics on Optimization*

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INTRODUCTION

The p-Center Problem, also known as the minimax location problem, is one of the best-known NP-hard discrete location problems. Its purpose is to determine the optimal location of exactly *p* facilities, minimizing the maximum distance between any demand node and the nearest facility. Simply put, this problem seeks to answer the following question: if we have to open exactly *p* facilities (for example, fire stations or ambulances), can we ensure that the client furthest away receives the best possible service under the given conditions? Unlike other models that attempt to minimize average cost or average distance, the p-Center Problem focuses on the worst-case scenario. That is, it doesn't focus so much on what happens "on average," but rather on ensuring that no one is too far from the service. This is why it is known as a minimax model: it minimizes the maximum distance between any demand node and its nearest facility. [1] [3]

The importance of this topic lies primarily in its focus on equity. In many real-world situations, especially in emergency services, it is unacceptable to neglect any one area simply because the overall average is fine. For example, when locating fire stations, ambulance bases, or disaster relief centers, the most important thing is that even the client living farthest away can receive help within a reasonable timeframe. In these cases, reducing the most critical response time can literally save lives. [2]

Furthermore, the problem presents different variations that model distinct real-world contexts. For example, the vertex-center p-problem assumes that facilities can only be located at specific nodes in a network, while the absolute p-problem allows them to be placed at any point along the edges (Kariv and Hakimi, 1979). Weighted versions also exist where nodes have varying importance or associated population, which is useful in urban and regional applications. Regarding its mathematical modeling, several formulations have been proposed to represent the problem efficiently. Elloumi, Labbé, and Pochet (2004) developed a new linear integer formulation that strengthens the classic model and improves the lower bounds, allowing for better computational performance in certain cases. These contributions demonstrate that, although the problem's structure is conceptually simple, its mathematical treatment requires careful analysis. [2]

In this proposal, we will investigate the behavior of the p-Center Problem from a computational perspective, analyzing its mathematical formulation, its fundamental constraints, and how it can be implemented for specific instances. We will also study how to evaluate the quality of a solution and how to build a model that adequately represents real-world service location situations. Our aim is to understand not only the formal structure of the problem but also its practical relevance and its impact on applications where ensuring equitable coverage is a priority.

SECTION 2: PROBLEM DESCRIPTION

In this section, we present the formal description of the p-Center problem. While the introduction explained a general, intuitive idea, it is now necessary to define it with greater mathematical precision. In optimization, a problem must clearly specify four fundamental elements: the input data (what information is known), the decisions (what we must choose), the objective function (what we want to optimize), and the constraints (what conditions must be met for a solution to be feasible). We will now explain each of these components for the p-Center problem.

1.- DATA INPUT: First, the input data consists of a set of demand nodes, representing the customers or users requiring service, and a set of candidate locations where the facilities could be installed. In many cases, both sets coincide and correspond to the nodes of a network. Additionally, a distance matrix d_{ij} is available, indicating the distance between each customer i and each possible location j. Finally, the value of p is known; this value is fixed and represents the exact number of facilities that must be located. [1]

I = set of demand nodes, $I = \{1, \dots, N\}$,
J = set of candidate facility sites, $J = \{1, \dots, \mathrm{M}\}$,
d_{ij} = distance between demand node $i \in I$ and candidate site $j \in J$,
P = number of facilities to be located,

2.- DECISIONS: Secondly, the model's decisions involve determining which locations to select for opening facilities and how each customer will be assigned to an open facility. Binary variables are used to represent this mathematically. The variable w_j takes the value 1 if a facility is located at candidate site j, and 0 otherwise. Similarly, the variable Y_{ij} takes the value 1 if demand node (customer) i is assigned to an open facility j, and 0 if they are not. These variables allow the real-world decisions to be translated into a mathematical language that can be processed by an algorithm. [1]

$$w_j = \begin{cases} 1 & \text{if a facility is located at candidate site } j \in J, \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{ij} = \begin{cases} 1 & \text{if demand node } i \in I \text{ is assigned to an open facility at candidate site } j \in J, \\ 0 & \text{otherwise,} \end{cases}$$

3.- OPTIMIZATION: Third, the objective function must reflect the main purpose of the problem: to minimize the maximum distance between any demand node and the facility serving them. It is important to understand that the goal here is not to minimize the sum of distances or the average, but the maximum value. For this reason, the model introduces an additional variable, D, which represents this maximum distance. The objective is to minimize D. Conceptually, this means that we are trying to minimize the worst-case scenario (i.e., the customer farthest from the service).

4.- CONSTRAINS: Regarding the constraints, these ensure that the solution is practically feasible. First, each customer must be assigned to exactly one facility. Second, a customer can only be assigned to a facility that is actually open. Third, the total number of open facilities must be exactly p. Finally, the variable D must be greater than or equal to the distance between each customer and their assigned facility, ensuring that D accurately represents the maximum distance in the system. With all this, the mathematical model should be as follows:

(1) Minimise D
subject to:

(2) $\sum_{j \in J} Y_{ij} = 1 \qquad \forall i \in I,$

(3) $Y_{ij} \leq w_j \qquad \forall i \in I,\ j \in J,$

(4) $\sum_{j \in J} w_j = P,$

(5) $D \geq \sum_{j \in J} d_{ij} Y_{ij} \qquad \forall i \in I,$

(6) $w_j,\ Y_{ij} \in \{0, 1\} \qquad \forall i \in I,\ j \in J,$

The objective function (1) minimises the maximum distance between each demand node and its closest open facility. Constraint (2) ensures that each demand node is assigned to exactly one facility, while constraints (3) restrict demand nodes to be assigned to open facilities. Constraint (4) stipulates that P facilities are to be located. Constraints (5) define the maximum distance between any demand node i and the nearest facility at node j . Finally, constraints (6) refer to integrality constraints. [1]

SECTION 3: PROBLEM EXAMPLE

In this section, we present the formal description of the p-Center problem. While the introduction explained a general, intuitive idea, it is now necessary to define it with greater mathematical precision. In optimization, a problem must clearly specify four fundamental elements: the input data (what information is known), the decisions (what we must choose), the objective function (what we want to optimize), and the constraints (what conditions must be met for a solution to be feasible). We will now explain each of these components for the p-Center problem.

Now we will apply the algorithm to the next problem example:

1.- INPUT

A weighted network of n demand nodes and edges with non-negative travel distances, plus an integer p (number of facilities).

Given:

- $N = \{P, Q, R, S, T, U, V, W\}$
- $n = 8$

- $p = 1$ (parts a and b)
- $p = 2$ (parts c)

2.- DECISIONS

The location of p facilities on the network at any node or interior point of an edge (absolute), or at nodes only (vertex).

- Where to place P* (parts a, b) or P1 and P2 (part c) on the tree.

3.- OPTIMIZATION

Minimize the coverage radius $D = MAX\, d_{ij}$ over all demand nodes i: the worst-case distance from any node to its nearest facility.

- $D = 23$ (part a) $D = 26$ (part b) $D = 18.5$ (part c)

4.- CONSTRAINTS

Exactly p facilities must be placed. Each node is assigned to its nearest facility. Locations must lie on the network.

- p = 1 or p = 2 facilities on the 8-node tree.

To illustrate the p-Center Problem, we construct a small custom instance: a tree of 8 nodes (labeled P through W) connected by 7 weighted edges, where each weight represents the travel distance between two adjacent nodes. Because the network is a tree connected graph with no cycles there is exactly one path between any pair of nodes, so the distance between them is simply the sum of the edge weights along that path.

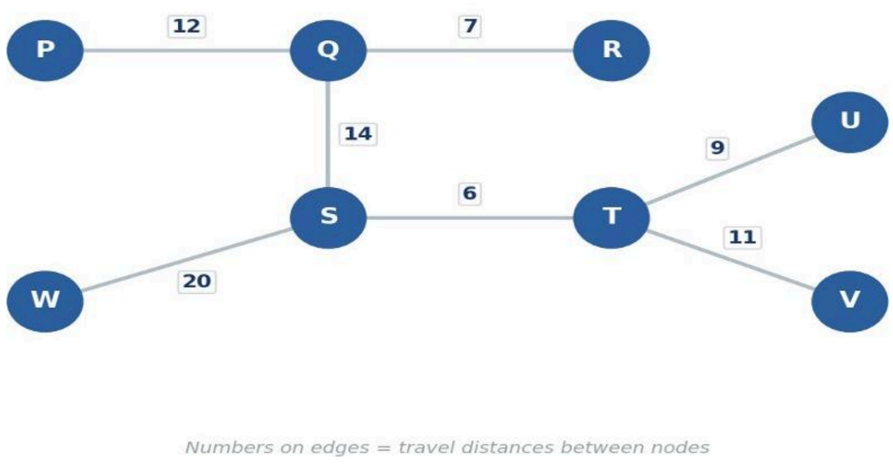

FIGURE 1. NETWORK INSTANCE.

d_{ij}	P	Q	R	S	T	U	V	W
P	0	12	19	26	32	41	43	46
Q	12	0	7	14	20	29	31	34
R	19	7	0	21	27	36	38	41
S	26	14	21	0	6	15	17	20
T	32	20	27	6	0	9	11	26
U	41	29	36	15	9	0	20	35
V	43	31	38	17	11	20	0	37
W	46	34	41	20	26	35	37	0

TABLE 1. MATRIX DISTANCE.

PART (A).

The Absolute 1-Center asks: where on the tree (at any node or any interior point of an edge) should a single facility be placed to minimize the worst-case distance to any node? Formally, if x is any point on the tree, we want to minimize $D(x) \ = maxd_{xj}$

Theorem (Daskin, 1995): On a weighted tree, the absolute 1-center is the midpoint of the diameter — the longest shortest-path between any two nodes. This follows because D(x) is convex along any path in the tree, so its minimum is at the midpoint of the diameter [4].

From Table 1. , the diameter is d(P, W) = 12+14+20 = 46 units, achieved along path P-Q-S-W. The optimal facility location P* sits at the midpoint of this path, 46/2 = 23 units from each endpoint. Tracing from P: after 12 units we reach Q, after 26 units we reach S — so the midpoint at 23 units falls 11 units past Q on edge Q–S, leaving 3 units to S.

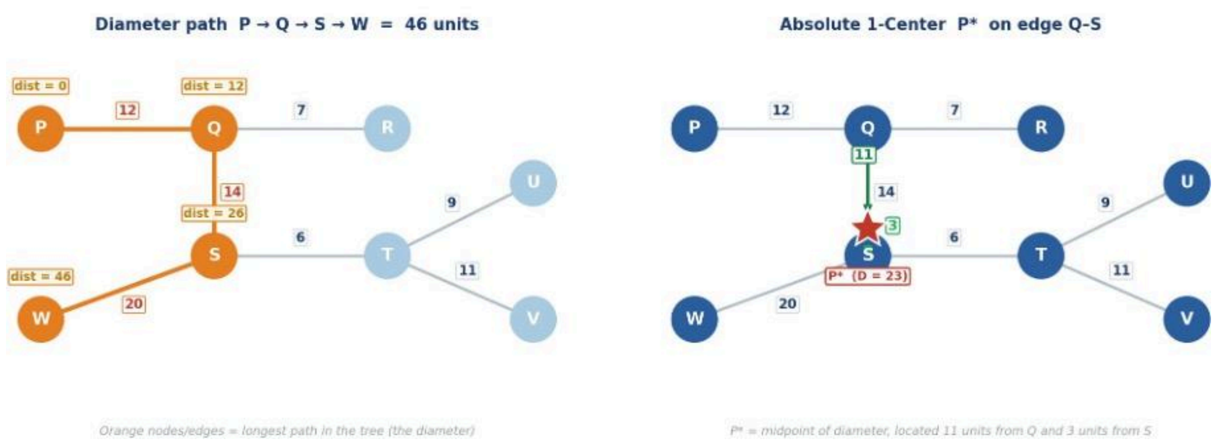

FIGURE 2. FINDING THE DIAMETER AND PLACING P*.

To confirm the radius, we check the distance from P* to every node by routing through the nearest endpoint of its edge (Q at distance 11, or S at distance 3). Nodes P and W are each exactly 23 units away, confirming the radius. All other nodes are closer: Q (11), R (18), S (3), T (9), U (18), V (20).

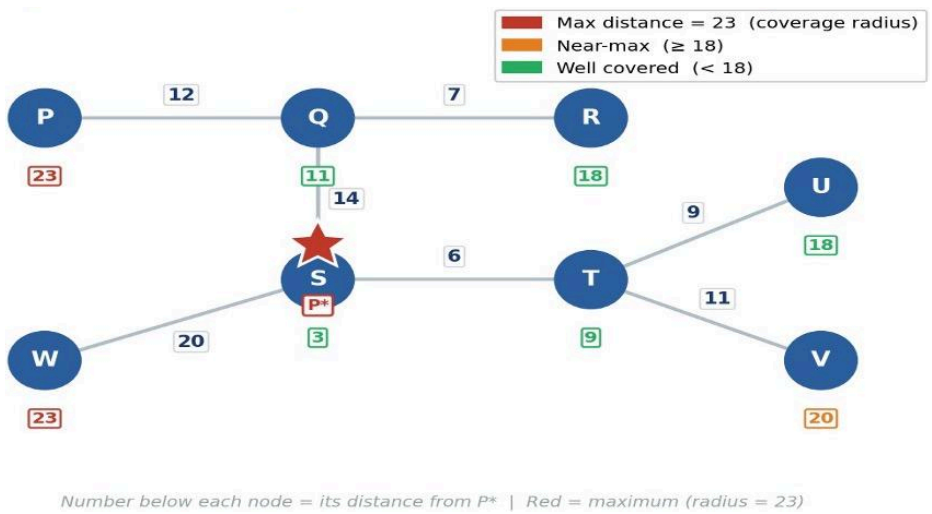

FIGURE 3. COVERAGE RADIUS OF P*.

Result (a): The absolute 1-center is P* on edge Q–S, 11 units from Q. Optimal radius = D = 23 units.

PART (B).

In the Vertex 1-Center variant, the facility must be placed at one of the eight nodes. For each candidate node v, its eccentricity e(v) $= max_{i \epsilon N}$d(v,i) measures the worst-case coverage if v is chosen as the facility. The vertex 1-center is the node with the smallest eccentricity.

Scanning the rows of Table 3.1, node S has eccentricity 26 (its farthest node is P or W at distance 26), which is the minimum across all nodes. Every other node would leave at least one demand point farther than 26 units away. Compared to the absolute 1-center (D=23), the vertex solution is 3 units worse because the optimal location P* lies strictly inside edge Q–S, unavailable to the vertex formulation.

PART (C).

With two facilities, each node is served by its nearest facility. The key result for trees is that the optimal solution always partitions the tree into two connected subtrees by removing exactly one edge.

Theorem (Daskin, 1995): On a weighted tree, the optimal 2-center solution splits the tree into two connected subtrees by removing one edge. It therefore suffices to test all 7 edges as candidate cuts and pick the one that minimizes max(R1, R2), where Ri is the 1-center radius of subtree i [4].

CUT	T1 (NODES)	R1	T2 (NODES)	R2	MAX(R1,R2)
P-Q	{P}	0	{Q,R,S,T,U,V, W}	20.5	20.5
Q-R	{R}	0	{P,Q,S,T,U,V, W}	23	23
Q-S	**{P,Q,R}**	**9.5**	**{S,T,U,V,W}**	**18.5**	**18.5**
S-T	{P,Q,R,S,W}	23	{T,U,V}	10	23
T-U	{U}	0	{P,Q,R,S,T,V, W}	23	23
T-V	{V}	0	{P,Q,R,S,T,U, W}	23	23
S-W	{W}	0	{P,Q,R,S,T,U, V}	21.5	21.5

TABLE 2. ALL 7 EDGE CUTS.

The optimal cut is edge Q–S, splitting the tree into T1 = {P, Q, R} (diameter 19, center P1 on edge P–Q at 9.5 fromP) and T2 = {S, T, U, V, W} (diameter 37 along path V-T-S-W, center P2 on edge S–W at 1.5 from S). Both radii equal 18.5.

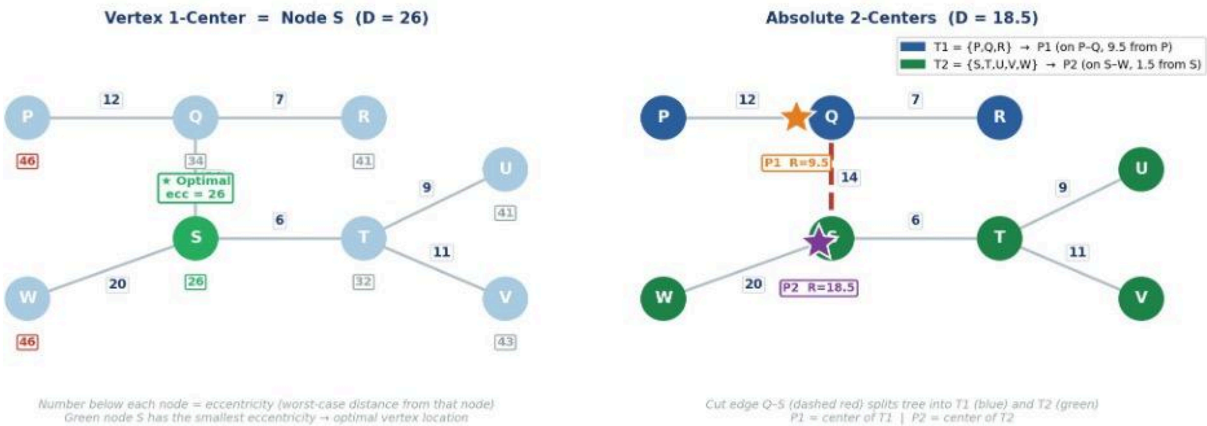

FIGURE 4. VERTEX 1-CENTER AND ABSOLUTE 2-CENTERS.

- **Result (b):** Vertex 1-center = Node S, D = 26 units.
- **Result (c):** P1 on edge P–Q (9.5 from P) and P2 on edge S–W (1.5 from S). Optimal radius = D = 18.5 units.

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