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Heuristic Solution Methods for Two Location Problems with Unreliable Facilities

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In this paper, the p-median and p-centre problems are generalized by considering the possibility that one or more of the facilities may become inactive. The unreliable p-median problem is defined by introducing the probability that a facility becomes inactive. The (p, q)-centre problem is defined when p facilities need to be located but up to q of them may become unavailable at the same time. An heuristic procedure is presented for each problem. A rigorous procedure is discussed for the (p, q)-centre problem. Computational results are presented.

Key words: heuristics, location, stochastic

INTRODUCTION

The p-median[1–4] and the p-centre[3,5] problems have been investigated quite extensively. Locations for p facilities among n demand points are required. The objective of the p-median problem is to minimize the sum of weighted distances between each demand point and its closest facility. The p-centre problem is similar. Here the maximal weighted distance between demand points and their closest facility is to be minimized. The p-median problem is an extension of the minisum Weber problem (sometimes called one-median by this terminology),[6–7] and the p-centre problem is the extension of the single-facility minimax problem (also called the one-centre problem by this terminology).[9–11]

A problem of locating p facilities, like the p-median or p-centre problems, is considered here. It is assumed that the facilities are unreliable and may fail to render the service. In the case of failure of a facility, the customer (demand point) must resort to the 'second best' facility.

The unreliable p-median problem is defined by assuming that a facility has a given probability of becoming inactive. In practice it may occur because of physical problems with the facility in an unexpected fashion, or it may represent a situation where a facility is idle part of the time because of planned regular maintenance.

The p-centre problem is extended to the (p, q)-centre problem. There are p facilities to be located, and the system should provide for a situation where at most q facilities may become inactive. This may happen in one of three ways (the first two seem to be applicable only for $q = 1$): an unexpected breakdown of the facility; the facility may be servicing one customer and therefore is unavailable for the service of another (e.g. fire stations, ambulance services, etc.); normal planned maintenance. For example, if breakdown of facilities is independent, with a probability of 5%, then protecting against the breakdown of one facility should be included in the objective function, while a breakdown of two facilities (probability of 0.25%) can be ignored. If the probability of a breakdown is 20%, then breakdown of two facilities is still likely (4%) and should be protected against, while the probability of simultaneous breakdown of three facilities is 0.8% and can be ignored in the objective function. The case $q = 1$ seems to have the most applications. However, the problem is formulated for general q because it is not more difficult. $q = 2$ can be operational if, for example, there is always one facility that is going through a planned maintenance procedure and we would like to protect against the possibility that another facility becomes unavailable unexpectedly. We find the best location for the p facilities when q facilities become inactive simultaneously. It is possible, of course, that $q + 1$ or more facilities become inactive. However, we ignore such an event in our model. This is similar to assuming that $q = 0$ in the ordinary p-centre problem and ignoring the possibility of a positive q. The $(p, 0)$-centre problem is, of course, the ordinary p-centre problem.

In the following sections, we present a heuristic algorithm for the solution of the unreliable p-median problem, i.e. not guaranteeing an optimal solution. Then we analyse the (p, q)-centre problem by giving some simple examples, prove a theorem about the nature of the solution, discuss an algorithm for solving it, and present a heuristic algorithm that can solve relatively large problems without guaranteeing an optimum. Some computational results are presented.

THE UNRELIABLE p-MEDIAN PROBLEM

Assume that there is a known probability that facility j, $1 \leqslant j \leqslant p$, becomes inactive. Also assume that the probability that $1 \leqslant k \leqslant p$ facilities $j_1, j_2, \ldots, j_k$ become inactive is known. Normally one would assume independence of such events. We choose not to do so because, if the unreliability of a facility is caused in a planned manner, then the probability of two or more facilities becoming inactive is zero. We only ignore the possibility that all p facilities become inactive because, in such a case, the location of the p facilities is irrelevant to the objective function.

Let

- $\mathbf{X}$ be the location vector for p facilities;
- $d_{ij}(\mathbf{X})$ be the distance between demand point i and facility j by any given norm;
- w_i be the weight associated with demand point i (note that w_i is independent of the facility supplying the service; it can be the amount of service required by demand point i);
- $j(i, k, \mathbf{X})$ be the facility at the k'th least distance from demand point i, for $k = 1, \ldots, p$ for given locations of the p facilities $\mathbf{X}$ (this means $d_{i,j(i,1,\mathbf{X})}(\mathbf{X}) \leqslant d_{i,j(i,2,\mathbf{X})}(\mathbf{X})$; $\leqslant \ldots \leqslant d_{i,j(i,p,\mathbf{X})}$);
- $P(i, k, \mathbf{X})$ for $1 \leqslant k \leqslant p$ be the probability that facility $j(i, k, \mathbf{X})$ is active, while facilities $j(i, 1, \mathbf{X}), \ldots, j(i, k-1, \mathbf{X})$ are not active; for example, $P(i, 1, \mathbf{X})$ is the probability that the closest facility to demand point i is active.

Note that when all $P(i, 1, \mathbf{X}) = 1$, then the problem is the regular p-median problem.

The objective function $F(\mathbf{X})$ to be minimized is:

$$F(\mathbf{X}) = \sum_{i=1}^{n} w_i \sum_{k=1}^{p} P(i, k, \mathbf{X}) d_{i,j(i,k,\mathbf{X})}(\mathbf{X}), \tag{1}$$

where $F(\mathbf{X})$ measures the expected sum of weighted distances between demand points and their closest available facility.

The heuristic algorithm presented here is based on Cooper's[1,2] ideas for location–allocation problems. Cooper solves the p-median problem by selecting p locations for the new facilities. A set of demand points is assigned to each new facility. This set consists of all demand points that are closest to the particular new facility. p 1-median problems are solved (one for each new facility), and the solution points define new locations for the new facilities. The process is repeated until no change occurs in any of the sets. Further improvements might be possible by applying the ideas in Love and Juel.[4] The algorithm is based on the translation of the objective function into a sum of one-median objective functions. Let $k(i, j, \mathbf{X})$ be the inverse of $j(i, k, \mathbf{X})$, i.e. the position of the distance $d_{ij}(\mathbf{X})$ in the sorted vector of distances between demand point i and all p facilities. By definition $j(i, k(i, j, \mathbf{X}), \mathbf{X}) = j$. Rearranging (1) yields

$$F(\mathbf{X}) = \sum_{j=1}^{p} F_j(\mathbf{X}), \tag{2}$$

where

$$F_j(\mathbf{X}) = \sum_{i=1}^{n} w_i d_{ij}(\mathbf{X}) P(i, k(i, j, \mathbf{X}), \mathbf{X}). \tag{3}$$

This suggests the following heuristic algorithm.

Heuristic algorithm for the solution of the unreliable p-median problem

(1) Start with p distinct locations for p new facilities as the vector $\mathbf{X}^{(0)}$. Set the iteration counter r to zero.
(2) For each demand point i, find $k(i, j, \mathbf{X}^{(r)})$ for all j, $1 \leqslant j \leqslant p$.
(3) If $r > 0$: if $k(i, j, \mathbf{X}^{(r)}) = k(i, j, \mathbf{X}^{(r-1)})$ for all i and j, stop with $\mathbf{X}^{(r)}$ as the solution.
(4) Otherwise, for each j, solve a one-median problem[6–8] defined by (3) using $\mathbf{X} = \mathbf{X}^{(r)}$ in the definition of the probabilities. Each such solution defines the location of facility j in the new location vector $\mathbf{X}^{(r+1)}$. Go to step 2 with $r = r + 1$.

If every one-median problem solved in step 4 of the algorithm possesses a unique solution, the algorithm must terminate with a solution. A unique solution exists, for example, for l_p distances for $p > 1$ when all demand points are not collinear. When a unique solution exists, the objective function decreases every iteration, and therefore the same solution cannot be encountered twice. In any case, when such uniqueness of the solution point cannot be guaranteed, a limit on the number of iterations always yields a solution point. As starting locations for the p facilities, one can use the p-median solution, if available, or generate one or more starting points and select the best solution obtained by the algorithm.

We tested the algorithm for Euclidean distances problems. We randomly generated demand points in a one-by-one square. For each such problem we generated 10 different starting points, and compared the final solutions for these starting points. The results are summarized in Table 1. The minimum, average and maximum cost and run times obtained in these 10 different solutions are presented. The cost for each problem is normalized by the best solution obtained for that problem. We used $w_i = 1$ $P(i, 1, \mathbf{X}) = 0.95$ and $P(i, 2, \mathbf{X}) = 0.05$ for all i. These particular probabilities entail there being a 95% chance that the closest facility is active, and a 5% chance that the closest facility is inactive but the second closest is active. In this case, when the first facility is inactive, then the second one is always available, and demand points never resort to the third closest facility. Different values for these parameters do not make it more difficult computationally. The optimal locations depend on the probability values. If $P(i, 1, \mathbf{X}) = 1$ for all i, then the problem is the p-median problem. In the other extreme, if $P(i, k, \mathbf{X}) = 1/p$ for all i and k, then the problem is equivalent to the 1-median problem, and all p facilities are located at the 1-median solution point.

The program was written in FORTRAN IV on the Amdahl 580 computer at the University of Michigan, Ann Arbor.

THE (p, q)-CENTRE PROBLEM

The same notation that was used in the formulation of the unreliable p-median problem can be used for the formulaton of the (p, q)-centre problem. Since at most q facilities may become unavailable simultaneously, the worst case for demand point i is having to use the $q + 1$'th distanced facility. This distance is defined as $d_{i,j(i,q+1,\mathbf{X})}(\mathbf{X})$. Therefore, the (p, q)-centre problem is

$$\operatorname*{Min}_{\mathbf{X}} \left\{ \operatorname*{Max}_{i} \left\{ w_i d_{i,j(i,q+1,\mathbf{X})}(\mathbf{X}) \right\} \right\}. \tag{4}$$

TABLE 1. *Heuristic for the unreliable p-median problem*

		Objective function		Run times (sec)		
n	p	$F_{ave.}/F_{min.}$	$F_{max.}/F_{min.}$	Min.	Ave.	Max.
50	2	1.061	1.140	0.009	0.013	0.017
50	5	1.093	1.143	0.033	0.061	0.095
50	10	1.132	1.335	0.037	0.071	0.114
100	2	1.053	1.158	0.017	0.022	0.026
100	5	1.057	1.106	0.063	0.148	0.256
100	10	1.058	1.137	0.078	0.136	0.186
200	2	1.055	1.128	0.034	0.044	0.055
200	5	1.023	1.063	0.200	0.367	0.804
200	10	1.024	1.053	0.205	0.309	0.396
500	2	1.038	1.072	0.086	0.098	0.106
500	5	1.027	1.046	0.804	1.127	1.763
500	10	1.013	1.060	0.892	1.314	1.900

In order to get an intuitive feel for the properties of the solution, we first check several examples. For simplicity of explanation, we assume Euclidean distances and all $w_i = 1$.

Consider the problem where four demand points are located at the corners of a square. The $(1, 0)$-centre solution is located at the centre of the square. The $(2, 0)$-centre solution points are located at the centres of two opposite sides of the square. The $(3, 0)$-centre problem yields the same solution. The third facility can be put anywhere on the plane, and the objective function remains the same. The $(4, 0)$-centre solution points are located at the four corners of the square. Additional facilities do not improve the solution. We now turn to unreliable facilities. The $(2, 1)$-centre solution points are both located at the centre of the square (see theorem 1 below). The $(3, 1)$-centre problem is more interesting. However, all the solution points are again located at the centre of the square. The same solution is obtained if two facilities are put at the centre and the third one is put anywhere on the plane. One of the $(4,1)$-centre solutions is a pair of facilities located at each of the centres of opposite sides of the square [like the $(2, 0)$-centre solution]. Another optimal solution is to locate the four facilities at the centres of the sides of the square.

Similar configurations are obtained when demand points are spread all over the interior of a circle and its circumference. It can be shown that all the $(3, 1)$-solution points are located at the centre of the circle, and in fact only two facilities there will suffice.

A different configuration is obtained when there are three demand points located at the vertices of an equilateral triangle. In this case, the $(3, 1)$-centre solution points are located at the centres of the side of the triangle. A removal of any facility will retain a maximal distance of half the side of the triangle. The same solution is obtained when demand points are spread all over the triangle (including the three vertices). In fact, a $(p, 1)$-centre solution for demand points located at the vertices of a simple polygon of p sides is at the centres of these sides.

These examples demonstrate that sometimes a solution is redundant because fewer facilities yield the same objective function, and sometimes all facilities are contributing to the solution. We also observe that, in some cases, the solution to the (p, q)-centre problem is obtained by 'doubling' solutions for an r-centre problem for $r < p$.

Theorem 1

A new facility which is part of a $(p, p - 1)$-centre problem solution must be located at a solution point to the 1-centre problem.

Proof

If $p - 1$ facilities are unavailable, the remaining facility must serve all the demand points with the minimal maximum distance, and therefore must be located at a 1-centre solution point. This argument holds for all new facilities, and therefore each of them must be located at a solution point to the 1-centre problem, QED.

We examined many $(4, 1)$-centre problems, and in most cases the solution was a 'doubled' $(2, 0)$-centre solution. (Other solutions with the same value of the objective function may exist, though.) It was very difficult to find an example where it is not so. One contrived example for which there is *no* solution to the $(4, 1)$-centre problem which is a doubling of a $(2, 0)$-centre solution is given in the Appendix.

A procedure that finds the optimal solution to the (p, q)-centre Euclidean problem can be constructed along the lines described in Drezner.[5] The only change in the formulation is in the set-covering problem. The right-hand side of these equations must be $q + 1$ rather than 1 because each demand point must be covered by $q + 1$ facilities rather than one facility. Such an algorithm has the same complexity bound of $0(n^{2p+1} \log n)$ given there. [Solving the covering-set problem by total enumeration requires $0(n^{2p+1})$ time, and $0(\log n)$ such problems must be solved by a bisection approach.] This restiсts the usefulness of the algorithm to small values of n and p. Therefore, we suggest a heuristic approach.

We generalize the ideas that were presented by Cooper[1,2] and applied to the p-centre problem by Drezner.[5] For a given transformation $j(i, k, \mathbf{X})$, define the set $I(j, k, \mathbf{X})$:

$$I(j, \mathbf{X}) = \{i \,|\, k(i, j, \mathbf{X}) \leqslant q + 1\}. \tag{5}$$

TABLE 2. *Heuristic for the (p, 1)-centre problem*

		Objective function		Run times (sec)		
n	p	$F_{ave.}/F_{min.}$	$F_{max.}/F_{min.}$	Min.	Ave.	Max.
50	3	1.000	1.000	0.002	0.003	0.004
50	5	1.039	1.056	0.004	0.005	0.007
50	10	1.185	1.594	0.004	0.009	0.016
100	3	1.000	1.000	0.004	0.006	0.009
100	5	1.027	1.067	0.008	0.011	0.013
100	10	1.071	1.243	0.015	0.022	0.034
200	3	1.000	1.000	0.010	0.012	0.015
200	5	1.013	1.036	0.014	0.022	0.028
200	10	1.039	1.124	0.042	0.050	0.068
500	3	1.000	1.000	0.025	0.034	0.045
500	5	1.018	1.051	0.043	0.057	0.097
500	10	1.043	1.137	0.107	0.148	0.186

Then problem (4) is equivalent to

$$\underset{\mathbf{X}}{\text{Min}}\,\{\underset{j}{\text{Max}}\,\{F_j(\mathbf{X})\}\}, \tag{6}$$

where

$$F_j(\mathbf{X}) = \underset{i \in I(j, \mathbf{X})}{\text{Max}}\,\{w_i d_{ij}(\mathbf{X})\}. \tag{7}$$

Heuristic algorithm for the solution of the (p, q)*-centre problem*

(1) Start with p distinct locations for p new facilities, $\mathbf{X}^{(0)}$, and set the iteration counter r to zero.
(2) For each facility j, find $I(j, \mathbf{X}^{(r)})$ by (5).
(3) If $r > 0$: if $I(j, \mathbf{X}^{(r-1)}) = I(j, \mathbf{X}^{(r)})$ for all j, stop with $\mathbf{X}^{(r)}$ as the solution vector.
(4) Otherwise, solve the 1-centre problem[9–11] of minimizing $F_j(\mathbf{X})$ of (7), for the given sets $I(j, \mathbf{X}^{(r)})$. Go to step 2 with $r = r + 1$.

We tested the algorithm on randomly generated problems, as in the unreliable p-median case. We tested Euclidean distances and $q = 1$ only. Few changes in the results are expected for $q > 1$. The results are summarized in Table 2.

APPENDIX

The (4, 1)-centre solution of the problem in Figure 1 is not a 'doubled' (2, 0)-centre solution. The problem consists of many points densely arranged on the circumference of a circle of radius R, centred at 0, and the points D, E and F that are a little outside the circle and form an equilateral triangle. Each of the points A, B and C is distant R from a pair of points selected from D, E or F, and they also form an equilateral triangle.

The (4, 1)-centre solution points are located at A, B, C and O. This solution's objective function is R. We prove that there is no (2, 0)-centre solution with a value of the objective function of R or less. Indeed, if one facility is located at 0, then D, E and F cannot be covered by another

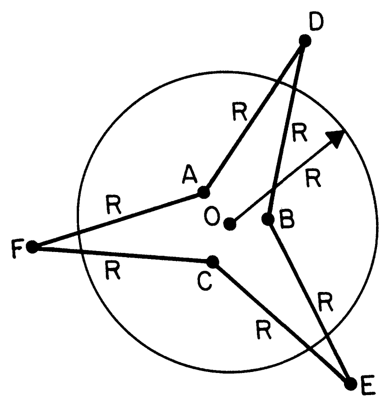

FIG. 1. *A (4, 1)-centre problem.*

facility within a distance R. On the other hand, if a facility is not located at 0, it covers *less* than one half of the circumference of the circle. Therefore, two facilities cannot cover the whole circumference of the circle within a radius of R or less if none is located at 0. This completes the proof.

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