THE LOCATION OF EMERGENCY SERVICE FACILITIES

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This paper views the location of emergency facilities as a set covering problem with equal costs in the objective. The sets are composed of the potential facility points within a specified time or distance of each demand point. One constraint is written for each demand point requiring 'cover,' and linear programming is applied to solve the covering problem, a single-cut constraint being added as necessary to resolve fractional solutions.

DETERMINING 'good' locations for facilities on a network has received a reasonable amount of attention in the last decade. Many of the approaches to this class of problems are indicated in CABOT ET AL.,[1] and REVELLE ET AL.[5] This paper is concerned with a facility-location problem with the special aspect that the maximum time or distance that separates a user from his closest service is a crucial parameter. As such, the problem is seen as most applicable to the location of emergency services such as fire stations, although one may equally well apply it to the location of ordinary services, such as schools, libraries, etc.

If an upper limit is placed on the response time or distance to any user node, consideration can be given to determining the minimum-cost spatial arrangement of service facilities that adequately serves the entire user region. If costs (determined in any manner desired) are identical for all possible facility locations, then an equivalent problem is to minimize the total number of service facilities required to meet the response time or distance standards for each of the users. The solution to this problem will indicate both the number and location of the facilities that provide the desired service.

The location of fire stations might be approached according to the structure just described. The limit on response time is imposed to ensure that no more than a specified time period will elapse before a response will occur to any fire. In applications, the definition of response time may not be unique. Clearly, its definition must be such that it indicates the spatial effects of distributing the emergency service; but, beyond this requirement, the definition of response time depends primarily upon the available data. Once a response time s is specified, then for each point of demand there must be a fire station located within s time units. (It is assumed here that each facility has response capability at all times.) The desired solution to this problem locates the minimum number of fire stations that satisfies the response-time requirement.

In order to achieve a more tractable problem structure, several abstractions are required. First, it will be assumed that the user demands can be represented as occurring at a finite set of points and that the potential locations for service facilities are also a finite set of points. Second, it is assumed that the minimum distance or minimum response time between any user-node/service-facility pair is known. Third, it will be assumed that the user-demand points and the possible facility-location points constitute the same set of points. As will be seen shortly, this final assumption is not essential in structuring the solution technique. It has been included here to bring the formulation in line with earlier work in the same area. Under these simplifications, the problem has now been reduced to a problem of 'covering' each of the user nodes with one of the facility nodes.

HAKIMI[3,4] was the earliest to consider similar problems. A part of his first study[3] was directed toward the location of the center of a network, where the center of a network is the point of the network from which the distance to the furthest point is a minimum. Hakimi[4] later generalized the concept of a center, and, using Boolean functions, he sought the minimum number of centers (chosen from a discrete set) that covered all demand points within a specified maximum distance. The resulting method requires an enumeration of all feasible solutions, and, as problem size grows, the effort of determining the minimum number of facilities can be expected to grow rapidly. The technique given in the following section avoids enumeration, so that the growth in problem size has a less significant effect upon computational requirements.

PROBLEM FORMULATION AND SOLUTION

RECALLING THE assumptions that all user points may also be emergency-facility locations and that the minimum time or distance between any pair of user nodes is known, the problem will now be structured as an integer programming problem.

If the maximum response time s has been decided upon, then, for any node i, only the set of nodes within s of i can provide acceptable emergency service to i; this set will be denoted as N_i. If d_{ji} is the response time or distance from any node j to node i, the set N_i can be defined as $N_i=\{j|d_{ji} \leqq s\}$. (One may equally well define the set N_i as the nodes within s_i of node i, where s_i may be different for each node i. Problems with this feature have been examined and do not appear to add any difficulties to the solution process.) If there are n user nodes, there will be n sets N_i, and each set will have at least one member, if d_{ii} is taken to be zero. It is important to notice that the definition of N_i does not depend on the nature of the points j that may be used to provide emergency service to i. Therefore, the potential facility locations may be both user and other locations. The theoretical requirement is that d_{ji} be known and that the number of potential facility locations be finite; if this number is too enormous, then, of course, the proposed solution method will not be practicable. The choice of which points should be taken as potential facility locations is up to the analyst.

To structure the mathematical program, the following decision variables are now defined:

$$x_j=\begin{cases}0, \text{ if no facility is established at point } j,\\ 1, \text{ if a facility is established at point } j.\end{cases} \qquad (j=1, 2, \cdots, n)$$

Thus, x_j is a zero-one integer variable. Values of x_j other than zero or one will not be acceptable in a solution.

As discussed in the first section, any user node i must have at least one facility within s. Recalling that the set of potential facility locations within s of i is N_i and using the decision variables, we can write the service requirement for user node i as

$$\textstyle\sum_{j\epsilon N_i} x_j \geqq 1. \qquad (1)$$

The objective z that is to be minimized is the total number of facilities used:

$$z=\textstyle\sum_{j=1}^{j=n} x_j. \qquad (2)$$

Even if the x_j's are only restricted to be nonnegative, none of them will be greater than one in an optimal solution to (1) and (2). If any one x_j were not, it could be reduced in value without violating a constraint. This reduction would cause a corresponding reduction in the objective function, indicating that the optimum was not at hand. The entire program can be written:

$$\begin{array}{llll} \text{Minimize} & z=\sum_{j=1}^{j=n} x_j & & (2)\\ \text{(I)}\quad \text{subject to:} & \sum_{j\epsilon N_i} x_j \geqq 1, & (i=1, 2, \cdots, n) & (1)\\ & x_j=(0, 1). & (j=1, 2, \cdots, n) & (3) \end{array}$$

The structure is that of the set-covering problem with inequality constraints; the set-covering problem has received extensive treatment in the literature, both as a problem in its own right and as a special class of the integer programming problem. A recent review of many of the approaches to the problem is found in GARFINKLE.[2]

Three approaches to this problem seem most favored; these are linear-programming and cutting-plane techniques, reduction techniques, and implicit enumeration. Some investigators combine these approaches. The technique utilized here is linear programming supplemented by the addition of a single cut constraint. The technique has not yet failed to yield all zero-one variables.

The simplicity of this program is particularly important. It should be observed that only one constraint is written for each of the user points to be served, and it should also be noted that only one variable is associated with each of the potential facility locations. These two items, coupled with the capacity of the current mathematical programming systems, should permit the solution of problems with several hundred or more user points and potential locations. A second important feature of the model is the ease with which a facility location can be forced in or out of the solution by specifying its x_j to be either one or zero. This feature may be important in situations where a facility already exists or where the possibility of prohibiting a facility at a given location is to be examined.

The actual implementation of (I) requires two steps. The first is determination of the sets $N_i(i=1, 2, \cdots, n)$ for a given value of s and a given matrix of shortest distances D. The second step is solution of (I) using the sets established in step one.

We formed the sets using a Fortran program, and they were recorded on disk storage in a form acceptable for use by MPS/360 (the mathematical programming code available on IBM S/360 Equipment). The solution then was obtained using the linear programming algorithm of MPS/360 for the sets produced by the Fortran program. This coupling of procedures saves the tedious operation of forming and punching the sets. It should be noted that s is not explicitly contained in (I). Consequently, it is not possible to obtain a set of solutions parameterized on s using only the mathematical programming code. Both the set determination and mathematical programming procedures must be used for this operation. The potential inefficiency of this circumstance is reasonably surmounted by generating several different-data sets corresponding to a sequence of values of s, and then solving each of the associated mathematical programming problems.

TABLE I

The Minimum-Distance Matrix for Locations in New York State
(The distances are in miles)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	0	244	140	128	281	196	181	51	248	167	338	54	203	146	295	211	295	78	169	33	167	112	71	220	157	16	135	7	90	165
2	244	0	158	359	37	111	66	268	60	112	101	278	272	328	51	222	77	200	106	281	332	263	294	33	284	233	109	248	161	164
3	140	158	0	202	194	56	92	170	117	46	215	137	256	170	209	206	160	62	114	177	279	105	136	136	239	129	78	144	92	148
4	128	359	202	0	395	258	294	179	319	248	416	90	331	61	410	339	361	176	290	100	295	106	70	337	285	143	254	133	211	293
5	281	37	194	395	0	145	102	305	92	148	69	317	309	366	19	259	74	236	143	318	369	299	330	70	321	272	146	285	198	201
6	196	111	56	258	145	0	60	229	61	34	159	189	269	226	162	219	104	118	112	233	315	161	192	100	274	185	91	200	128	161
7	181	66	92	294	102	60	0	208	67	47	157	220	225	262	117	175	114	134	59	218	279	197	228	46	237	170	49	185	101	117
8	51	268	170	179	305	229	208	0	275	195	366	105	152	197	319	180	322	108	186	81	116	163	124	242	106	41	159	48	107	175
9	248	60	117	319	92	61	67	275	0	87	111	254	292	287	111	242	56	175	126	285	346	222	253	60	304	237	116	252	168	184
10	167	112	46	248	148	34	47	195	87	0	185	179	235	216	163	185	130	93	79	204	281	151	182	90	240	156	57	171	94	127
11	338	101	215	416	69	159	157	366	111	185	0	348	373	381	88	323	55	273	207	375	433	316	351	134	385	327	206	342	258	265
12	54	278	137	90	317	189	220	105	254	179	348	0	257	95	329	250	293	86	205	65	221	58	20	254	211	69	169	61	126	204
13	203	272	256	331	309	269	225	152	292	235	373	257	0	349	323	66	339	236	168	233	60	315	274	239	46	193	178	200	171	108
14	146	328	170	61	366	226	262	197	287	216	381	95	349	0	379	343	326	179	284	144	313	69	75	306	303	161	248	151	219	297
15	295	51	209	410	19	162	117	319	111	163	88	329	323	379	0	273	93	251	157	332	383	314	345	84	335	284	160	299	212	215
16	211	222	206	339	259	219	175	180	242	185	323	250	66	343	273	0	289	192	118	243	126	293	270	189	92	198	128	213	129	58
17	295	77	160	361	74	104	114	322	56	130	55	293	339	326	93	289	0	218	173	332	393	261	296	106	351	284	163	299	215	231
18	78	200	62	176	236	118	134	108	175	93	273	86	236	179	251	192	218	0	130	118	221	124	106	178	194	67	94	82	64	146
19	169	106	114	290	143	112	59	185	126	79	207	205	168	284	157	118	173	130	0	206	228	219	225	73	180	156	36	171	79	60
20	33	281	177	100	318	233	218	81	285	204	375	65	233	144	332	243	332	118	206	0	191	123	76	257	187	52	172	37	127	202
21	167	332	279	295	369	315	279	116	346	281	433	221	60	313	383	126	393	221	228	191	0	279	238	299	52	157	230	162	187	168
22	112	263	105	106	299	161	197	163	222	151	316	58	315	69	314	293	261	124	219	123	279	0	60	241	269	127	183	119	164	247
23	71	294	136	70	330	192	228	124	253	182	351	20	274	75	345	270	296	106	225	76	238	60	0	272	228	86	189	76	146	224
24	220	33	136	337	70	100	46	242	60	90	134	254	239	306	84	189	106	178	73	257	299	241	272	0	251	209	85	224	135	131
25	157	284	239	285	321	274	237	106	304	240	385	211	46	303	335	92	351	194	180	187	52	269	228	251	0	147	188	154	147	120
26	16	233	129	143	272	185	170	41	237	156	327	69	193	161	284	198	284	67	156	52	157	127	86	209	147	0	124	15	77	152
27	135	109	78	254	146	91	49	159	116	57	206	169	178	248	160	128	163	94	36	172	230	183	189	85	188	124	0	139	52	70
28	7	248	144	133	285	200	185	48	252	171	342	61	200	151	299	213	299	82	171	37	162	119	76	224	154	15	139	0	92	167
29	90	161	92	211	198	128	101	107	168	94	258	126	171	219	212	129	215	64	79	127	187	164	146	135	147	77	52	92	0	83
30	165	164	148	293	201	161	117	175	184	127	265	204	108	297	215	58	231	146	60	202	168	247	224	131	120	152	70	167	83	0

COMPUTATIONAL EXPERIENCE AND THE ELIMINATION OF FRACTIONAL RESULTS

THE SOLUTION of (I) requires zero-one variables, but the use of a linear-programming code admits the possibility that fractional solutions may be produced.

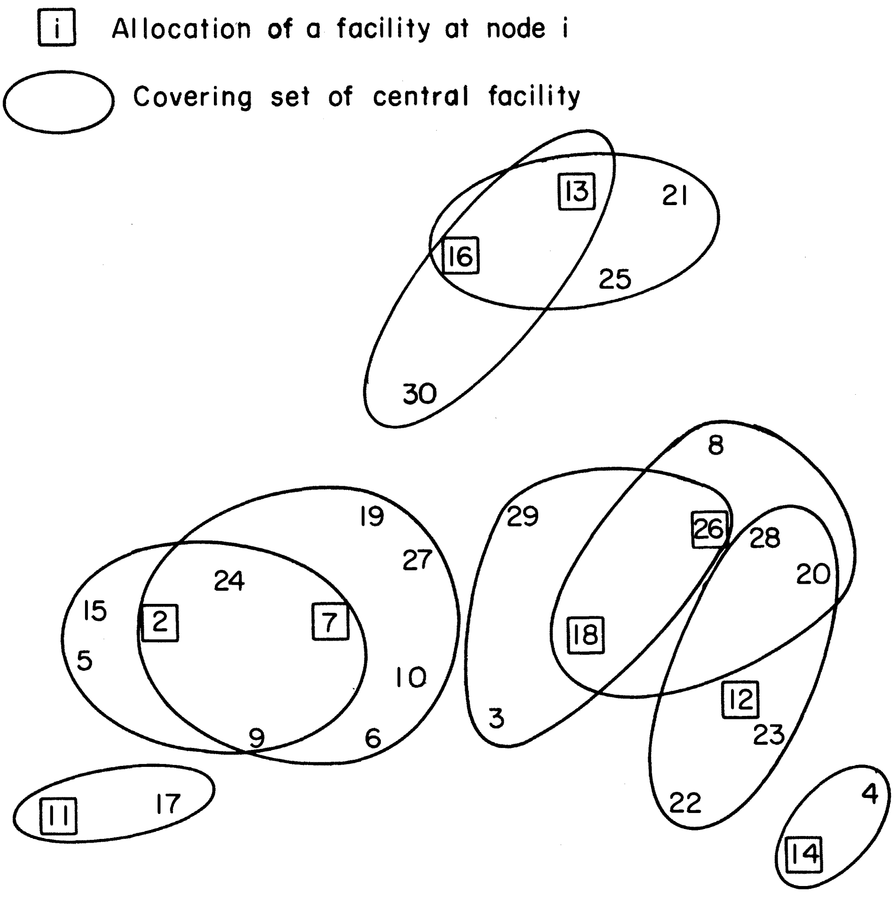

Fig. 1. Allocation of service facilities in an example problem.

To examine the properties of solutions generated by such a code, a 30-node problem was examined for many values of s, ranging from the minimum distance between nodes, 7, to the maximum distance between any two nodes, 275. These nodes represent major areas in New York State and the distance matrix between them is shown in Table I.

An example solution, obtained with $s=68$, is shown in Fig. 1. For a number of the user nodes, two service facilities are within s. This feature may be desirable in many applications.

Figure 2 indicates the number of service facilities required as a function of the maximum response distance. Notice that, for $s=69$, the number of centers used is noninteger, with a value of $8\frac{1}{2}$. The corresponding solution is shown in Fig. 3. Although each user node is covered by at least one service facility, many of the facilities are at level $\frac{1}{2}$, which is invalid according to the definition of x_j. To

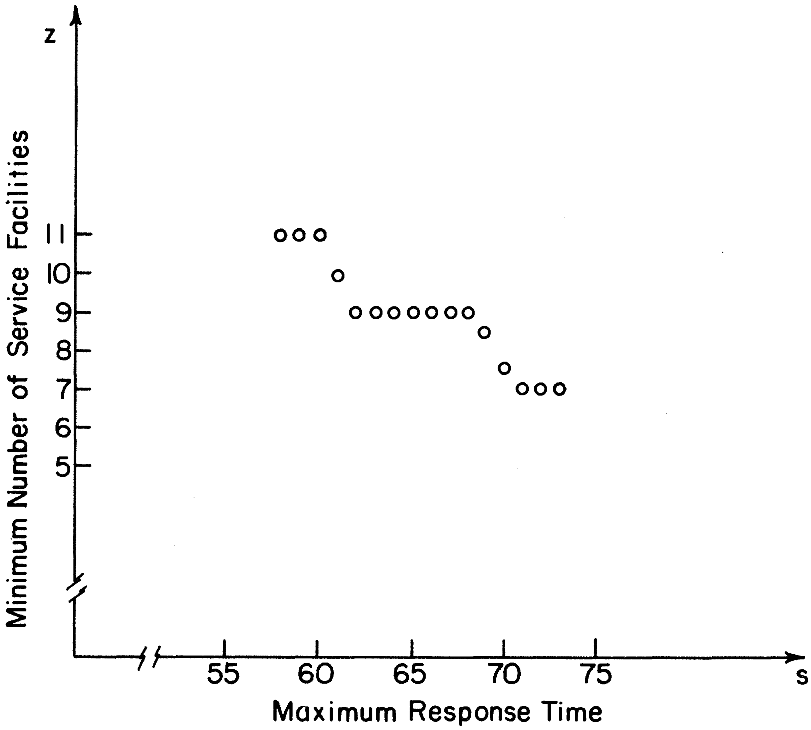

Fig. 2. Minimum number of service facilities as a function of maximum response time s.

achieve a solution with all x_j either zero or one, the facility-sharing patterns must be broken up.

All noninteger solutions observed in execution of the covering problem have been resolved with the addition of a single cut constraint, which is a direct result of the integer requirements. Suppose that m^0 is the optimal objective value obtained from a fractional linear programming solution to (I). If m^0 is noninteger, then, in any integer solution, the minimum number of servers must be at least as great as the least integer greater than m^0. Therefore, one adds the cut

$$\sum_{j=1}^{j=n} x_j \geqq [m^0]+1, \qquad (4)$$

where $[m^0]$ is the integer part of m^0. Although it is possible to have a fractional solution where m^0 is integer, no such condition has yet occurred. Should such a case occur, it appears that (4) may still be useful.

The use of (4) can be illustrated by referring to the noninteger solution already discussed. In that example, m^0 had the value $8\frac{1}{2}$. The application of (4) indi-

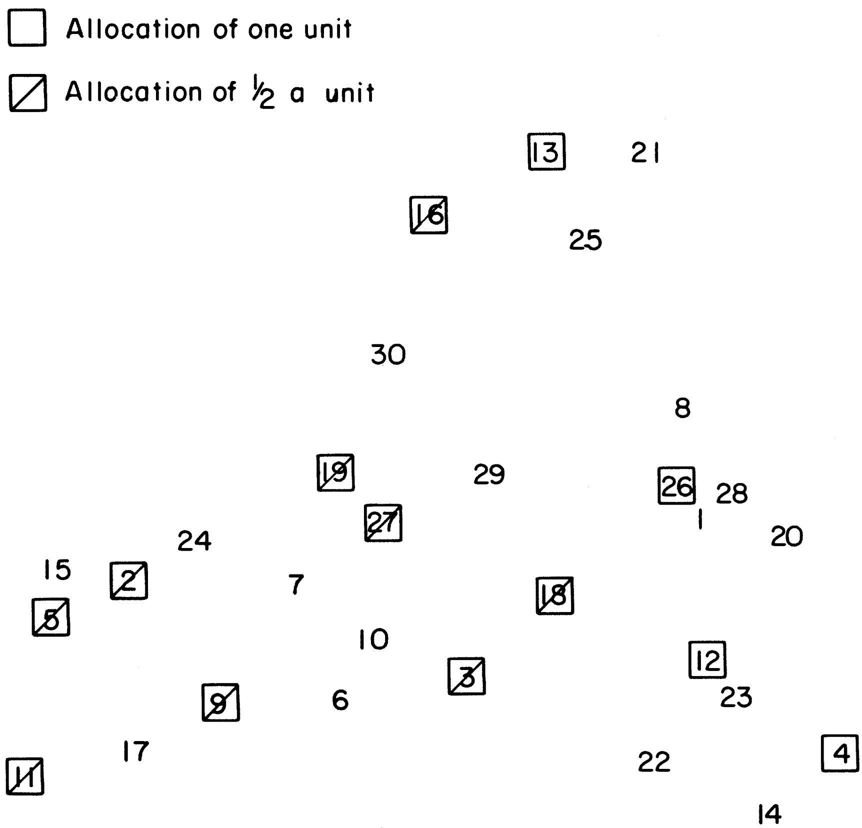

Fig. 3. Noninteger solution of the example problem for the value of $s = 69$.

cates that any integer solution must have at least 9 facilities. The constraint $\sum_{j=1}^{j=30} x_j \geqq 9$ was added to (I), and the problem was solved again. The resulting solution, shown in Fig. 4, is integer, as were all solutions in which (4) was included to eliminate noninteger results.

Several observations about the properties of solutions to this problem can be made.

First, if the minimum number of servers is m^* for two different values of s, s_1, and s_2, then the minimum number of servers is m^* for any s in $[s_1, s_2]$.

Second, for any s greater than the maximum distance entry in the distance matrix, only one server is required.

Third, for any s less than the minimum distance $d_{ij}(i \neq j)$, all nodes must have a service facility.

We have solved this problem at sizes up to 50 nodes over 150 times with a standard linear programming code. In the few cases of fractional results,

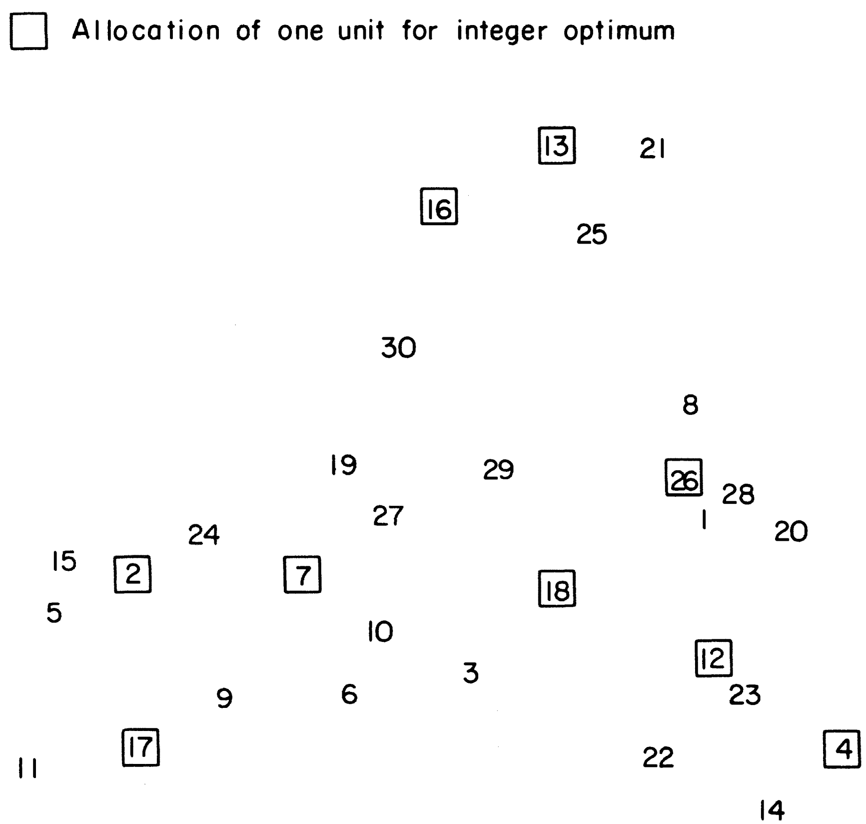

Fig. 4. Integer solution of the example problem for the value of s = 69.

the addition of the cut (4) to the linear program always resulted in an all zero-one solution.

THE RELATION OF THE PROBLEM TO THE MODIFIED p-MEDIAN PROBLEM

THE SOLUTIONS provided from (I) can be used to provide valuable information for a modified form of the p-median problem.[6]

In the original version of the p-median problem, there is no restriction on the distance that a user group might be required to travel to the closest open facility. A limit on the maximum distance any user group may travel can be imposed in the same manner as it is imposed in (I). The same definition of N_i as given previously is used in this modified version of the p-median problem.

Fig. 5. Average travel distance as a function of the maximal distance travelled.

To structure the problem, the following additional definitions are provided:

x_{ij} = the fraction of population of node i that receives service at node j;
d_{ij} = the distance from node i to node j;
a_i = the user population at node i;
p = the number of facilities to be established.

Model I utilized distances d_{ji} and considered (at least implicitly) movement from facility to user (e.g., fire engines). Here, the orientation of this model is shifted, reflecting movement from user to facility (e.g., schools), hence the use of d_{ij} rather than d_{ji}. The subscript reversal is irrelevant for the mathematics.

The problem is then written:

$$\text{Minimize} \qquad \sum_{i=1}^{i=n} \sum_{j\epsilon N_i} a_i d_{ij} x_{ij}, \tag{5}$$

$$\text{subject to} \qquad \sum_{j\epsilon N_i} x_{ij}=1, \qquad (i=1, 2, \cdots, n) \tag{6}$$

$$\text{(II)} \qquad x_{jj} \geqq x_{ij}, \qquad (i=1, 2, \cdots, n; \quad j\epsilon N_i; \quad i\neq j) \tag{7}$$

$$\sum_{j=1}^{j=n} x_{jj}=p, \tag{8}$$

$$x_{ij}\geqq 0 \quad \text{for all} \quad i, j, \quad \text{and} \quad x_{jj}=(0, 1) \quad \text{for all} \quad j.$$

In this formulation, $x_{jj}=1$ is taken to indicate that a facility is established at node j. The objective, as expressed in (5), is to minimize the total user distance travelled by users to their closest open facilities. The constraints represented by (6) ensure that the user population of each node is assigned to a service facility. Constraints (7) require that users are assigned only to open facilities, and (8) requires that exactly p facilities be established.

It is important to note that for some (s, p) combinations, no feasible solution to (II) may exist. In addition, given a particular value of p, there is a maximum value of s beyond which the solution of (II) will not differ from the solution without distance constraints. Figure 5 indicates the effect that s has upon the solution of (II) for a particular value of p.

The solutions obtained from the formulation (I) provide the value of $s_{\min}$, the lowest feasible limit on maximum distance, for a particular value of p. For a particular value of p, say p^*, $s_{\min}$ is the least value of s for which problem (I) has an objective value of p^*. No indication of either s^*, or the values of s in $[s_{\min}, s^*]$ at which the objective of (II) changes, is provided by the solutions to (I).

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