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Kidney exchange: An egalitarian mechanism

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Abstract

Kidney exchange programs utilize both deceased-donor and live-donor kidneys. One of these programs, a two-way *kidney paired donation* (*KPD*), involves two patients exchanging their live donors' kidneys. Another possibility is a *list exchange* (*LE*): a living incompatible donor provides a kidney to a candidate on the deceased-donor waitlist and in return the intended recipient of this donor receives a priority on the waitlist. By taking into consideration the fact that transplants from live donors have a higher chance of success than those from cadavers, we characterize the set of efficient and egalitarian exchanges involving the KPD's and LE's.

JEL classification: C71; C78; D02; D63; I10

Keywords: Mechanism design; Matching; Kidney exchange; Random assignment; Lorenz dominance

1. Introduction

Transplantation is the preferred treatment for the most serious forms of kidney disease. Unfortunately, there is a considerable shortage of deceased-donor kidneys: as of June 13, 2008, there are 76,313 patients waiting for kidney transplants in the US, with the median waiting time of over 3 years, and in 2007, there were only 10,587 transplants of deceased-donor kidneys. The cadaveric kidneys are not the only sources for transplantation. Since healthy people have two kidneys

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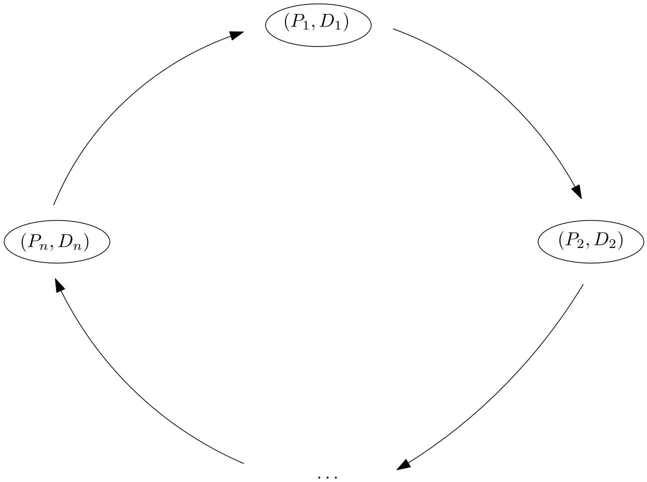

Fig. 1. An n-way KPD.

and can remain healthy on one, it is also possible for a kidney patient to receive a live-donor transplant. In 2007, there were 6,038 transplants of live-donor kidneys. Our goal is to design an efficient and fair mechanism utilizing these two sources of kidneys.

The two sources of kidneys enable the medical authorities to develop different programs to increase the number of transplantations. One of these programs is a *kidney paired donation* (*KPD*). A two-way KPD involves two patient-donor couples, for each of whom a transplant from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other couple [15,16]. This pair of couples can then exchange donated kidneys. Multiple-way exchanges, in which multiple pairs participate, can also be utilized (Fig. 1). To expand the opportunity for the KPD, optimal matching algorithms have been designed to identify maximal sets of compatible donor/recipient pairs from a registry of incompatible pairs.

Another possibility is a *list exchange* (*LE*). In an LE-chain of length two, a living incompatible donor provides a kidney to a candidate on the deceased-donor (DD) waitlist and in return the intended recipient of this donor receives a priority on the DD-waitlist. This improves the welfare of the patient in the couple, compared to having a long wait for a compatible cadaver kidney, and it benefits the recipient of the live kidney, and other on the DD-waitlist who benefit from the increase in the kidney supply due to an additional living donor. Through April 2006, 24 list exchanges have been performed. The LE in which more than one additional pair participates can also be considered. An LE with n pairs is depicted in Fig. 2.

In utilizing these two protocols, an important distinction is that transplants from live donors have a higher chance of success than those from cadavers. This fact is underlined by medical authorities and is supported by the data on the difference between the patient survival rate for live-donor transplants and for cadaveric transplants performed between 1997 and 2004 in the US (Table 1).

As the data shows, the gap between the patient survival rates increases over the years post transplant. The comparison between the graft survival rates is actually more striking (Table 2).

While transplants from live donors have a higher chance of success than those from cadavers, the experience of American surgeons suggests that patients should be indifferent among kidneys

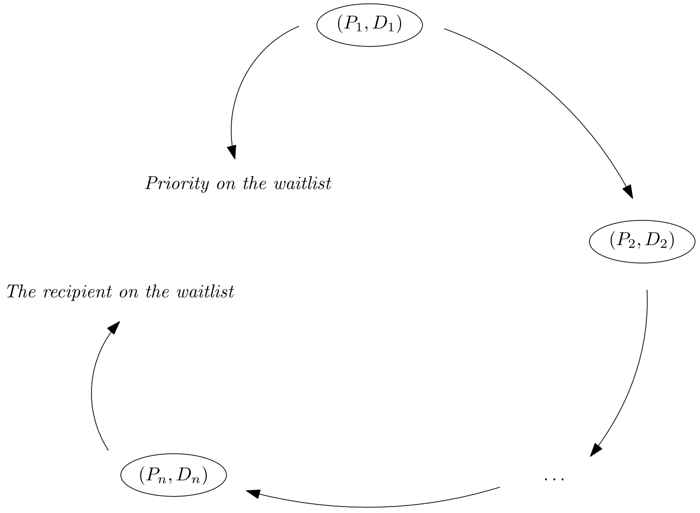

Fig. 2. An LE with n pairs.

Table 1
Kidney patient survival rates.

Donor type	Years post transplant	Number alive/ functioning	Survival rate
Cadaveric	1 year	23,735	94.5
Living	1 year	18,026	97.9
Cadaveric	3 year	24,436	88.3
Living	3 year	18,198	94.3
Cadaveric	5 year	19,042	82.0
Living	5 year	12,642	90.2

Table 2
Kidney graft survival rates.

Donor type	Years post transplant	Number alive/ functioning	Survival rate
Cadaveric	1 year	22,753	89.0
Living	1 year	17,649	95.0
Cadaveric	3 year	23,075	77.8
Living	3 year	17,556	87.9
Cadaveric	5 year	17,629	66.5
Living	5 year	12,033	79.7

from healthy donors that are blood type and immunologically compatible with the patient. This is because, in the US, transplants of compatible live kidneys have about equal graft survival probabilities, regardless of the closeness of tissue types between patient and donor [6,11]. In accordance with this medical findings, we assume that, while patients' preferences over the set of live-donor kidneys are such 0-1 preferences (we refer to such preferences as *dichotomous*), they prefer a live-donor kidney transplant to a cadaveric-kidney transplant.

Our goal is to explore how to organize kidney exchange by integrating the KPD's and LE's, under the assumptions of dichotomous preferences of the patients, and that the success rates of transplants from live donors are higher than those from cadavers. Kidney exchange is an assignment problem with donors (resources) to be allocated to the patients (agents), and under the dichotomous preferences assumption, the characterization of the set of the efficient exchanges reduces to the characterization the maximal cardinality matchings in the corresponding bipartite graph, in which there is a link connecting a patient and a donor if and only if there are no medical incompatibilities between them.[2] However, kidney exchange has more structure: each agent (patient) has a private endowment (donor). The fairness implication of this structure and the distinction between the success rates of transplants from live donors and those from cadavers is that, if the donor of an intended recipient donates to a patient in the patient-donor couples pool, then that intended recipient should have a priority in receiving live-donor kidney transplant.[3] In random matchings, this wisdom is elevated to equating the difference between the probability of a patient's receiving a live-donor kidney transplant and the probability of his donor's donating her kidney to someone in the patient-donor couples pool, as much as possible among all patient-donor couples in the pool. Our contribution is the characterization of the set of efficient and fair (fairness to be formally defined capturing the idea just described) exchanges involving the KPD's and LE's.

2. Related literature

While the transplantation community approved the use of the KPD's and LE's to increase kidney donations, it has provided little guidance about how to organize such exchanges. Roth, Sönmez, and Ünver [17] suggested that, by modeling kidney exchange as a mechanism design problem, integrating the KPD's and LE's may benefit additional candidates.[4] This approach turns out to be very successful and is supported by the medical community. Since then, a centralized mechanism for kidney exchange based on these two protocols has been used in the regional exchange program in New England (The United Network for Organ Sharing-UNOS-Region 1). In terms of integrating the KPD's and LE's, their paper is closest to the present work.

Bogomolnaia and Moulin [5] assumed dichotomous preferences and considered two-sided matching such that an agent on one side of the market can only be matched with an agent on the other side. (This problem can be represented as a bipartite graph derived from the underlying (dichotomous) preferences of the agents.) They define the egalitarian solution as the one that picks an efficient matching equalizing as much as possible the individual probabilities of being matched, and show that the corresponding profile of utilities first-order stochastically

[2] The maximum cardinality matching problem is well analyzed in the graph theory literature. More specifically, the Gallai [9,10] and Edmonds [8] Decomposition Lemma characterizes the set of maximum cardinality matchings. We make use of this result in constructing an efficient exchange.

[3] Note that, a patient-donor couple can always go to the DD-waitlist to obtain a priority in receiving a deceased-donor kidney, and the incompatible patient-donor pairs register the centralized clearinghouse with the expectation of receiving a live-donor kidney transplant for the patient.

[4] See also Roth, Sönmez, and Ünver [19]. The kidney exchange problem has some common features with the assignment problem with private endowments and/or a social endowment. (See for example Abdulkadiroğlu and Sönmez [1,2], Hylland and Zeckhauser [13], Shapley and Scarf [23], Sönmez and Ünver [24], Yılmaz [27,28].)

dominates any other feasible profile of utilities arranged increasingly, a property known as *Lorenz-dominance*.[5]

Roth, Sönmez, and Ünver [18] assumed dichotomous preferences and considered the constrained kidney exchange problem, in which only the two-way KPD's are allowed. They show that, in the constrained problem, efficient and strategy-proof mechanisms exist. These mechanisms include a deterministic mechanism based on the priority setting that organ banks currently use for the allocation of cadaver kidneys, and a stochastic mechanism motivated by the fairness considerations. The results of Roth, Sönmez, and Ünver [18] on the egalitarian mechanism generalize the corresponding results of Bogomolnaia and Moulin [5] to general (not necessarily bipartite) graphs.[6] Recently, Sönmez, and Ünver [25] characterize the set of efficient matchings for the constrained kidney exchange model with altruistic pairs under the dichotomous preferences assumption, where an altruistic pair is a compatible patient-donor pair agreeing to take part in a two-way KPD for purely altruistic reasons, i.e. so that the number of patients receiving a transplant increases in the kidney exchange program which they contribute. Also, Yılmaz [29] characterizes the set of matchings with the maximum number of transplants in a constrained kidney exchange model where only the LE's with two pairs and two-way KPD's are possible.

We explore how to organize kidney exchanges without any constraint on the number of patients taking part in a KPD or an LE. In this unconstrained problem, we integrate the KPD's and LE's and take into consideration the fact that transplants from live donors have a higher chance of success than those from cadavers. Our contribution is the characterization of the set of efficient and egalitarian matchings. While the characterization of the efficient set follows from the Gallai–Edmonds Decomposition Lemma [8–10], we need another elegant result from graph theory in order to characterize the set of egalitarian kidney exchanges (see Theorem 2 in Appendix A.2).

3. The model

Let P be a finite set of patients each of whom has an incompatible donor, and D be the set of these donors. We denote the donor of patient $p \in P$ by d_p, and the patient whose donor is $d \in D$ by p_d. For expositional convenience, we assume that all patients are male and all donors are female.

For each $p \in P$, $D_p \subseteq D$ denotes the set of compatible donors for patient p. For each $S \subseteq P$, we write $D_S = \bigcup_{p \in S} D_p$ for the set of donors compatible with at least one patient in S. Also, for each $d \in D$, P_d denotes the set of patients for whom donor d is compatible. For each $F \subseteq D$, we write $P_F = \bigcup_{d \in F} P_d$ for the set of patients for each of whom there is at least one compatible donor in F.

5 Another work that uses the same criterion for an egalitarian allocation is by Dutta and Ray [7]. They show that, for convex cooperative games, the egalitarian allocation is unique and it is in the core.

6 Roth, Sönmez, and Ünver [20] also explore that, for a specific preference profile of the patients (this profile is constructed according to the medical facts on the blood-type compatibilities), when multiple-way KPD's are feasible, three-way KPD's as well as two-way KPD's will have a substantial effect (and larger than three-way KPD's have less impact) on the number of transplants that can be arranged. Also, Ünver [26] introduces a dynamic kidney exchange model and analyzes efficient matching mechanisms in this dynamic setting.

Each patient evaluates each donor as compatible or incompatible and is indifferent between all compatible donors and between all incompatible donors. He prefers each compatible donor to the waitlist option w, and w to each incompatible donor. Thus, for each patient p,

$$d, d' \in D_p \text{ and } d'', d''' \notin D_p \quad \text{imply } d \sim_p d' \succ_p w \succ_p d'' \sim_p d'''.$$

Note that the set D_p fully describes the preferences of patient p.

A **kidney exchange problem**, or simply a **problem** is a triple $(P, D, (D_p)_{p\in P})$.

Let

$$c_{p,d} = \begin{cases} 1 & \text{if } d \in D_p, \\ 0 & \text{otherwise.} \end{cases}$$

Each problem $(P, D, (D_p)_{p\in P})$ induces a $|P| \times |D|$ **compatibility matrix** $C = [c_{p,d}]_{p\in P, d\in D}$. We refer to the triple (P, D, C) as the **reduced problem of** $(P, D, (D_p)_{p\in P})$. Throughout the paper, we fix a problem $(P, D, (D_p)_{p\in P})$, and the reduced problem (P, D, C) of $(P, D, (D_p)_{p\in P})$.

A **deterministic matching** is an *injective partial function* μ from P into D, that is, for each $d \in D$, there is at most one patient p such that $\mu(p) = d$. An unmatched patient receives high priority on the cadaver queue. By definition of a deterministic matching, the number of unmatched patients is equal to the number of unmatched donors. Thus, for each patient p receiving high priority on the cadaver queue, there is a donor d (not necessarily d_p) who donates her kidney to someone on the queue. A deterministic matching is represented as a $|P| \times |D|$ matrix with entries 0 or 1, and at most one nonzero entry per row and one per column. A deterministic matching μ is **individually rational** if, for each patient $p \in P$, $\mu(p) = d$ implies $d \in D_p$. Let $\mathcal{M}$ denote the set of all individually rational deterministic matchings.[7] Let $P_\mu \equiv \{p \in P\colon\ \mu(p) \in D\}$, the set of patients matched by μ. We call $|P_\mu|$ as the **cardinality of matching** μ.

Let $\lambda = (\lambda_\mu)_{\mu\in\mathcal{M}}$ be a **lottery** that is, a probability distribution over $\mathcal{M}$. Let $\Delta\mathcal{M}$ denote the set of all lotteries. Each lottery $\lambda \in \Delta\mathcal{M}$ induces a **random matching (matrix)** $Z(\lambda) = [z_{p,d}(\lambda)]_{p\in P, d\in D}$, where $z_{p,d}(\lambda)$ is the probability that patient p is matched to donor d, that is, the probability that λ selects a deterministic matching μ such that $\mu(p) = d$. Thus, for each $\lambda \in \Delta\mathcal{M}$, the $|P| \times |D|$ matrix $Z(\lambda)$ is substochastic, that is to say, it is nonnegative and the sum of each row (each column) is at most one. Let Z be a non-negative and substochastic matrix such that $z_{p,d} > 0$ implies $d \in D_p$. The set of all such random matching matrices is denoted by $\mathcal{Z}$.

For patient $p \in P$, the aggregate probability that he receives a live-donor transplant, is the canonical utility representation of his preferences over random matchings. Thus, given a random matching $Z \in \mathcal{Z}$, the utility of patient p is defined as the sum of the entries in the pth row of Z:

$$u_p(Z) = \sum_{d\in D} z_{p,d},$$

and the utility profile is defined as the non-negative real vector $\mathbf{u}(Z) = (u_p(Z))_{p\in P}$. We denote by $\mathcal{U}$ the set of all feasible utility profiles. That is, $\mathcal{U} = \{\mathbf{u}(Z)\colon\ Z \in \mathcal{Z}\}$.

Given a random matching $Z \in \mathcal{Z}$, the probability that kidney of donor d is transplanted to someone in the exchange pool, $t_d(Z)$, is the sum of the entries in the dth column of Z:

$$t_d(Z) = \sum_{p\in P} z_{p,d},$$

and the transplantation probability profile is defined as the non-negative real vector $\mathbf{t}(Z) = (t_{d_p}(Z))_{p\in P}$.

[7] Throughout the rest of the paper, we consider only individually rational matchings.

A variant of the Birkhoff–von Neumann Theorem [3,30], implies that each substochastic matrix $Z \in \mathcal{Z}$ obtains as a (in general not unique) lottery $\lambda \in \Delta\mathcal{M}$.[8] Since, for each patient, two lotteries resulting in the same random matching yield the same aggregate probability of receiving a live-donor transplant, we do not distinguish them. Thus, a **random solution to** (P, D, C) is a matrix $Z \in \mathcal{Z}$.

4. Efficiency

A deterministic matching $\mu \in \mathcal{M}$ is **Pareto efficient** if there exists no other matching $\eta \in \mathcal{M}$ such that $P_\eta \supsetneq P_\mu$, i.e. if P_μ is inclusion maximal. Let $\mathcal{E}$ be the set of Pareto efficient matchings. A well-known property of matchings states that each Pareto efficient matching matches the same number of patients. For the sake of completeness, we repeat a result from abstract algebra which implies this property[9]:

A *matroid* is a pair $(X, \mathcal{I})$ such that X is a set and $\mathcal{I}$ is a collection of subsets of X such that

M1. if I is in $\mathcal{I}$ and $J \subseteq I$ then J is in $\mathcal{I}$; and
M2. if I and J are in $\mathcal{I}$ and $|I| > |J|$ then there exists an $i \in I \setminus J$ such that $J \cup \{i\}$ is in $\mathcal{I}$.

Proposition 1. *Let $\mathcal{I}$ be the sets of simultaneously matchable patients, i.e. $\mathcal{I} = \{I \subseteq P\colon \exists \mu \in \mathcal{M}$ such that $I \subseteq P_\mu\}$. Then, $(P, \mathcal{I})$ is a matroid.*

The following property follows immediately from the second property of matroids:

Lemma 1. *For each pair of Pareto efficient matchings $\mu, \eta \in \mathcal{E}$, $|P_\mu| = |P_\eta|$.*

4.1. The Gallai–Edmonds Decomposition

The *Gallai–Edmonds Decomposition* (GED) of bipartite graphs, a well-known result in graph theory, further clarifies the structure of Pareto efficient deterministic matchings.[10]

Lemma 2 *(The Gallai–Edmonds Decomposition). Given a reduced problem (P, D, C), there is a unique pair of partitions $\{P^o, P^f, P^u\}$ of P and $\{D^u, D^f, D^o\}$ of D such that*:

(i) *D^u is only compatible with P^o, and D^u is underdemanded by P^o*:

$$P_{D^u} = P^o$$

and

$$\text{for each } S \subseteq P^o\colon \quad \left|D_S \cap D^u\right| > |S|;$$

(ii) *there is a full match between P^f and D^f, that is, all patients in P^f can be matched with all donors in D^f*:

$$\text{for each } S \subseteq P^f\colon \quad \left|D_S \cap D^f\right| \geqslant |S|;$$

[8] The Birkhoff–von Neumann Theorem holds for bistochastic matrices. This result is generalized to substochastic matrices by Bogomolnaia and Moulin [4].

[9] This result is also stated by Roth, Sönmez, and Ünver [18].

[10] All the results in this section are also stated by Bogomolnaia and Moulin [5].

(iii) *P^u is only compatible with D^o, and D^o is overdemanded by P^u:*

$$D_{P^u} = D^o$$

and

$$\text{for each } F \subseteq D^o\text{:}\quad \left|P_F \cap P^u\right| > |F|.$$

Note in particular that $|P^o| < |D^u|$, $|P^f| = |D^f|$, and $|P^u| > |D^o|$. The GED Lemma states that it is possible to match the patients in P^o with the donors in D^u such that each patient in P^o receives a live-donor transplant from the set D^u. In this case, there are $|D^u| - |P^o|$ donors in D^u, each of whom donates her kidney to someone on the queue. Also, the patients in P^u can be matched only with the donors in D^o. But, there are not enough donors in D^o such that each patient in P^u receives a live-donor transplant. Thus, if each patient in P^u is matched with a donor in D^o, then there are $|P^u| - |D^o|$ patients in P^u, each of whom receives high priority on the cadaver queue rather than a live-donor transplant. Note that $|D^u| - |P^o| = |P^u| - |D^o|$.

As shown before, finding a Pareto efficient deterministic matching reduces to finding a maximum cardinality matching. The GED Lemma characterizes the set of maximum cardinality matchings.

Lemma 3. *A deterministic matching $\mu \in \mathcal{M}$ is Pareto efficient if and only if exactly $|P^o| + |P^f| + |D^o|$ patients are matched by μ. Moreover, at each Pareto efficient matching, patients in P^o are matched to donors in a proper subset of D^u, patients in a proper subset of P^u are matched to donors in D^o, and there is a full match between P^f and D^f.*

We now turn our attention to random matchings. A lottery λ is **ex post efficient** if its support is a subset of the set of Pareto efficient deterministic matchings, that is, if $\lambda_\mu > 0$ implies $\mu \in \mathcal{E}$. A random matching Z is **ex ante efficient** if there exists no other random matching Z' such that $\mathbf{u}(Z') \geqslant \mathbf{u}(Z)$ and for some $p \in P$, $u_p(Z') > u_p(Z)$. We denote the set of ex ante efficient random matchings by $\mathcal{Z}^e$. A utility profile $\mathbf{u} \in \mathcal{U}$ is **efficient** if there exists no other utility profile $\mathbf{v} \in \mathcal{U}$ such that $\mathbf{v} \geqslant \mathbf{u}$ and for some $p \in P$, $v_p > u_p$. We denote the set of efficient utility profiles by $\mathcal{U}^e$.

The GED Lemma is also key to the characterization of the efficient utility profiles.

Lemma 4.

(i) *A lottery is ex post efficient if and only if, with probability one, it matches exactly $|P^o| + |P^f| + |D^o|$ patients.*
(ii) *A random matching is ex ante efficient if and only if the sum of its entries is $|P^o| + |P^f| + |D^o|$.*
(iii) *A random matching is ex ante efficient if and only if $z_{p,d} > 0$ implies $(p, d) \in (P^o, D^u) \cup (P^f, D^f) \cup (P^u, D^o)$, and its restriction to (P^o, D^u) is row-stochastic, to (P^f, D^f) is bistochastic, and to (P^u, D^o) is column-stochastic.*

Throughout the rest of the paper, we consider only efficient matchings.

5. Stochastic exchange

Given a random matching Z, and a patient p, the difference between his utility and the probability that the kidney of his donor d_p is transplanted to someone in the exchange pool (we call it as the **u–t difference for patient p**) is important in the sense of fairness: if the donor d_p donates her kidney to someone in the exchange pool, then it is plausible to think that patient p should have the priority in receiving a live-donor kidney transplantation in exchange for his donor's contribution to the pool. But, there may be several patients whose donors donate their kidneys to the pool, yet there are not enough compatible donors in the pool to donate their kidneys to these patients. Thus, for a random matching Z, the vector $\mathbf{u}(Z) - \mathbf{t}(Z) = (u_p(Z) - t_{d_p}(Z))_{p \in P}$ is key to evaluating its fairness; equalizing the u–t differences as much as possible is very plausible from an equity perspective. We use the Lorenz criterion as the partial ordering of the matchings. The *Lorenz dominance* is the following partial orderings of vectors in $\mathbb{R}^{|P|}$: $\mathbf{v}$ *Lorenz dominates* $\mathbf{y}$ if upon rearranging their $|P|$ coordinates increasingly as $\mathbf{v}^*$ and $\mathbf{y}^*$, we have

$$\text{for each } k = 1, \ldots, |P|: \quad \sum_{i=1}^{k} \left(v_i^* - y_i^*\right) \geqslant 0.$$

If a matching $Z \in \mathcal{Z}^e$ is such that $\mathbf{u}(Z) - \mathbf{t}(Z)$ is Lorenz dominant in the set $\{\mathbf{u}(Z') - \mathbf{t}(Z'){:}\ Z' \in \mathcal{Z}^e\}$, then it has a very strong claim to fairness within the set of efficient matchings. It achieves the maximum over $\{\mathbf{u}(Z') - \mathbf{t}(Z'){:}\ Z' \in \mathcal{Z}^e\}$ of any collective welfare function averse to inequality in the sense of the Pigou–Dalton transfer principle. Also, it maximizes not only the leximin ordering but also any collective welfare function $\sum_P f(u_p - t_p)$ for each increasing and concave function f. (See Moulin [14] and Sen [22] for these results and more on Lorenz dominance.) This leads to the following definition.

Definition 1. A random matching $Z \in \mathcal{Z}^e$ is **egalitarian** if the vector $\mathbf{u}(Z) - \mathbf{t}(Z)$ is Lorenz dominant in the set $\{\mathbf{u}(Z') - \mathbf{t}(Z'){:}\ Z' \in \mathcal{Z}^e\}$.

Since the Lorenz dominance is a partial order, an egalitarian random matching may not exist in general. It turns out that it exists and our main result is to characterize the set of the egalitarian random matchings.

Let $P^{u;u}$ denote the set of underdemanded patients whose donors are underdemanded, $P^{u;f,o}$ denote the set of underdemanded patients whose donors are fully demanded or overdemanded. Also, let $P^{u;1}(Z)$ denote the set of underdemanded patients who receive live-donor transplantations with probability one in the random matching Z. Similarly, $D^{u;u}$ denotes the set of underdemanded donors of underdemanded patients, and $D^{u;f,o}$ denotes the set of underdemanded donors of fully demanded or overdemanded patients. Note that $P^u = P^{u;u} \cup P^{u;f,o}$ and $D^u = D^{u;u} \cup D^{u;f,o}$. Also, let $P^{f,o;u}$ denote the set of fully demanded or overdemanded patients whose donors are underdemanded.

To convey the idea in our characterization result, let us consider the special case where $P^f = D^f = \emptyset$, and $P^{u;f,o} = D^{u;f,o} = \emptyset$, that is, each underdemanded patient has an underdemanded donor. For an overdemanded patient p, at each efficient random matching $Z \in \mathcal{Z}^e$, the probability of both him receiving a live donor kidney and also his donor donating her kidney someone in the exchange pool is one, thus, $u_p(Z) - t_{d_p}(Z) = 0$. For an underdemanded patient, on the other hand, the u–t difference may be negative or positive. Also, for each $Z \in \mathcal{Z}^e$, $\sum_{p \in P}(u_p(Z) - t_{d_p}(Z)) = 0$. (Note that this equality holds for the general case as well.) By efficiency, the underdemanded patients will be matched to the overdemanded donors and the underdemanded donors will be matched to the overdemanded patients. The difficulty of the problem of equating the u–t differences as much as possible in the sense of Lorenz-dominance is that these two matching problems cannot be considered separately: when a set of underdemanded patients are matched (randomly) to the overdemanded donors, the matching of their donors to the overdemanded patients has to be considered simultaneously. Thus, there are two different matching problems, connected to each other by the patient-donor type private ownership relation and the notion of the u-t difference.

Since an egalitarian random matching necessarily maximizes the leximin ordering of the u-t differences vectors, first step in finding such a matching is to find the maximum possible first coordinate of the vector upon rearranging their coordinates increasingly. For each $E \subseteq D$, define $P(E) \equiv \{p \in P\colon\ D_p \subseteq E\}$ as the set of patients whose compatible donors are only in E. Let $S \subseteq P^u$ be a set of patients. Our goal is to find a random matching such that the u-t difference for each patient in S is the same and as maximum as possible. For each F' such that $\{d_p\colon\ p \in S\} \subseteq F' \subseteq D^u$, $P(F') \subseteq P^o$; and each patient in $P(F')$ receives a live-donor kidney transplantation with probability one. By the GED Lemma, at an efficient random matching, the patients in S can receive at most $|D_S|$ live-donor kidney transplantations, and also, at best, their donors donate only to $|P(F')| - |F' \setminus \{d_p\colon\ p \in S\}|$ patients in P^o. Then, if the u-t difference for each patient in S is the same, then its maximum possible value can not be greater than

$$f\big(S, F'\big) = \frac{|D_S| - (|P(F')| - |F' \setminus \{d_p\colon\ p \in S\}|)}{|S|}.$$

Since, given F', this number is an upper bound, to find the maximum possible u–t difference, we need to take the minimum of this function over all such sets. Let

$$F = Arg \min_{F'\colon\ F' \supseteq \{d_p\colon\ p \in S\}} \frac{|D_S| - (|P(F')| - |F' \setminus \{d_p\colon\ p \in S\}|)}{|S|}.$$

But, the problem is that we don't know whether there is a match such that each donor in $F \setminus \{d_p\colon\ p \in S\}$ donates to a patient in $P(F)$. It turns out that there is such a match and it follows from Hall's Theorem:

Hall's Theorem. *(See [12].) There exists a matching such that each donor in $F \setminus \{d_p\colon\ p \in S\}$ donates to a patient in $P(F)$ if and only if*

$$\text{for each } E \subseteq F \setminus \{d_p\colon\ p \in S\}\colon\quad |E| \leqslant \big|\big\{p \in P(F)\colon\ D_p \cap E \neq \emptyset\big\}\big|.$$

Suppose there does not exist a matching such that each donor in $F \setminus \{d_p\colon\ p \in S\}$ donates to a patient in $P(F)$. Then, by Hall's Theorem, there is a set $E \subseteq F \setminus \{d_p\colon\ p \in S\}$ such that

$$|E| > \big|\big\{p \in P(F)\colon\ D_p \cap E \neq \emptyset\big\}\big|.$$

This is equivalent to

$$|E| > \big|P(F)\big| - \big|\big\{p \in P(F)\colon\ D_p \cap E = \emptyset\big\}\big|.$$

Consider now the set $F \setminus E$. Note that $P(F \setminus E) = \{p \in P(F)\colon\ D_p \cap E = \emptyset\} > |P(F)| - |E|$. Thus,

$$\begin{aligned} f(S, F \setminus E) &= \frac{|D_S| - |P(F \setminus E)| + |(F \setminus E) \setminus \{d_p\colon\ p \in S\}|}{|S|} \\ &= \frac{|D_S| - |P(F \setminus E)| + |F \setminus \{d_p\colon\ p \in S\}| - |E|}{|S|} \end{aligned}$$

$$< \frac{|D_S| - (|P(F)| - |E|) + |F \setminus \{d_p\colon\ p \in S\}| - |E|}{|S|}$$
$$= f(S, F).$$

Since $F \setminus E \supseteq \{d_p\colon\ p \in S\}$, this contradicts with the definition of the set F. Thus, the maximum possible u–t difference for each patient in S can be achieved by matching each donor in $F \setminus \{d_p\colon\ p \in S\}$ to a patient in $P(F)$, and the donors in $\{d_p\colon\ p \in S\}$ to the remaining patients in $P(F)$.

Since $f(S, F)$ is an upper bound for the set S, and we need to take the minimum of this function over all subsets of P^u, we conclude that the maximum value of the first coordinate of the leximin ordering can not be greater than

$$\lambda^* = \min_{S'\colon\ S' \subseteq P^u} \left\{ \min_{F'\colon\ F' \supseteq \{d_p\colon\ p \in S'\}} \frac{|D_{S'}| - (|P(F')| - |F' \setminus \{d_p\colon\ p \in S'\}|)}{|S'|} \right\}.$$

The question is whether there exists a random matching such that the u–t difference for each patient is at least λ^*. Our main result shows that there exists such a matching. Next, we generalize our findings here and present a recursive construction of the egalitarian random matchings. In the egalitarian random matchings, the characterization of the minimum positive u–t difference is slightly different than the characterization of the minimum non-positive u–t difference.

5.1. *The egalitarian mechanism: recursive construction of the egalitarian random matchings*

First, the donors in D^f are matched to the patients in P^f, such that they are fully matched to each other. Let $P_1^{u;u} = P^{u;u}$, $P_1^{u;f,o} = P^{u;f,o}$, $D_1^{u;u} = D^{u;u}$, $D_1^{u;f,o} = D^{u;f,o}$, $D_1^o = D^o$ and $P_1^o = P^o$.

Step 1: For each $S \subseteq P_1^u$, $F \subseteq D_1^u$, define a real-valued function f_1 through

$$f_1(S, F) = \frac{|D_S| - |P_1^o(F)| - |S| + |F|}{|S|}.$$

Let

$$\lambda_1 = \min_S \left\{ \min_{F\colon\ \{d_p\colon\ p \in S \cap P_1^{u;u}\} \subseteq F} f_1(S, F) \right\}$$

and S_1, and F_1 be the largest sets in the sense of inclusion[11] such that

$$\lambda_1 = f_1(S_1, F_1).$$

Let

$$P_2^{u;u} = P_1^{u;u} \setminus \left(S_1 \cup \left\{p \in P_1^u\colon\ d_p \in \left(F_1 \cap D_1^{u;u}\right) \setminus \left\{d_p\colon\ p \in S_1 \cap P_1^{u;u}\right\}\right\}\right),$$
$$P_2^{u;f,o} = \left(P_1^{u;f,o} \setminus S_1\right) \cup \left\{p \in P_1^u\colon\ d_p \in \left(F_1 \cap D_1^{u;u}\right) \setminus \left\{d_p\colon\ p \in S_1 \cap P_1^{u;u}\right\}\right\},$$

$D_2^{u;u} = D_1^{u;u} \setminus F_1$, $D_2^{u;f,o} = D_1^{u;f,o} \setminus F_1$, $D_2^o = D_1^o \setminus D_{S_1}$, and $P_2^o = P_1^o \setminus P_1^o(F_1)$. Let $Z^1 \subseteq \mathcal{Z}^e$ denote the set of all random matchings Z such that for each patient $p \in S_1$, $u_p(Z) - t_{d_p}(Z) = \lambda_1 \leqslant 0$, and $p \in P \setminus S_1$, $u_p(Z) - t_{d_p}(Z) > \lambda_1$.

[11] As we show in Appendix A, these largest sets are well defined.

Step k: For each $S \subseteq P_k^u$, $F \subseteq D_k^u$, define a real-valued function f_k through

$$f_k(S,F)=\frac{|D_S\cap D_k^o|-|P_k^o(F)|-|S|+|F|}{|S|}.$$

Let

$$\lambda_k=\min_S\Big\{\min_{F:\ \{d_p:\ p\in S\cap P_k^{u;u}\}\subseteq F} f_k(S,F)\Big\}$$

and S_k, and F_k be the largest sets in the sense of inclusion such that

$$\lambda_k=f_k(S_k,F_k).$$

Let

$$P_{k+1}^{u;u}=P_k^{u;u}\big\backslash\big(S_k\cup\big\{p\in P_k^u:\ d_p\in\big(F_k\cap D_k^{u;u}\big)\big\backslash\big\{d_p:\ p\in S_k\cap P_k^{u;u}\big\}\big\}\big),$$
$$P_{k+1}^{u;f,o}=\big(P_k^{u;f,o}\backslash S_k\big)\cup\big\{p\in P_k^u:\ d_p\in\big(F_k\cap D_k^{u;u}\big)\big\backslash\big\{d_p:\ p\in S_k\cap P_k^{u;u}\big\}\big\},$$

$D_{k+1}^{u;u}=D_k^{u;u}\setminus F_k$, $D_{k+1}^{u;f,o}=D_k^{u;f,o}\setminus F_k$, $D_{k+1}^o=D_k^o\setminus D_{S_k}$, and $P_{k+1}^o=P_k^o\setminus P_k^o(F_k)$. Let $Z^k\subseteq Z^{k-1}$ denote the set of all random matchings Z such that for each patient $p\in S_k$, $u_p(Z)-t_{d_p}(Z)=\lambda_k\leqslant 0$, and $p\in P\setminus\bigcup_{i=1}^k S_i$, $u_p(Z)-t_{d_p}(Z)>\lambda_k$.

Let Step K be such that $\lambda_K\leqslant 0$ and $\lambda_{K+1}>0$. For each $Z\in Z^K$, let $P^{u;1}(Z)\equiv P_{K+1}^{u;1}\equiv P_{K+1}^{u;f,o}$.

Step $K+1$: For each $T\subseteq P_{K+1}^{u;u}\cup P_{K+1}^{u;1}$, and $H\subseteq D_{K+1}^u$, define a real-valued function g_1 through

$$g_1(T,H)=\frac{|D_T\cap D_{K+1}^o|-|P_{K+1}^o(H)|-|T|+|H|}{|H|}.$$

Let

$$\beta_1=\min_H\Big\{\min_{T:\ \{d_p:\ p\in T\cap P_{K+1}^{u;u}\}\subseteq H} g_1(T,H)\Big\}$$

and T^1, and H^1 be the largest sets in the sense of inclusion such that

$$\beta_1=g_1\big(T^1,H^1\big).$$

Let $P_{K+2}^{u;u}=P_{K+1}^{u;u}\setminus\{p:\ d_p\in H^1\}$,

$$P_{K+2}^{u;1}=\big(P_{K+1}^{u;1}\cup\big\{p:\ d_p\in D_{K+1}^{u,u}\cap H^1\big\}\big)\big\backslash T^1,$$

$D_{K+2}^u=D_{K+1}^u\setminus H^1$, $P_{K+2}^o=P_{K+1}^o\setminus P_{K+1}^o(H^1)$, and $D_{K+2}^o=D_{K+1}^o\setminus D_{T^1}$. Let $Z^{K+1}\subseteq Z^K$ denote the set of all random matchings Z such that for each patient $p\in T^1$, $u_p(Z)-t_{d_p}(Z)=\beta_1>0$, and $p\in P\setminus(\bigcup_{i=1}^K S_i\cup T^1)$, $u_p(Z)-t_{d_p}(Z)>\beta_1$.

Step $K+m$: For each $T\subseteq P_{K+m}^{u;u}\cup P_{K+m}^{u;1}$, and $H\subseteq D_{K+m}^u$, define a real-valued function g_m through

$$g_m(T,H)=\frac{|D_T\cap D_{K+m}^o|-|P_{K+m}^o(H)|-|T|+|H|}{|H|}.$$

Let

$$\beta_m=\min_H\Big\{\min_{T:\ \{d_p:\ p\in T\cap P_{K+m}^{u;u}\}\subseteq H} g_m(T,H)\Big\}$$

and T^m, and H^m be the largest sets in the sense of inclusion such that

$$\beta_m = g_m(T^m, H^m).$$

Let $P^{u;u}_{K+m+1} = P^{u;u}_{K+m} \setminus \{p\colon d_p \in H^m\}$,

$$P^{u;1}_{K+m+1} = \left(P^{u;1}_{K+m} \cup \{p\colon d_p \in D^{u,u}_{K+m} \cap H^m\}\right) \setminus T^m,$$

$D^u_{K+m+1} = D^u_{K+m} \setminus H^m$, $P^o_{K+m+1} = P^o_{K+m} \setminus P^o_{K+m}(H^m)$, and $D^o_{K+m+1} = D^o_{K+m} \setminus D_{T^m}$. Let $Z^{K+m} \subseteq Z^{K+m-1}$ denote the set of all random matchings Z such that for each patient $p \in T^m$, $u_p(Z) - t_{d_p}(Z) = \beta_m > 0$, and $p \in P \setminus (\bigcup_{i=1}^K S_i \bigcup_{i=1}^m T^i)$, $u_p(Z) - t_{d_p}(Z) > \beta_m$.

Let $K + M$ be the last step of the construction, that is $P_{K+M+1} = D_{K+M+1} = \emptyset$.

5.2. *Main result*

At the end of Step 1, each donor in $F_1 \setminus \{d_p\colon p \in S_1 \cap P^{u;u}_1\}$ donates her kidney to someone in $P^o_1(F_1)$ with probability one. Each donor in D_{S_1} donates her kidney to someone in S_1 with probability one. The next step continues with the remaining patients and donors. But there is a change in the decomposition of the patients: Note that each donor in $F_1 \setminus \{d_p\colon p \in S_1 \cap P^{u;u}_1\}$ donates her kidney in the current step with probability one. Thus, in the next step, they are fully demanded or overdemanded donors of the underdemanded patients $\{p \in P^u_1\colon d_p \in (F_1 \cap D^{u;u}_1) \setminus \{d_p\colon p \in S_1 \cap P^{u;u}_1\}\}$. Thus, each such patient switches from being a member of $P^{u;u}_1$ to being a member of $P^{u;f,o}_2$.

At the end of Step K, there is a matching such that the u–t difference for each remaining patient is positive. Thus, to maximize the leximin ordering among the remaining patients, we now have to consider all the patients including the overdemanded patients. (In the previous steps, since efficiency implies that the u–t difference for each overdemanded patient is positive, we ignored them.)

At the end of Step $K + 1$, for each $H \subseteq D^u_{K+1}$, the patients in $\{p\colon d_p \in H \cap D^{u;f,o}_{K+1}\}$ are overdemanded. For each T such that $\{d_p\colon p \in T \cap P^{u;u}_{K+1}\} \subseteq H$, the patients in $\{p\colon d_p \in H \cap D^{u;u}_{K+1}\}$ can receive at most

$$\left|D_T \cap D^o_{K+1}\right| - \left|T \cap P^{u;1}_{K+1}\right| + \left|H \cap D^{u;u}_{K+1}\right| - \left|T \cap P^{u;u}_{K+1}\right|$$

donors. Thus, together with the patients in $\{p\colon d_p \in H \cap D^{u;f,o}_{K+1}\}$, they can receive at most

$$\begin{aligned}
&\left|D_T \cap D^o_{K+1}\right| - \left|T \cap P^{u;1}_{K+1}\right| + \left|H \cap D^{u;u}_{K+1}\right| - \left|T \cap P^{u;u}_{K+1}\right| + \left|\{p\colon d_p \in H \cap D^{u;f,o}_{K+1}\}\right| \\
&\quad = \left|D_T \cap D^o_{K+1}\right| - \left|T \cap P^{u;1}_{K+1}\right| + \left|H \cap D^{u;u}_{K+1}\right| - \left|T \cap P^{u;u}_{K+1}\right| + \left|H \cap D^{u;f,o}_{K+1}\right| \\
&\quad = \left|D_T \cap D^o_{K+1}\right| - |T| + |H|
\end{aligned}$$

donors. Also, efficiency implies that their donors are matched to at least $|P^o_{K+1}(H)|$ patients. Thus, the upper bound for the lowest u–t difference for the patients in $\{p\colon d_p \in H\}$ is

$$\frac{|D_T \cap D^o_{K+1}| - |P^o_{K+1}(H)| - |T| + |H|}{|H|}.$$

Since H and T are arbitrarily chosen, to determine the upper bound for the lowest u–t difference, we need to consider each such pair of sets such that this upper bound as specified above is the minimum. Thus, $g_1(T^1, H^1)$ is the upper bound for the value of the first coordinate of the leximin ordering for the remaining patients. As we show in Appendix A, there is actually a matching such that the first coordinate of the leximin ordering for the remaining patients is equal to $g_1(T^1, H^1)$.

The set of random matchings obtained by the egalitarian mechanism is the set Z^{K+M}. For each patient p, the u–t difference is the same in all the random matchings in the set Z^{K+M}. Our main result is to show that the set Z^{K+M} coincides with the set of the egalitarian random matchings.

Theorem 1. *A random matching Z is egalitarian if and only if $Z \in Z^{K+M}$.*

The proof of this result is relegated to Appendix A and it highly relies on an elegant result from graph theory (see part (i) of Theorem 2 in Appendix A.2), which gives the necessary and sufficient condition for the existence of flows along the arcs in a directed graph, where for each vertex, the difference between the inflow and the outflow is specified.

Our main result has an implication in terms of the condition under which an efficient matching with no inequality (i.e. an efficient matching where the u–t difference is zero for each patient) exists.

Corollary 1. *For each kidney exchange problem $(P, D, (D_p)_{p\in P})$, the following are equivalent:*

(i) *For each $S \subseteq P^u$ and $F \supseteq \{d_p\colon\ p \in S \cap P^{u;u}\}$, $|F| - |P^o(F)| \geqslant |S| - |D_S|$.*
(ii) *There exists a matching with no inequality.*
(iii) *There exists a deterministic matching with no inequality.*
(iv) *There exists a deterministic egalitarian matching.*[12]

While the set Z^{K+M} is not necessarily a singleton, the corresponding u–t difference vector is unique. However, the random matchings in Z^{K+M} may correspond to different utility vectors. This point is particularly important for the analysis of the strategic properties.

6. Concluding remarks

Roth, Sönmez, and Ünver [17] proposed efficient kidney exchange mechanisms that integrates the KPD and LE. Roth, Sönmez, and Ünver [18] later suggested an alternative mechanism which involves only two-ways KPD's and no LE's, and assumes that each patient is indifferent between all compatible kidneys. In addition to this latter assumption, we also adopt the assumption that each patient prefers each compatible live-donor kidney to each deceased-donor kidney; and allow multiple-ways KPD's and as well as LE's as in the mechanism proposed by Roth, Sönmez, and Ünver [17]. Our contribution is to construct a stochastic kidney exchange mechanism that is efficient and egalitarian. Although we consider only the kidney exchange problem, the same mechanism applies to the assignment problems with private endowments where the endowment of each agent is ranked at the bottom of his preference ordering, and there is an outside option that is always feasible.

[12] See Appendix A.3 for the proof of this result, which follows directly from Theorem 1 and part (ii) of Theorem 2 in Appendix A.2.

Appendix A

A.1. The egalitarian mechanism: an example

Let (p_k, d_k) be the k-th incompatible patient-donor pair. The set of underdemanded patients and donors are as follows:

$$P^u = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\},$$
$$D^u = \{d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_{13}\}.$$

The compatibility of the underdemanded patients with the overdemanded donors, d_1^o, d_2^o, d_3^o, d_4^o, d_5^o, is given as follows[13]:

	d_1^o	d_2^o	d_3^o	d_4^o	d_5^o
p_1	0	1	0	0	0
p_2	1	1	0	0	0
p_3	1	0	1	0	1
p_4	0	0	0	1	0
p_5	1	0	1	0	0
p_6	1	0	1	1	1
p_7	1	1	0	0	0
p_8	0	1	0	0	0
p_9	1	0	1	0	0
p_{10}	0	1	0	0	0

The compatibility of the underdemanded donors with the overdemanded patients, p_1^o, p_2^o, p_3^o, p_4^o, p_5^o, p_6^o, p_7^o, is given as follows:

	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}
p_1^o	1	0	0	0	0	0	1	0	0	0	0	0
p_2^o	0	0	0	0	0	0	1	0	1	0	0	0
p_3^o	1	0	0	0	0	0	0	1	0	1	0	0
p_4^o	0	0	0	1	0	1	0	1	1	0	0	0
p_5^o	0	0	0	1	0	1	0	0	0	1	0	0
p_6^o	0	0	1	0	1	0	0	0	0	0	1	1
p_7^o	0	1	1	0	0	0	0	0	0	0	0	1

Step 1: The largest sets S_1 and F_1 such that

$$\min_S \left\{ \min_{F:\ \{d_p:\ p \in S \cap P_1^{u;u}\} \subseteq F} f_1(S, F) \right\} = f_1(S_1, F_1)$$

are as follows: $S_1 = \{p_1, p_8, p_{10}\}$ and $F_1 = \{d_2, d_8, d_{10}\}$. This implies that the patients in S_1 and $P_1^o(F_1)$ are matched to $D_{S_1} = \{d_2^o\}$ and F_1 respectively, such that the donor in $F_1 \setminus \{d_p\colon\ p \in$

[13] Since the egalitarian matchings are efficient, the only part that matters is how the underdemanded patients and donors are going to be matched and we do not include the rest of the patients and donors.

$S_1 \cap P_1^{u;u}\} = \{d_2\}$ donates her kidney to someone in $P_1^o(F_1)$ with probability one. Moreover, for each $p \in S_1$, the u–t difference is

$$\lambda_1 = f_1(S_1, F_1) = \frac{|D_{S_1}| - |P_1^o(F_1)| - |S_1| + |F_1|}{|S_1|} = \frac{1-3+3-2}{3} = -\frac{1}{3}.$$

The following is such an assignment of probabilities[14]:

	d_2^o
p_1	$\frac{2}{3}$
p_8	$\frac{1}{6}$
p_{10}	$\frac{1}{6}$

	d_2	d_8	d_{10}
p_1^o	1	0	0
p_2^o	0	$\frac{1}{2}$	$\frac{1}{2}$

Step 2: Since donor d_2 donates with probability 1 in the previous step, patient p_2 is an underdemanded patient now with a fully demanded (or overdemanded) donor. The largest sets S_2 and F_2 such that

$$\min_S \left\{ \min_{F:\ \{d_p:\ p \in S \cap P_2^{u;u}\} \subseteq F} f_2(S, F) \right\} = f_2(S_2, F_2)$$

are as follows: $S_2 = \{p_2, p_5, p_7, p_9\}$ and $F_2 = \{d_5, d_7, d_9, d_{11}\}$. This implies that the patients in S_2 and $P_2^o(F_2)$ are matched to $D_{S_2} \cap D_2^o = \{d_1^o, d_3^o\}$ and F_2 respectively, such that the donor in $F_2 \setminus \{d_p:\ p \in S_2 \cap P_2^{u;u}\} = \{d_{11}\}$ donates her kidney to someone in $P_2^o(F_2)$ with probability one. Moreover, for each $p \in S_2$, the u–t difference is

$$\lambda_2 = f_2(S_2, F_2) = \frac{|D_{S_2} \cap D_2^o| - |P_2^o(F_2)| - |S_2| + |F_2|}{|S_2|} = \frac{2-4+4-3}{4} = -\frac{1}{4}.$$

The following is such an assignment of probabilities:

	d_1^o	d_3^o
p_2	$\frac{3}{4}$	0
p_5	0	$\frac{1}{2}$
p_7	$\frac{1}{4}$	0
p_9	0	$\frac{1}{2}$

	d_5	d_7	d_9	d_{11}
p_3^o	0	0	0	1
p_4^o	0	$\frac{1}{4}$	$\frac{3}{4}$	0
p_5^o	$\frac{3}{4}$	$\frac{1}{4}$	0	0

Step 3: The remaining underdemanded patients and donors are $P_3^u = \{p_3, p_4, p_6\}$ and $D_3^u = \{d_3, d_4, d_6, d_{12}, d_{13}\}$, respectively. The remaining overdemanded donors and patients are $D_3^o =$

[14] Remember that, while the u–t difference vector of an egalitarian matching is uniquely determined by definition, there could be multiple stochastic matchings giving the same u–t difference vector.

$\{d_4^o, d_5^o\}$ and $P_3^o = \{p_6^o, p_7^o\}$, respectively. Since for each $S \subseteq P_3^u$ and for each F such that $\{d_p \in D_3^u\colon\ p \in S \cap P_3^{u;u}\} \subseteq F$,

$$\frac{|D_S \cap D_3^o| - |P_3^o(F)| - |S| + |F|}{|S|} > 0,$$

we conclude that Step 3 is the first step such that for each remaining pair, the u–t difference is positive. The largest sets T^1 and H^1 such that

$$\min_H \left\{ \min_{T\colon \{d_p\colon\ p \in T \cap P_3^{u;u}\} \subseteq H} g_1(T, H) \right\} = g_1(T^1, H^1)$$

are as follows: $T^1 = \{p_3, p_4, p_6\}$ and $H^1 = \{d_3, d_4, d_6, d_{12}, d_{13}\}$. This implies that the patients in T^1 and $P_3^o(H^1)$ are matched to $D_{T^1} \cap D_3^o = \{d_4^o, d_5^o\}$ and H^1 respectively. Moreover, for each p such that $d_p \in H^1$, the u–t difference is

$$\beta_1 = g_1\big(T^1, H^1\big) = \frac{|D_{T^1} \cap D_3^o| - |P_3^o(H^1)| - |T^1| + |H^1|}{|T^1|} = \frac{2-3+5-2}{5} = \frac{2}{5}.$$

The following is such an assignment of probabilities:

	d_4^o	d_5^o
p_3	0	$\frac{3}{5}$
p_4	$\frac{3}{5}$	0
p_6	$\frac{2}{5}$	$\frac{2}{5}$

	d_3	d_4	d_6	d_{12}	d_{13}
p_6^o	0	0	$\frac{2}{5}$	$\frac{3}{5}$	0
p_7^o	$\frac{1}{5}$	$\frac{1}{5}$	0	0	$\frac{3}{5}$

The random matching obtained is given by two substochastic matrices. The first one is for the matching of the underdemanded patients with the overdemanded donors:

	d_1^o	d_2^o	d_3^o	d_4^o	d_5^o
p_1	0	$\frac{2}{3}$	0	0	0
p_2	$\frac{3}{4}$	0	0	0	0
p_3	0	0	0	0	$\frac{3}{5}$
p_4	0	0	0	$\frac{3}{5}$	0
p_5	0	0	$\frac{1}{2}$	0	0
p_6	0	0	0	$\frac{2}{5}$	$\frac{2}{5}$
p_7	$\frac{1}{4}$	0	0	0	0
p_8	0	$\frac{1}{6}$	0	0	0
p_9	0	0	$\frac{1}{2}$	0	0
p_{10}	0	$\frac{1}{6}$	0	0	0

The second one is for the matching of the overdemanded patients with the underdemanded donors:

	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}
p_1^o	1	0	0	0	0	0	0	0	0	0	0	0
p_2^o	0	0	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
p_3^o	0	0	0	0	0	0	0	0	0	1	0	0
p_4^o	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{3}{4}$	0	0	0	0
p_5^o	0	0	0	$\frac{3}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0
p_6^o	0	0	0	0	$\frac{2}{5}$	0	0	0	0	0	$\frac{3}{5}$	0
p_7^o	0	$\frac{1}{5}$	$\frac{1}{5}$	0	0	0	0	0	0	0	0	$\frac{3}{5}$

A.2. Directed graphs: preliminaries

A **directed graph**, or **digraph** is a pair $G = (V, A)$, consisting of a set of **vertices** V and a set of ordered pairs of vertices, A, called **arcs**. For each $U \subseteq V$, an arc $a = (u, v)$ is said to *leave* U if $u \in U$ and $v \notin U$; it is said to *enter* U if $u \notin U$ and $v \in U$. We denote the set of arcs of G entering U by $\delta^{in}(U)$ and the set of arcs leaving U by $\delta^{out}(U)$.

Let $k : A \to \mathbb{R}_+$ be a function which associates each arc $a = (u, v)$ a nonnegative real number $k(a)$ called the **capacity of the arc**.

Let $b : V \to \mathbb{R}$ be a function. A function $f : A \to \mathbb{R}$ is called a **b-transshipment** if for each $u \in V$,

$$\sum_{a\in\delta^{in}(\{u\})} f(a) - \sum_{a\in\delta^{out}(\{u\})} f(a) = b(u).$$

The next result is the key for the proof of our main result.[15]

Theorem 2. *Let $G = (V, A)$ be a digraph and let $k : A \to \mathbb{R}$ and $b : V \to \mathbb{R}$ with $\sum_{v\in V} b(v) = 0$.*

(i) *Then, there exists a b-transshipment f satisfying that for each $a \in A$, $0 \leqslant f(a) \leqslant k(a)$ if and only if*

$$\text{for each} \quad U \subseteq V\colon \sum_{u\in U} b(u) \leqslant \sum_{a\in\delta^{in}(U)} k(a). \tag{1}$$

(ii) *Moreover, if b and k are integer-valued, then f can be taken integer-valued.*

A.3. Proofs

Proof of Theorem 1.

Lemma 5. *Consider the first step of the egalitarian mechanism.*[16] *Suppose the sets $Y_1, Y_2 \subseteq P^{u;u}$, $Z_1, Z_2 \subseteq P^{u;f,o}$, $K_1, K_2 \subseteq D^{u;u}$, and $L_1, L_2 \subseteq D^{u;f,o}$ are such that*

$$f_1(Y_1 \cup Z_1, K_1 \cup L_1) = f_1(Y_2 \cup Z_2, K_2 \cup L_2) = \lambda_1.$$

[15] For more on this result, see Schrijver [21].

[16] The result directly applies to steps $2, \ldots, K$ as well.

Then,

$$f_1(Y_1 \cup Z_1 \cup Y_2 \cup Z_2, K_1 \cup L_1 \cup K_2 \cup L_2) = \lambda_1$$

as well.

Proof. For $i = 1, 2$, define

$$n_i = |Y_i \cup Z_i|, \qquad d_i = \left|D_{Y_i \cup Z_i} \cap D^o\right|,$$
$$m_i = \left|P(K_i \cup L_i)\right| - \left|(K_i \cup L_i) \setminus \{d_p\colon\ p \in Y_i\}\right|.$$

Also, define

$$n_3 = \left|\bigcap_{i=1,2}(Y_i \cup Z_i)\right|, \qquad n_4 = \left|\bigcup_{i=1,2}(Y_i \cup Z_i)\right|,$$
$$d_3 = \left|D_{(Y_1 \cup Z_1)\cap(Y_2 \cup Z_2)} \cap D^o\right|, \qquad d_4 = \left|D_{(Y_1 \cup Z_1)\cup(Y_2 \cup Z_2)} \cap D^o\right|,$$

and

$$m_3 = \left|P\left(\bigcap_{i=1,2}(K_i \cup L_i)\right)\right| - \left|\left(\bigcap_{i=1,2}(K_i \cup L_i)\right) \Big\backslash \{d_p\colon\ p \in Y_1 \cap Y_2\}\right|,$$
$$m_4 = \left|P\left(\bigcup_{i=1,2}(K_i \cup L_i)\right)\right| - \left|\left(\bigcup_{i=1,2}(K_i \cup L_i)\right) \Big\backslash \{d_p\colon\ p \in Y_1 \cup Y_2\}\right|.$$

By definition, we have

$$n_1 + n_2 = n_3 + n_4, \quad \text{and} \quad |Z_1| + |Z_2| = |Z_3| + |Z_4|.$$

Also,

$$d_1 + d_2 \geqslant d_3 + d_4.$$

This is because, not only the compatible kidneys of the patients in $\bigcap_{i=1,2}(Y_i \cup Z_i)$ are counted twice, but also two patients, one in $Y_1 \cup Z_1$, the other in $Y_2 \cup Z_2$, may reveal the same donor as compatible.

By definition of $P(\cdot)$, the patients in $P_1(\bigcap_{i=1,2}(K_i \cup L_i))$ are the only double counted patients in $\bigcup_{i=1,2} P_1(K_i \cup L_i)$. Moreover, a patient who is neither in $P_1(K_1 \cup L_1)$ nor in $P_1(K_2 \cup L_2)$, may be in $P_1(\bigcup_{i=1,2}(K_i \cup L_i))$. Thus,

$$\sum_{i=1,2}\left|P_1(K_i \cup L_i)\right| \leqslant \left|P_1\left(\bigcup_{i=1,2}(K_i \cup L_i)\right)\right| + \left|P_1\left(\bigcap_{i=1,2}(K_i \cup L_i)\right)\right|.$$

For each $i = 1, 2$, $d \in (\bigcap_{i=1,2}(K_i \cup L_i)) \setminus \{d_p\colon\ p \in Y_1 \cap Y_2\}$ and $p_d \in Y_i$ implies $d \in ((K_{-i} \cup L_{-i}) \setminus \{d_p\colon\ p \in Y_{-i}\})$, but $d \notin ((K_i \cup L_i) \setminus \{d_p\colon\ p \in Y_i\})$. Thus, the only double counted donors in the set

$$\bigcup_{i=1,2}\big((K_i \cup L_i) \setminus \{d_p\colon\ p \in Y_i\}\big)$$

are the donors in

$$\left(\left(\bigcap_{i=1,2}(K_i \cup L_i)\right) \Big\backslash \{d_p\colon\ p \in Y_1 \cap Y_2\}\right) \Big\backslash (Y_1 \cup Y_2).$$

Thus,

$$\sum_{i=1,2}\left|(K_i \cup L_i) \setminus \{d_p\colon\ p \in Y_i\}\right| = \left|\Bigl(\bigcap_{i=1,2}(K_i \cup L_i)\Bigr)\Bigm\backslash\{d_p\colon\ p \in Y_1 \cap Y_2\}\right| + \left|\Bigl(\bigcup_{i=1,2}(K_i \cup L_i)\Bigr)\Bigm\backslash\{d_p\colon\ p \in Y_1 \cup Y_2\}\right|.$$

Thus,

$$m_1 + m_2 \leqslant m_3 + m_4.$$

By definition of λ_1,

$$\lambda_1 = \frac{d_1 - m_1 - |Z_1|}{n_1} = \frac{d_2 - m_2 - |Z_2|}{n_2} \leqslant \frac{d_3 - m_3 - |Z_1 \cap Z_2|}{n_3},$$

thus,

$$\lambda_1 n_1 = d_1 - m_1 - |Z_1|,$$
$$\lambda_1 n_2 = d_2 - m_2 - |Z_2|,$$
$$\lambda_1 n_3 \leqslant d_3 - m_3 - |Z_1 \cap Z_2|.$$

Adding the first two lines and subtracting the third line,

$$\lambda_1 \underbrace{(n_1 + n_2 - n_3)}_{=n_4} \geqslant \underbrace{(d_1 + d_2 - d_3)}_{\geqslant d_4} - \underbrace{(m_1 + m_2 - m_3)}_{\leqslant m_4} - \underbrace{\bigl(|Z_1| + |Z_2| - |Z_1 \cap Z_2|\bigr)}_{=|Z_1 \cup Z_2|};$$

thus,

$$\lambda_1 \geqslant \frac{d_4 - m_4 - |Z_1 \cup Z_2|}{n_4} = f_1(Y_1 \cup Z_1 \cup Y_2 \cup Z_2, K_1 \cup L_1 \cup K_2 \cup L_2).$$

Since λ_1 is the minimum value of f_1 among all possible sets as defined in the solution,

$$f_1(Y_1 \cup Z_1 \cup Y_2 \cup Z_2, K_1 \cup L_1 \cup K_2 \cup L_2) = \lambda_1. \qquad \square$$

Lemma 6. *Consider Step $K+1$ of the egalitarian mechanism.*[17] *Suppose the sets $Y_1, Y_2 \subseteq P^{u;u}_{K+1} \cup P^{u;1}_{K+1}$, and $L_1, L_2 \subseteq D^u_{K+1}$ are such that*

$$g_1(Y_1, L_1) = g_1(Y_2, L_2) = \beta_1.$$

Then,

$$g_1(Y_1 \cup Y_2, L_1 \cup L_2) = \beta_1.$$

Proof. For $i = 1, 2$, define

$$n_i = |Y_i|, \qquad d_i = \left|D_{Y_i} \cap D^o_{K+1}\right|,$$
$$m_i = \left|P^o_{K+1}(L_i)\right|.$$

Also, define

[17] The result directly applies to steps $K+2, \ldots, K+M$ as well.

$$n_3 = |Y_i \cup Y_2|, \qquad n_4 = |Y_1 \cup Y_2|,$$
$$d_3 = \left|D_{Y_1 \cap Y_2} \cap D^o_{K+1}\right|, \qquad d_4 = \left|D_{Y_1 \cup Y_2} \cap D^o_{K+1}\right|,$$

and

$$m_3 = \left|P^o_{K+1}\bigl(\{d_p\colon\ p \in Y_1 \cap Y_2\}\bigr)\right|,$$
$$m_4 = \left|P^o_{K+1}\bigl(\{d_p\colon\ p \in Y_1 \cup Y_2\}\bigr)\right|.$$

By definition, we have $n_1 + n_2 = n_3 + n_4$. Also,

$$d_1 + d_2 \geqslant d_3 + d_4.$$

This is because, not only the compatible kidneys of the patients in $\bigcap_{i=1,2} Y_i$ are counted twice, but also two patients, one in Y_1, the other in Y_2, may reveal the same donor as compatible.

By definition of $P(\cdot)$, the patients in $P^o_{K+1}(\{d_p\colon\ p \in Y_1 \cap Y_2\})$ are the only double counted patients in $\bigcup_{i=1,2} P^o_{K+1}(\{d_p\colon\ p \in Y_i\})$. Moreover, a patient who is not in the latter set may be in $P^o_{K+1}(\{d_p\colon\ p \in Y_1 \cup Y_2\})$. Thus,

$$m_1 + m_2 \leqslant m_3 + m_4.$$

By applying the same technics as in the previous lemma, we obtain

$$g_1(Y_1 \cup Y_2, L_1 \cup L_2) = \beta_1. \qquad \square$$

Thus, in the mechanism, the largest sets minimizing f_k and g_m in the sense of inclusion are well defined for each step $k \in \{1, \dots, K\}$ and $K + m$ for $m \in \{1, \dots, M\}$.

The set of random matchings obtained by the egalitarian mechanism is Z^{K+M}. First, we need to show that $Z^{K+M} \subseteq \mathcal{Z}^e$.

Lemma 7. *The set Z^{K+M} is non-empty and contains only efficient random matchings.*

Proof. We prove by induction.

Step 1: We claim that there exists a random matching $Z \in \mathcal{Z}^e$ such that, for each $p \in P$, $u_p(Z) - t_{d_p}(Z) \geqslant \lambda_1$. We construct the following digraph $G = (V, A)$:

$$V = \bigl((P \cup D) \setminus \bigl(P^f \cup D^f\bigr)\bigr) \cup \{t\}.$$
$$A = \left(\bigcup_{p \in P^o \cup P^u} \bigl\{(p, d)\colon\ d \in D_p\bigr\}\right) \cup \bigl\{(d, p_d)\colon\ d \in D^{u:u}\bigr\} \cup \bigl\{(d, t)\colon\ d \in D^o \cup D^{u;f,o}\bigr\}.$$

Define the capacity function $k : A \to \mathbb{R}_+$ as follows:

$$k(a) = \begin{cases} \infty & \text{if } a \in (\bigcup_{p \in P^o \cup P^u} \{(p, d)\colon\ d \in D_p\}), \\ 1 & \text{if } a \in \{(d, p_d)\colon\ d \in D^{u;u}\} \cup \{(d, t)\colon\ d \in D^{u;f,o} \cup D^o\}. \end{cases}$$

Define $b : V \to \mathbb{R}$ as follows:

$$b(v) = \begin{cases} -1 & \text{if } v \in P^o, \\ 0 & \text{if } v \in D^u \cup D^o, \\ -\lambda_1 & \text{if } v \in P^{u;u}, \\ -1 - \lambda_1 & \text{if } v \in P^{u;f,o}, \\ |P^o| + \lambda_1 |P^{u;u}| + (1 + \lambda_1)|P^{u;f,o}| & \text{if } v = t. \end{cases}$$

Let $f : A \to \mathbb{R}$ be a function. The interpretation is that, for

$$a = (p', d') \in \left(\bigcup_{p \in P^o \cup P^u} \{(p, d)\colon d \in D_p\} \right),$$

$f(a)$ is the probability that patient p' receives kidney transplantation from donor d'. Similarly, for $a = (d'', p'') \in \{(d, p_d)\colon d \in D^{u;u}\}$, $f(a)$ is the probability that donor d'' of patient p'' donates her kidney to someone in the set P^o. The function b is specified to capture efficiency, that is, the GED. For $p \in P^o$, since by efficiency, patient p receives a live donor transplantation with probability one, $b(p) = -1$. Also, for each $p \in P^{u;u}$, the difference between the sum of the values of f of the arcs leaving vertex p and entering vertex p is the difference between the probability that patient p receives a live donor kidney transplantation and the probability that his donor donates her kidney someone in the pool. Our claim is that it is possible to stochastically match the patients to the donors such that for each underdemanded patient, the u–t difference is at least λ_1. Given the digraph $G = (V, A)$, the functions $k : A \to \mathbb{R}_+$ and $b : V \to \mathbb{R}$ constructed above, since the existence of a function $f : A \to \mathbb{R}$ such that for each $u \in V$,

$$\sum_{a \in \delta^{in}(\{u\})} f(a) - \sum_{a \in \delta^{out}(\{u\})} f(a) = b(u).$$

implies that there is an ex ante efficient random matching Z such that, for each $p \in P$, $u_p(Z) - t_{d_p}(Z) \geqslant \lambda_1$, we need to show that there exists a *b-transshipment* f satisfying that for each $a \in A$, $0 \leqslant f(a) \leqslant k(a)$. By Theorem 2, there exists such a *b-transshipment* f if and only if

$$\text{for each } U \subseteq V\colon \quad \sum_{u \in U} b(u) \leqslant \sum_{a \in \delta^{in}(U)} k(a).$$

First, note that the inequality is satisfied for $U = \{t\}$. Suppose not. Then,

$$|P^o| + \lambda_1 |P^{u;u}| + (1 + \lambda_1)|P^{u;f,o}| = b(t) > k(\delta^{in}(t)) = |D^o| + |D^{u;f,o}|,$$

which implies

$$\lambda_1 > \frac{|D^o| + |D^{u;f,o}| - |P^o| - |P^{u;f,o}|}{|P^u|}. \tag{2}$$

Consider $P^{u;u} \cup P^{u;f,o}$, and $F' = D^{u;u} \cup D^{u;f,o}$. By the GED Lemma, $D_{P^u} \cap D^o = D^o$. Then, by definition of λ_1,

$$\frac{|D^o| - (|P^o| + |P^{u;f,o}| - |(D^{u;u} \cup D^{u;f,o}) \setminus \{d_p\colon p \in P^{u;u}\}|)}{|P^{u;u}| + |P^{u;f,o}|} \geqslant \lambda_1$$

which is equivalent to

$$\frac{|D^o| - (|P^o| + |P^{u;f,o}| - |D^{u;f,o}|)}{|P^u|} \geqslant \lambda_1,$$

and this contradicts with (2).

Let $S' = R' \cup T'$ with $R' \subseteq P^{u;u}$ and $T' \subseteq P^{u;f,o}$ and consider a set $U \subseteq V$ such that $\{t\} \cup S' \subseteq U$. If for some $d \in D_{P^u \setminus S'} \cap D^o$, $d \in U$, then, since $k(\delta^{in}(U)) = \infty$, the inequality (1) is trivially satisfied. Thus, we need to check inequality (1) only for U such that $D^o \setminus D_{P^u \setminus S'} \subseteq U$. Similarly, if $D' \subseteq U$ for some $D' \subseteq D^u$, we need to check it only for U such that $P^o \setminus P(D^u \setminus D') \subseteq U$.

We claim that it is enough to check inequality (1) for $D' \subseteq D^u$ such that $D' \cap D^{u;u} \subseteq \{d_p:\ p \in R'\}$. Let $D' \subseteq U$ such that $D'' \subseteq D'$ where $D'' \subseteq D^{u;u} \setminus \{d_p:\ p \in R'\}$. As argued above, $P^o \setminus P(D^u \setminus D') \subseteq U$. Now, let us consider $U \setminus D''$. Since $P^o \setminus P(D^u \setminus D') \supseteq P^o \setminus P(D^u \setminus (D' \setminus D''))$, and for each $p \in P^o$, $b(p) = -1$, this implies that $\sum_{u \in U} b(u) \leqslant \sum_{u \in U \setminus D''} b(u)$. Also, the construction of the capacity function k implies that $\sum_{a \in \delta^{in}(U)} k(a) = \sum_{a \in \delta^{in}(U \setminus D'')} k(a)$. Thus, if inequality (1) is satisfied for $U \setminus D''$, then it is satisfied for U as well. Thus, it is enough to check inequality (1) for U where $D' \subseteq U$ implies $D' \cap D^{u;u} \subseteq \{d_p:\ p \in R'\}$.

Now, consider $U = \{t\} \cup S' \cup (D^o \setminus D_{P^u \setminus S'}) \cup F' \subseteq V$ such that $F' \cap D^{u;u} \subseteq \{d_p:\ p \in R'\}$. Let $R = P^{u;u} \setminus R'$, $T = P^{u;f,o} \setminus T'$, and $F = D^u \setminus F'$. Suppose $\sum_{u \in U} b(u) > \sum_{a \in \delta^{in}(U)} k(a)$. Thus,

$$\begin{aligned}
&\begin{pmatrix} -(|P^o| - |P(F)|) + (-\lambda_1)(|P^{u;u}| - |R|) \\ +(-1-\lambda_1)(|P^{u;f,o}| - |T|) + |P^o| + \lambda_1 |P^{u;u}| + (1+\lambda_1)|P^{u;f,o}| \end{pmatrix} \\
&\quad \geqslant |D_S| + \left|\left(F \cap D^{u;u}\right) \setminus \{d_p:\ p \in R\}\right| + \left|F \cap D^{u;f,o}\right|,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\lambda_1 &> \frac{|D_S| + |(F \cap D^{u;u}) \setminus \{d_p:\ p \in R\}| - |P(F)| - |T| + |F \cap D^{u;f,o}|}{|S|} \\
&= \frac{|D_S| + |\{d_p:\ p \in R\}| - |P(F)| - |T| + |F|}{|S|} \\
&= \frac{|D_S| + |R| - |P(F)| - |T| + |F|}{|S|} \\
&= \frac{|D_S| - |P(F)| - |S| + |F|}{|S|}.
\end{aligned}$$

This contradicts with the definition of λ_1. Thus, $\sum_{u \in U} b(u) \leqslant \sum_{a \in \delta^{in}(U)} k(a)$. Since U is arbitrarily chosen, this condition holds for each U. Then, by Theorem 2, there exists a *b-transshipment* f satisfying that for each $a \in A$, $0 \leqslant f(a) \leqslant k(a)$. Thus, there exists an ex ante efficient random matching $Z \in \mathcal{Z}^e$ such that, for each $p \in P$, $u_p(Z) - t_{d_p}(Z) \geqslant \lambda_1$. Let Z^1 be the set of all such ex ante efficient random matchings.

Step k: Let the sets S_{k-1}, and F_{k-1} be the largest sets in the sense of inclusion such that

$$\lambda_{k-1} = f_{k-1}(S_{k-1}, F_{k-1}).$$

The donors in F_{k-1} are matched only to the patients in $P^o_{k-1}(F_{k-1})$. The donors in $D^o \cap (D_{S_{k-1}} \setminus \bigcup_{n=1}^{k-2} D_{S_n})$ are matched only to the patients in S_{k-1}. The donors in $F_{k-1} \setminus \{d_p:\ p \in S_{k-1}\}$ and $D^o \cap (D_{S_{k-1}} \setminus \bigcup_{n=1}^{k-2} D_{S_n})$ are matched with probability one. The patients and donors in $S_{k-1} \cup F_{k-1}$ leave. Then, we construct the digraph with the remaining patients and donors, as in the previous step. By Theorem 2, there exists a *b-transshipment* f satisfying that for each $a \in A$, $0 \leqslant f(a) \leqslant k(a)$. Thus, there exists an ex ante efficient random matching $Z \in Z^{k-1}$ such that, for each remaining patient p, $u_p(Z) - t_{d_p}(Z) \geqslant \lambda_k$. Let Z^k be the set of all such ex ante efficient random matchings.

Step $K+1$: Let the sets S_K, and F_K be the largest sets in the sense of inclusion such that

$$\lambda_K = f_K(S_K, F_K).$$

The patients and donors in $S_K \cup F_K$ leave. If, among the remaining patients, there exists an underdemanded patient p such that his donor has left at an earlier stage, then for each $Z \in Z^K$,

$u_p(Z) - t_{d_p}(Z) \leqslant 0$. This contradicts with the definition of λ_K. Thus, the donor of a remaining underdemanded patient p is among the remaining donors. We construct the digraph $G = (V, A)$ where V is the set of remaining patients and donors together with the vertex t, that is,

$$V = \{t\} \cup D^u_{K+1} \cup D^o_{K+1} \cup P^o_{K+1} \cup P^u_{K+1}$$

and

$$A = \left(\bigcup_{p \in P^o_{K+1} \cup P^u_{K+1}} \{(p, d)\colon d \in D_p\} \right) \cup \{(d, p_d)\colon d \in D^{u;u}_{K+1}\}$$
$$\cup \{(d, t)\colon d \in D^o_{K+1} \cup D^{u;f,o}_{K+1}\}.$$

Define the capacity function $k : A \to \mathbb{R}_+$ as follows:

$$k(a) = \begin{cases} \infty & \text{if } a \in (\bigcup_{p \in P^o_{K+1} \cup P^u_{K+1}} \{(p, d)\colon d \in D_p\}), \\ 1 & \text{if } a \in \{(d, p_d)\colon d \in D^{u;u}_{K+1}\} \cup \{(d, t)\colon d \in D^{u;f,o}_{K+1} \cup D^o_{K+1}\}. \end{cases}$$

Define $b : V \to \mathbb{R}$ as follows:

$$b(v) = \begin{cases} -1 & \text{if } v \in P^o_{K+1} \cup P^{u;1}_{K+1}, \\ 0 & \text{if } v \in P^{u;u}_{K+1} \cup D^o_{K+1}, \\ -\beta_1 & \text{if } v \in D^{u;u}_{K+1}, \\ 1 - \beta_1 & \text{if } v \in D^{u;f,o}_{K+1}, \\ |P^o_{K+1}| + |P^{u;1}_{K+1}| + \beta_1 |D^{u;u}_{K+1}| - (1 - \beta_1)|P^{u;f,o}_{K+1}| & \text{if } v = t. \end{cases}$$

Let $T' \subseteq P^u_{K+1}$. By the same argument used in Step 1, it is enough to check the condition in Theorem 2 for $U \subseteq V$ such that $U \cap (D^{u;u}_{K+1} \setminus \{d_p\colon p \in T'\}) = \emptyset$. The definition of β_1 implies that for each $U \subseteq V$, $\sum_{u \in U} b(u) \leqslant \sum_{a \in \delta^{in}(U)} k(a)$. Then, Theorem 2 implies that there exists a *b-transshipment* f satisfying that for each $a \in A$, $0 \leqslant f(a) \leqslant k(a)$. Thus, there exists an ex ante efficient random matching $Z \in Z^K$ such that for each remaining patient p, $u_p(Z) - t_{d_p}(Z) \geqslant \beta_1$. Let Z^{K+1} be the set of all such ex ante efficient random matchings.

Step $K + m$: Let the sets T^{m-1} and H^{m-1} be the largest sets in the sense of inclusion such that

$$\beta_{m-1} = g_{m-1}\bigl(T^{m-1}, H^{m-1}\bigr).$$

The donors in $(D_{T^{m-1}} \setminus (\bigcup_{n=1}^K D_{S_n}) \setminus (\bigcup_{j=1}^{m-2} D_{T^j})) \cap D^o$ are matched to the patients in T^{m-1}, and the donors in $\{d_p\colon p \in T^{m-1}\} \cup H^{m-1}$ are matched to the patients in $P^o_{K+m-1}(\{d_p\colon p \in T^{m-1}\} \cup H^{m-1})$. These patients and donors leave and a digraph is constructed with the remaining patients and donors, as in the previous step. By Theorem 2, there exists a *b-transshipment* f satisfying that for each $a \in A$, $0 \leqslant f(a) \leqslant k(a)$. Thus, there exists an ex ante efficient random matching $Z \in Z^{K+m-1}$ such that, for each remaining patient p, $u_p(Z) - t_{d_p}(Z) \geqslant \beta_m$. Let Z^{K+m} be the set of all such ex ante efficient random matchings.

At the end of step $K + M$, the set Z^{K+M} is non-empty and contains only efficient random matchings. □

Lemma 8. *For each $k \in \{1, \ldots, K - 1\}$, $\lambda_k < \lambda_{k+1}$. For each $m \in \{1, \ldots, M - 1\}$, $\beta_m < \beta_{m+1}$.*

Proof. Let the sets S_k, F_k; and S_{k+1}, F_{k+1} be the largest sets in the sense of inclusion such that

$$\lambda_k = f_k(S_k, F_k),$$
$$\lambda_{k+1} = f_{k+1}(S_{k+1}, F_{k+1}).$$

Suppose

$$\lambda_{k+1} \leqslant \lambda_k.$$

Thus,

$$\frac{|(D_{S_{k+1}} \setminus (\bigcup_{n=1}^{k} D_{S_n})) \cap D^o| - (|P^o_{k+1}(F_{k+1})| + |S_{k+1}| - |F_{k+1}|)}{|S_{k+1}|} \leqslant \frac{|(D_{S_k} \setminus (\bigcup_{n=1}^{k-1} D_{S_n})) \cap D^o| - (|P^o_k(F_k)| + |S_k| - |F_k|)}{|S_k|}. \tag{3}$$

Now, consider $S_k \cup S_{k+1}$ and $F_k \cup F_{k+1}$. Note that

$$\left|\left(D_{S_k \cup S_{k+1}} \Big\backslash \left(\bigcup_{n=1}^{k-1} D_{S_n}\right)\right) \cap D^o\right| = \left|\left(D_{S_k} \Big\backslash \left(\bigcup_{n=1}^{k-1} D_{S_n}\right)\right) \cap D^o\right| + \left|\left(D_{S_{k+1}} \Big\backslash \left(\bigcup_{n=1}^{k} D_{S_n}\right)\right) \cap D^o\right|. \tag{4}$$

Also, F_k and F_{k+1} are mutually exclusive and this implies

$$\left|P^o_k(F_k \cup F_{k+1})\right| \geqslant \left|P^o_{k+1}(F_{k+1})\right| + \left|P^o_k(F_k)\right|. \tag{5}$$

Then, combining (3), (4) and (5), we obtain

$$f_k(S_k, F_k) \geqslant f_k(S_k \cup S_{k+1}, F_k \cup F_{k+1}).$$

This contradicts with the definition of S_k and F_k, that they are the largest sets in the sense of inclusion such that

$$\lambda_k = f_k(S_k, F_k).$$

Also, by construction $\beta_1 > 0$, and $\lambda_K \leqslant 0$. By using the same inequalities/equalities as above, we see that, for each $m \in \{1, \ldots, M-1\}$, $\beta_m < \beta_{m+1}$. □

Let $Z^* \in \mathcal{Z}^{K+M}$ be a random matching. For each patient p in S_1, $u_p(Z^*) - t_{d_p}(Z^*)$ is the lowest under Z^*, for each patient p in S_2, $u_p(Z^*) - t_{d_p}(Z^*)$ is the lowest among the remaining patients under Z^*, and so on. Note that $S_1 \subseteq P^u$, $F_1 \subseteq D^u$ such that $\{d_p\colon\ p \in S_1\} \cap P^{u;u} \subseteq F_1$, and

$$f_1(S_1, F_1) = \frac{|D_{S_1} \cap D^o| - (|P^o(F_1)| + |S_1| - |F_1|)}{|S_1|}.$$

By compatibility, the patients in S_1 can be matched to at most $|D_{S_1} \cap D^o|$ patients. Also, by efficiency, the patients in $P^o(F_1)$ are matched to the donors in F_1. At an efficient random matching, the least possible number of patients who are matched to $\{d_p\colon\ p \in S_1 \cap P^{u;u}\}$ is $|P^o(F_1)| - (|F_1| - |S_1 \cap P^{u;u}|)$; and this is possible only if each donor in $F_1 \setminus (\{d_p\colon\ p \in S_1\} \cap D^{u;u})$ is matched to a patient in $P^o(F_1)$ with probability one. After this matching of the donors in $F_1 \setminus (\{d_p\colon\ p \in S_1\} \cap D^{u;u})$, there are $|P^o(F_1)| - (|F_1| - |S_1 \cap P^{u;u}|)$ remaining patients in $P^o(F_1)$. By efficiency, each such patient is matched to the donors in $\{d_p\colon\ p \in S_1 \cap P^{u;u}\}$ with probability one. Also, by efficiency, the donor of each patient $p \in S_1 \cap P^{u;f,o}$ is matched with probability one. Thus, for each $Z \in \mathcal{Z}^e$,

$$\begin{aligned}\sum_{p \in S_1} \bigl(u_p(Z) - t_{d_p}(Z)\bigr) &\leqslant \bigl|D_{S_1} \cap D^o\bigr| - \bigl(\bigl|P^o(F_1)\bigr| - \bigl(|F_1| - \bigl|S_1 \cap P^{u;u}\bigr|\bigr)\bigr) - \bigl|S_1 \cap P^{u;f,o}\bigr| \\ &= \bigl|D_{S_1} \cap D^o\bigr| - \bigl|P^o(F_1)\bigr| + |F_1| - \bigl(\bigl|S_1 \cap P^{u;u}\bigr| + \bigl|S_1 \cap P^{u;f,o}\bigr|\bigr) \\ &= \bigl|D_{S_1} \cap D^o\bigr| - \bigl|P^o(F_1)\bigr| + |F_1| - |S_1|.\end{aligned}$$

We have already shown that Z^* is an ex ante efficient random matching and this upper bound is reached in an egalitarian way under Z^*, thus, for each patient $p \in S_1$, $u_p(Z^*) - t_{d_p}(Z^*) = \lambda_1$. Similarly, for the patients in S_2, the highest possible aggregate u–t difference achieved by the lowest u–t difference $|S_1| + 1$ patients in $S_1 \cup S_2$ is $|S_1| \times f_1(S_1, F_1) + f_2(S_2, F_2)$. Proceeding in this way, the upper bound for the patients in S_2 is reached in an egalitarian way, and so on. Thus, the vector $\mathbf{u}(Z^*) - \mathbf{t}(Z^*)$ Lorenz dominates any other vector in the set $\{\mathbf{u}(Z) - \mathbf{t}(Z)\colon Z \in \mathcal{Z}^e\}$.

Proof of Corollary 1. (i) $\Leftrightarrow$ (ii). It follows directly from Theorem 1.

(iv) $\Rightarrow$ (iii). By the GED Lemma, $|F| > |P^o(F)|$. Thus, by Theorem 1, the u–t difference of each patient is greater than -1 under the egalitarian matching. Thus, a deterministic egalitarian matching with inequality is not possible.

(ii) $\Rightarrow$ (iii). By definition, an efficient matching with no inequality is an egalitarian matching. Thus by Theorem 1, it is possible to construct a b-transshipment. Note that, since the u–t difference of each patient is zero, b is integer-valued. Then, by Theorem 2, f can be taken as integer-valued. Thus, there exists a deterministic matching with no inequality. □

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