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# An analytical comparison of different formulations of the travelling salesman problem

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A transformation technique is proposed that permits one to derive the linear description of the image $X$ of a polyhedron $Z$ under an affine linear transformation from the (given) linear description of $Z$. This result is used to analytically compare various formulations of the asymmetric travelling salesman problem to the standard formulation due to Dantzig, Fulkerson and Johnson which are all shown to be "weaker formulations" in a precise setting. We also apply this transformation technique to "symmetrize" formulations and show, in particular, that the symmetrization of the standard asymmetric formulation results into the standard one for the symmetric version of the travelling salesman problem.

*Key words*: Travelling salesman problem, problem formulations, convex polyhedral cones, polyhedra, affine transformations.

## Introduction

A *formulation* of a combinatorial optimization problem is a finite set of linear inequalities and/or equations in a finite set of variables, the integer (or mixed-integer) solutions of which are in one-to-one correspondence with the combinatorial configurations (stable sets, tours of a travelling salesman, spanning trees, etc.) over which we wish to minimize a linear objective function. As we are dealing with linear inequalities or equations, we obtain a polyhedron if we drop the integrality requirement and thus a "formulation" is a *polyhedron* in some finite-dimensional Euclidean vector-space. Its intersection with the lattice of integer (or mixed-integer) points in that space is in one-to-one correspondence with the set of the desired combinatorial configurations. Denote $D$ this discrete set of points. If $D$ is a finite set or if the data of the formulation involve only *rational* data, then the *convex hull* of $D$, $\mathrm{conv}(D)$, is a well-defined polyhedron and thus by Weyl's theorem [1935] there exists a *finite* set of linear inequalities and/or equations that describe $\mathrm{conv}(D)$ completely. We call such a complete linear description the *ideal formulation* of the

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underlying problem; see also Wolsey (1987). Of course, because of the NP-completeness characteristic of many combinatorial optimization problems the search for an ideal formulation *may* prove to be elusive.

For many combinatorial optimization problems there are typically different ways of "formulating" them. Given two different formulations $A$ and $B$ of a given problem that are stated in the *same* space of variables we have thus two polyhedra $X_A$ and $X_B$. If $X_A \subset X_B$ then clearly formulation $A$ is "better" than formulation $B$ since the optimization over $X_A$ brings us "closer" to the "true" optimum, i.e. the upper bound to the combinatorial optimization problem provided for by $X_A$ is *always* better than or equal to the upper bound provided for by $X_B$. (We hasten to point out that we exclude considerations regarding the "speed of calculation" or other criteria in this definition of what we consider to be "better" and concentrate solely on the *goodness* of the upper bound that is obtained from a "formulation".) The entire line of research that studies *facet-defining* inequalities of various polyhedra occurring in combinatorial optimization is devoted to finding *improved* formulations of the respective problems.

Different formulations of a given problem can also frequently be stated in terms of different sets of variables. So suppose that we have a formulation $C$ that models the same problem as formulation $A$, but in a higher-dimensional space. We thus have a polyhedron $Z_C$, say, and the polyhedron $X_A$. If we have an affine transformation $T$ that maps the integer (or mixed-integer) points of $Z_C$ *onto* the integer (or mixed-integer) points of $X_A$, then it makes sense to calculate the image $T(Z_C)$ of $Z_C$ under this transformation. If $T(Z_C) \supset X_A$ then formulation $A$ is evidently better than formulation $C$ as the former provides no new polyhedral information about the convex hull of integer (or mixed-integer) points of $X_A$, i.e. the ideal formulation. On the other hand, formulation $C$ is better than formulation $A$ if $T(Z_C) \subset X_A$. However, if $T(Z_C) \supset X_A$, then the polyhedron $T(Z_C)$ need not even be a "formulation" of the problem formulated by $X_A$ when general affine transformations are considered.

The use of "auxiliary" variables to arrive at a different formulation occurs already in a paper by Miller, Tucker and Zemlin (1960) who show that with the help of $n-1$ *real* variables the "size" of the formulation of the asymmetric travelling salesman problem can be reduced from exponentially many linear constraints to a mere $n^2-n+1$ linear constraints where $n$ denotes the number of cities involved. More recently, a more systematic attempt is made to investigate such "reformulations" of integer and mixed-integer programming problems in higher-dimensional spaces and we refer the reader to the survey by Wolsey (1987) for further references.

In this paper we study different formulations of the asymmetric travelling salesman problem (TSP). In Section 1 we state some properties of polyhedral convex cones that are needed throughout our work. In Section 2 we consider affine transformations of polyhedra and derive a linear description of the image $X$ of a polyhedron $Z$ from the linear description of $Z$. In Section 3 we compare three different formulations of the TSP (including the Miller–Tucker–Zemlin formulation) to the standard formulation due to Dantzig, Fulkerson and Johnson (1954). In Section 4 we use linear transformations to "symmetrize" several of these asymmetric formulations of the TSP. In Section 5 we draw some conclusions from our work.

## 1. Some properties of polyhedral cones

In this section we give some definitions and properties of convex cones that are essentially known from the literature, see e.g. Gale (1951), Gerstenhaber (1951), Burger (1956), Simonnard (1966), Stoer and Witzgall (1970) and Bachem and Grötschel (1982), and that are used extensively throughout other parts of this work.

A set $C \subseteq \mathbb{R}^n$ is called a cone if $v^1, v^2 \in C$ implies that $\lambda_1 v^1 + \lambda_2 v^2 \in C$ for all scalars $\lambda_1, \lambda_2 \geqslant 0$. A halfline (or ray) $(v)$ is the set of points $\{\lambda v \in \mathbb{R}^n \mid \forall \lambda \geqslant 0\}$. A halfline $(v)$ is called an extreme ray of $C$ if for any $v^1, v^2 \in C$, $v = \lambda v^1 + (1-\lambda) v^2$ with $0 < \lambda < 1$ implies that $v^1, v^2$ are positive multiplies of $v$. A set of generators of a cone is a set of halflines which spans $C$ and such that no halfline of the set is in the convex hull of the others. The class of cones of our interest are the ones for which there exists a *finite* set of generators, the so-called *polyhedral* cones. All cones considered in our work are polyhedral ones and thus we drop the adjective from now on.

It follows from Weyl's (1935) theorem that every cone that we consider is the intersection of finitely many halfspaces and thus we can write $C$ as

$$C = \{x \in \mathbb{R}^n \mid Ax \geqslant 0\},$$

where $A$ is an $m \times n$ matrix. In most of our work we are given a cone in matrix form and we wish to find a full system of generators of $C$. To this end we define the lineality space $L$ of $C$ to be the set of all vectors $x$ such that $x \in C$ and $-x \in C$ and hence

$$L = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

It follows that the dimension of $L$, dim $L$, is $n - \operatorname{rank}(A)$. If $\operatorname{rank}(A) = n$, the lineality space of $C$ consists of the origin only. In this case $C$ is a *pointed* cone, we say that $C$ has its *apex* at the origin and moreover, $C$ has a uniquely defined finite set of extreme rays. Whenever dim $L \geqslant 1$, the cone has no apex and no extreme rays in the sense of the definition given above. However, $C$ has a finite set of generators and among all such sets we will distinguish one that we continue to call "extreme". To do so define

$$L^{\perp} = \{y \in \mathbb{R}^n \mid xy = 0 \ \forall x \in L\},$$

$$C^0 = C \cap L^{\perp}.$$

$L^{\perp}$ is the orthogonal complement of $L$. It follows that the lineality space of the cone

$C^0$ is the origin and hence, $C^0$ is a pointed cone and possesses a uniquely defined finite set of extreme rays (in the sense of the above definition). Since $L$ and $C^0$ are orthogonal to each other, every point $x \in C$ can be written as $x = l + x_0$ where $l \in L$, $x_0 \in C^0$ and $l$ and $x_0$ are uniquely determined. We define the extreme rays of $C^0$ to be the "extreme rays" of $C$ since they are a unique set of generators for the *conical* part of $C$. If $B_L$ denotes a basis of $L$, then $B_L$ and $(-B_L)$ form a unique set of generators for the *lineal* part of $C$. In summary, letting $D$ be the set of extreme rays of $C^0$, a unique set $R$ of generators of any cone $C$ is given by

$$R = B_L \cup (-B_L) \cup D.$$

This way the concept of an "extreme ray" of any cone $C$ — even if $\dim L \geqslant 1$ — is unambiguously defined and in the case of a pointed cone we retrieve the original definition.

The following lemma and theorem from Burger (1956) state a criterion for $x \in C$ to define an extreme ray of $C$.

**Lemma 1.** *Let* $C = \{x \in \mathbb{R}^n \mid Ax \geqslant 0\}$ *be such that* $\operatorname{rank}(A) = n$ *and denote $M$ the index set of all rows of $A$. $(x)$ is an extreme ray of $C$ if and only if $x \in C$, $x \neq 0$ and there exists $I \subseteq M$ such that* (i) $|I| = n-1$, (ii) $a^i x = 0$ *for all* $i \in I$ *and* (iii) *the rows $a^i$ with $i \in I$ are linearly independent.*

**Proof.** Given a vector $x \in C$, let $I$ be a maximal subset of $M$ such that $a^i x = 0$ for all $i \in I$ and that the rows $a^i$ with $i \in I$ are linearly independent. Let $A_1$ be the submatrix of $A$ having rows $a^i$ for all $i \in I$ and $P$ be the subspace given by $P = \{x \in \mathbb{R}^n \mid A_1 x = 0\}$. It follows that $\dim P = n - |I|$ and since $I$ is maximal it follows that $\dim P \cap C = \dim P$.

Suppose now that $|I| = n-1$, but that $x$ is not an extreme ray of $C$. Since $\operatorname{rank}(A) = n$ we have $C = C^0$ and hence there exist $x^1 \in C^0$ and $x^2 \in C^0$ such that $(x^1) \neq (x) \neq (x^2)$ and $x = \lambda x^1 + (1-\lambda)x^2$ with $0 < \lambda < 1$. From $a^i x^j \geqslant 0$ for $j = 1, 2$, we obtain $a^i x^j = 0$ for $i \in I$ and $j = 1, 2$ and hence, $x^j \in P$ for $j = 1, 2$. Since $x^1, x^2 \in P \cap C$ and $\dim P = 1$, it follows that $x^1 = \mu x^2$ for some $\mu > 0$. Consequently, $(x) = (x^2)$ which is a contradiction.

On the other hand, suppose $x \neq 0$ is an extreme ray of $C$ and let $|I| = k$. Then clearly $k \leqslant n-1$ since otherwise $\dim P = 0$ and $x = 0$. Suppose that $k < n-1$. Then $\dim P \cap C = n - k \geqslant 2$ and there exist $n-k$ linearly independent vectors $x^i \in P \cap C$ such that each $x^i$ annuls at least one row of $A$ that is not in $A_1$ and that is linearly independent of the rows of $A_1$ since $\operatorname{rank}(A) = n$ and where $i = 1, \ldots, n-k$. Consequently, $x \neq x^i$ and $x$ is a nonnegative linear combination of $x^1, \ldots, x^{n-k}$ having at least two coefficients positive. Since $C = C^0$ it follows that $x = \lambda x^1 + (1-\lambda)x^2$ for some $x^1 \neq x \neq x^2$, $x^1, x^2 \in C^0$ and $0 < \lambda < 1$ which is a contradiction. The lemma follows. □

**Theorem 1.** *Let* $C=\{x\in\mathbb{R}^n \mid Ax\geq 0\}$, $L=\{x\in\mathbb{R}^n \mid Ax=0\}$ *and* $\dim L=d$. *A halfline* $(x)$ *is an extreme ray of* $C$ *if and only if* (i) $x\in C^0$ *and* (ii) *there exist exactly* $n-d-1$ *linearly independent rows* $a^i$ *of* $A$ *such that* $a^i x=0$.

**Proof.** Since $\dim L=d$, it follows that $\operatorname{rank}(A)=n-d$. If $d=0$, we are in the case stated in previous lemma and we are done. Let $d\geq 1$. Then we can write $L^{\perp}=\{x\in\mathbb{R}^n \mid Bx=0\}$ where $B$ is a $d\times n$ matrix of rank $d$ whose rows correspond to a basis of $L$. It follows that $C^0=\{x\in\mathbb{R}^n \mid Ax\geq 0,\ Bx=0\}$. We claim that the rank of the constraint matrix of $C^0$ equals $n$. Let $A_1$ be a $(n-d)\times n$ submatrix of $A$ having a full rank and suppose that there exist $\lambda\in\mathbb{R}^{n-d}$, $\mu\in\mathbb{R}^d$ such that $\lambda\neq 0$, $\mu\neq 0$ and $\lambda A_1+\mu B=0$, equivalently $\lambda A_1=-\mu B$. Consequently, $\lambda A_1 A_1^{\mathrm{T}}=-\mu B A_1^{\mathrm{T}}$ where $BA_1^{\mathrm{T}}=\mathbf{0}$ is a null-matrix. Since $A_1A_1^{\mathrm{T}}$ is a nonsingular matrix of size $n-d$, it follows that $\lambda=0$. Furthermore, since $\operatorname{rank}(B)=d$, we conclude likewise that $\mu=0$ and hence the claim follows. We can thus apply the previous lemma to $C^0$ and the theorem follows. □

From a computational point of view it will be sometimes more convenient to determine *nonzero solutions of minimal support*, i.e. a nonzero solution with the *least* number of nonzero components, to equation systems satisfying the requirement (ii) of Theorem 1 if we want to determine a generator system for the conical part of $C$. From any such solution $x\in C$, say, that satisfies $x\notin L$ one obtains an extreme ray of $C$ as follows: Given a basis $B$ of the lineality space $L$ of $C$ one calculates the projection $x^1$ of $x$ by the least-squares formula

$$x^1=x-B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}Bx. \tag{1}$$

Since $x\notin L$ it follows that $x^1\neq 0$ and moreover, $x^1\in C^0$. The vector $x^1$ annuls the same rows of $A$ as does $x$. Consequently every nonzero solution of minimal support to an equation system satisfying the requirement (ii) of Theorem 1 that is not in $L$ yields via this projection an extreme ray of $C$. On the other hand, let $x\in C$ be any extreme ray of $C$. By Theorem 1 there exist $n-d-1$ linearly independent rows of $A$ that are annulled by $x$ and since $x\neq 0$ this system possesses a nonzero solution of minimal support. Since $x\notin L$ it follows by a standard linear algebra argument that it possesses a nonzero solution of minimal support that is not in $L$. Consequently, we have the following remark which we will use repeatedly later on:

**Remark 1.** A full generator system for the conical part of a cone $C$ with $\dim L=d$ can be obtained by determining for each system of $n-d-1$ linearly independent rows of $A$ a nonzero solution of minimal support to $Ax\geq 0$ that is not in $L$ provided it exists.

In most of our applications of the preceding material the matrix $A$ defining the cone $C$ is a *block diagonal* matrix. It is therefore important to note that we can work on the lower dimensional cones defined by the blocks of $A$ in order to find a full generator system for $C$. More precisely, we have an "intersection property" of cones given by the following proposition:

**Proposition 1.** *Let*

$$C_1=\{x^1\in\mathbb{R}^p \mid A_1x^1\geqslant 0\},\qquad C_2=\{x^2\in\mathbb{R}^q \mid A_2x^2\geqslant 0\}$$

*and $C_3$ be the intersection cone of cones $C_1$ and $C_2$ embedded naturally in the $\mathbb{R}^{p+q}$, i.e.*

$$C_3=\{(x^1,x^2)\in\mathbb{R}^{p+q} \mid A_1x^1\geqslant 0,\ A_2x^2\geqslant 0\}.$$

*Denote $R_i$ the set of generators of $C_i$ for $i=1,2,3$. Then*

$$R_3=\{(u,0)\in\mathbb{R}^{p+q} \mid u\in R_1\}\cup\{(0,v)\in\mathbb{R}^{p+q} \mid v\in R_2\}.$$

**Proof.** Denote $d_i=\dim L_i$ the dimension of the lineality space of $C_i$, $B_i$ any basis of $L_i$ and $D_i$ the set of extreme rays of $C_i$ for $i=1,2,3$. It follows that $d_3=d_1+d_2$ and consequently

$$B_3=\{(u,0)\in\mathbb{R}^{p+q} \mid u\in B_1\}\cup\{(0,v)\in\mathbb{R}^{p+q} \mid v\in B_2\}$$

since the respective vectors are linearly independent. Let $u\in D_1$. Then $u\in C_1^0$ and thus $(u,0)\in C_3^0$. Moreover, $(u,0)$ annuls precisely $p-d_1-1+q-d_2=p+q-d_3-1$ linearly independent rows of the constraint system defining $C_3$. Thus by Theorem 1 we have $(u,0)\in D_3$. By symmetry, we have $(0,v)\in D_3$ for all $v\in D_2$. Hence we have

$$\{(u,0)\in\mathbb{R}^{p+q} \mid u\in R_1\}\cup\{(0,v)\in\mathbb{R}^{p+q} \mid v\in R_2\}\subseteq R_3.$$

To show equality suppose that there exists a generator $d$ of $C_3$ such that $d=(d^1,d^2)$ with $d^1\neq 0\neq d^2$. We write $d=(d^1,0)+(0,d^2)$. Consequently, $d^i\in C_i$ for $i=1,2$ and thus $d^i$ can be expressed as a nonnegative linear combination of the elements of $R_i$ for $i=1,2$ and thus $d=\sum\lambda_j^1(u^j,0)+\sum\lambda_j^2(0,v^j)$ for $u^j\in R_1$, $v^j\in R_2$ and $\lambda_j^1,\lambda_j^2\geqslant 0$. This contradicts the fact that $d\in R_3$ and the proposition follows. □

## 2. Affine transformations of polyhedra

In this section we consider affine transformations of full rank that map $\mathbb{R}^n$ into $\mathbb{R}^m$ where $m\leqslant n$. It is well-known that an affine transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ sends a polyhedron in $\mathbb{R}^n$ into a polyhedron in $\mathbb{R}^m$, see e.g. Bachem and Grötschel (1982, Theorem 2.12). Given the linear description of a polyhedron $Z\subseteq\mathbb{R}^n$ we are interested in finding a linear description of the image of $Z$ under a given affine transformation. To achieve somewhat greater generality we restrict the "feasible" points in the image of $Z$ to be in some set $Q\subseteq\mathbb{R}^m$. In most cases we will have $Q=\mathbb{R}^m$.

Let

$$x=f+Lz \tag{2}$$

be the affine transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$, i.e. $f \in \mathbb{R}^m$ and $L$ is an $m \times n$ matrix having full row rank. It will be convenient to partition $L$ into $L_1$ and $L_2$ such that $L_1$ is an $m \times m$ matrix of rank $m$ that corresponds to the first $m$ columns of $L$. We will assume that the affine transformation is given along with a fixed partitioning. We note that for $f = 0$ we have a linear transformation and since a "projection" is a special case of a linear transformation the following development generalizes statement (2.1) of Balas and Pulleyblank (1987). Let

$$Z = \{z \in \mathbb{R}^n \mid Az = b,\ Dz \leqslant d\} \tag{3}$$

and

$$X = \{x \in Q \mid \exists z \in Z \text{ such that } x = f + Lz\} \tag{4}$$

where $A$ is a $p \times n$ matrix, $D$ is a $q \times n$ matrix, $f \in \mathbb{R}^m$, $Q \subseteq \mathbb{R}^m$ is an arbitrary set, $L = (L_1, L_2)$ is an $m \times n$ matrix having full row rank and $A$ and $D$ are partitioned as $A = (A_1, A_2)$ and $D = (D_1, D_2)$ according to the partitioning of $L$. Define

$$C = \{(u, v) \in \mathbb{R}^{p+q} \mid u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2) = 0,\ v \geqslant 0\}, \tag{5}$$

$$X_C = \{x \in Q \mid (uA_1 + vD_1) L_1^{-1} x \leqslant ub + vd + (uA_1 + vD_1) L_1^{-1} f \ \forall (u, v) \in C\}. \tag{6}$$

**Theorem 2.** $X = X_C$.

**Proof.** Let $x \in X$ and thus there exists a $z \in Z$ such that $x = f + Lz$. For any $v \geqslant 0$ we have in the partitioning induced by $(L_1, L_2)$,

$$(uA_1 + vD_1) z^1 + (uA_2 + vD_2) z^2 \leqslant ub + vd.$$

Consequently, we have equivalently

$$(uA_1 + vD_1) L_1^{-1}(L_1 z^1 + L_2 z^2) + (u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2)) z^2$$
$$\leqslant ub + vd.$$

Adding $(uA_1 + vD_1) L_1^{-1} f$ to both sides of the last inequality, we obtain

$$(uA_1 + vD_1) L_1^{-1}(f + Lz) + (u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2)) z^2$$
$$\leqslant ub + vd + (uA_1 + vD_1) L_1^{-1} f.$$

Since $x = f + Lz$ and since for $(u, v) \in C$ the second term of the expression is zero, we have

$$(uA_1 + vD_1) L_1^{-1} x \leqslant ub + vd + (uA_1 + vD_1) L_1^{-1} f.$$

Hence, $X \subseteq X_C$.

To show $X_C \subseteq X$, let $x \notin X$. If $x \notin Q$, we then immediately have $x \notin X_C$. Otherwise, there exists no $z \in Z$ such that $x = f + Lz$, i.e. the linear system

$$A_1 z^1 + A_2 z^2 = b, \tag{7}$$

$$D_1 z^1 + D_2 z^2 \leqslant d, \tag{8}$$

$$L_1 z^1 + L_2 z^2 = x - f, \tag{9}$$

is inconsistent. By Farkas' lemma there exist $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$ and $\xi \in \mathbb{R}^m$ satisfying

$$uA_1 + vD_1 + \xi L_1 = 0, \tag{10}$$

$$uA_2 + vD_2 + \xi L_2 = 0, \tag{11}$$

$$ub + vd + \xi(x - f) < 0, \tag{12}$$

$$v \geqslant 0.$$

Since $L_1$ is nonsingular, from (10) we have $\xi = -(uA_1 + vD_1)L_1^{-1}$. Substituting $\xi$ in (11) we obtain

$$u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2) = 0.$$

It follows that $(u, v) \in C$. Substituting $\xi$ in (12), we get

$$ub + vd < (uA_1 + vD_1)L_1^{-1}(x - f),$$

i.e.

$$ub + vd + (uA_1 + vD_1)L_1^{-1} f < (uA_1 + vD_1)L_1^{-1} x.$$

Hence $x \notin X_C$ and thus $x \in X_C$ implies $x \in X$. The theorem follows. □

Since many combinatorial problems include explicit nonnegativity constraints in their formulations, we state affine transformations of the associated polyhedra for ease of reference in the following corollary:

**Corollary 1.** *Let*

$$Z = \{z \in \mathbb{R}^n \mid Az = b,\ Dz \leqslant d,\ z \geqslant 0\}$$

*where $A$ is a $p \times n$ matrix and $D$ is a $q \times n$ matrix and let $X$ be defined as in* (4). *Then $X = X_C$, where*

$$X_C = \{x \in Q \mid (uA_1 + vD_1 - w)L_1^{-1} x \leqslant ub + vd + (uA_1 + vD_1 - w)L_1^{-1} f\ \forall (u, v, w) \in C\} \tag{13}$$

*and*

$$C = \{(u, v, w) \in \mathbb{R}^{p+q+m} \mid u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2) + wL_1^{-1} L_2 \geqslant 0,\ v \geqslant 0,\ w \geqslant 0\}. \quad \square$$

Since every $(u, v) \in C$ can be written as a linear combination of the elements of a basis of the lineality space $L$ of $C$ plus a nonnegative combination of the extreme rays of $C$, it follows that in the linear description (6) and (13) of the polyhedron $X$ we can restrict ourselves to any finite generator system of $C$. That is, we can replace the requirement "for all $(u, v) \in C$" in (6) and (13) by the requirement "for all $(u, v)$ in a generator system of $C$". This way we get a finite system of inequalities for $X$. In particular, since $l \in L$ implies that $-l \in L$, we get a system of equations for $X$ from any basis of $L$. While the linear system describing $X$ is finite, it may very well be exponential in the parameters $m$ or $n$. Thus it is interesting to note that for any $x \in \mathbb{R}^m$ the *constraint identification* problem for $X$, see e.g. Hoffman and Padberg (1985), can be stated as the following linear program: Find

$$z(x) = \max\{u(A_1 L_1^{-1}(x-f) - b) + v(D_1 L_1^{-1}(x-f) - d) \,|\, (u, v) \in C\} \tag{14}$$

where $x \in \mathbb{R}^m$ is given. It follows that $x \in X$ if and only if $z(x) = 0$; otherwise, the solution to this linear program yields a constraint for $X$ that is violated by $x$. Also, by the duality theory of linear programming the question of *testing* $x \notin X$ amounts e.g. in the case of (6) to showing that the linear system (7), (8), (9) is inconsistent, i.e. that this system does not have a feasible solution for the given $x \in \mathbb{R}^m$. Consequently, if the original problem over $Z$ has an input-length that is polynomial in $n$, then the constraint-identification problem over $X$ is automatically polynomial in $n$ as well since problem (14) is solvable in time polynomial in $n$; see Khachiyan (1979) and Karmarkar (1984). Of course, we *assume* in this statement that either $Q = \mathbb{R}^m$ or that $x \in Q$ can be checked separately in polynomial time as well.

The following proposition gives a sufficient condition for linear programs over $Z$ and $X$, respectively, to be "comparable".

**Proposition 2.** *Let $Z$, $X$ be defined as in* (3) *and* (4). *If $c = dL$, then* $\min\{cz \,|\, z \in Z\} = \min\{dx \,|\, x \in X\} - df$.

**Proof.** If $c = dL$, it follows that

$$\begin{aligned}\min\{cz \,|\, z \in Z\} &= \min\{dLz \,|\, z \in Z\} \\ &= \min\{d(f + Lz) \,|\, z \in Z\} - df \\ &= \min\{dx \,|\, \exists z \in Z \text{ such that } x = f + Lz\} - df \\ &= \min\{dx \,|\, x \in X\} - df. \qquad \square\end{aligned}$$

For ease of reference, we state next the projection result of Balas and Pulleyblank (1987) separately. It follows from our development by choosing $f = 0$, $L_1 = I$ and $L_2 = 0$. To be specific, let

$$Z = \{(z^1, z^2) \in \mathbb{R}^n \,|\, A_1 z^1 + A_2 z^2 = b,\ D_1 z^1 + D_2 z^2 \leqslant d,\ -z^1 \leqslant 0,\ -z^2 \leqslant 0\}$$

and

$$X = \{z^1 \in Q \,|\, \exists z^2 \in \mathbb{R}^{n-m} \text{ such that } (z^1, z^2) \in Z\},$$

where $A = (A_1, A_2)$ is a $p \times n$ matrix and $D = (D_1, D_2)$ is a $q \times n$ matrix. As we are "truncating" the vector $z$ we can without loss of generality insert the condition $z^1 \in Q$ in the constraint set of $Z$. Since $L_1 = I$, $L_2 = 0$, the associated cone $C'$ is then defined as follows:

$$C' = \{(u, v, w) \in \mathbb{R}^{p+q+m} \,|\, uA_2 + vD_2 \geqslant 0,\ v \geqslant 0,\ w \geqslant 0\}.$$

The inequalities $w \geqslant 0$ give rise to extreme rays of the form $(\mathbf{0}, \mathbf{0}, w^i)$ where $w^i$ is the $i$th unit vector of $\mathbb{R}^m$. By Corollary 1 these extreme rays yield the nonnegativity constraints $z^1 \geqslant 0$. Furthermore, by the intersection property of cones we can work with the smaller cone

$$C = \{(u, v) \in \mathbb{R}^{p+q} \mid uA_2 + vD_2 \geqslant 0,\ v \geqslant 0\}. \tag{15}$$

It follows from Corollary 1 that

$$X = \{z^1 \in Q \mid z^1 \geqslant 0,\ (uA_1 + vD_1)z^1 \leqslant ub + vd \text{ for all } (u, v) \in C\} \tag{16}$$

as stated in Balas and Pulleyblank (1987).

## 3. Four different formulations of the travelling salesman problem

The (standard) travelling salesman problem (TSP) is to find a shortest way from a home city to visit a given set of cities exactly once and then return to the home city. Dantzig, Fulkerson and Johnson published a seminal paper on the TSP in 1954 and formulated the problem as a zero-one linear program involving $O(n^2)$ variables and $O(2^n)$ linear constraints. Since their formulation involves an exponential number of constraints, various researchers have proposed formulations of the TSP that involve only a polynomial number of constraints usually at the expense of increasing the variables; see e.g. Miller, Tucker and Zemlin (1960), Fox, Gavish and Graves (1980) and Claus (1984). The issue we are addressing here is whether or not these *compact* formulations provide better characterizations of the travelling salesman polytope than the standard formulation due to Dantzig, Fulkerson and Johnson. In other words, we investigate the question whether or not the solvability of the TSP is improved by these other formulations when used in connection with linear-programming-based solution methods such as e.g. branch-and-bound, Lagrangean relaxation or branch-and-cut.

We use the following standard notation:

$$\delta^+(i) = \{(i, j) \in E \mid \forall j \in V\} \quad \text{for } i \in V,$$

$$\delta^-(i) = \{(j, i) \in E \mid \forall j \in V\} \quad \text{for } i \in V,$$

$$\delta(i) = \delta^+(i) \cup \delta^-(i) \quad \text{for } i \in V,$$

$$\bar{S} = V - S \quad \text{for } S \subseteq V,$$

$$E(S) = \{(i, j) \in E \mid \forall i, j \in S\} \quad \text{for } S \subseteq V,$$

$$(S_1 : S_2) = \{(i, j) \in E \mid \forall i \in S_1, j \in S_2\} \quad \text{for } S_1 \subseteq V \text{ and } S_2 \subseteq V - S_1,$$

$$x(E') = \sum_{e \in E'} x_e \quad \text{for } E' \subseteq E.$$

Furthermore, we write sometimes $\mathbb{R}^V$ and $\mathbb{R}^E$ rather than $\mathbb{R}^{|V|}$ and $\mathbb{R}^{|E|}$ and for any vectors $u \in \mathbb{R}^V$ and $w \in \mathbb{R}^E$ we denote $u^S$ and $w^F$ the characteristic vectors of $S \subseteq V$ and $F \subseteq E$ respectively, i.e. $u^S$ and $w^F$ are defined by

$$u_i^S = \begin{cases} 1 & \forall i \in S, \\ 0 & \forall i \in \bar{S}, \end{cases} \qquad w_e^F = \begin{cases} 1 & \forall e \in F, \\ 0 & \forall e \in E - F. \end{cases}$$

We use $e_m$ to denote the vector of size $m$ with all components equal to 1 and $u^i$ (or sometimes $w^i$, $\xi^i$) to denote the $i$th unit vector. The support graph of $x \in \mathbb{R}^E$ is defined as $G(x) = (N, S(x))$ where $S(x) = \{e \in E \mid x_e \neq 0\}$ and $N \subseteq V$ is the set of nodes spanned by $S(x)$.

### 3.1. *The Dantzig-Fulkerson-Johnson (DFJ) formulation*

Dantzig, Fulkerson and Johnson (1954) formulate the standard problem as a zero-one linear program on a graph $G = (V, E)$ as follows:

$$\min \quad \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} = 1, \ j = 1, \ldots, n, \tag{17}$$

$$\sum_{i=1}^{n} x_{ji} = 1, \ j = 1, \ldots, n, \tag{18}$$

$$\sum_{i,j \in S} x_{ij} \leq |S| - 1 \ \ \forall S \subseteq V \text{ and } 2 \leq |S| \leq n-1, \tag{19}$$

$$x_{ij} \geq 0 \ \ \forall i, j, \tag{20}$$

$$x_{ij} \text{ integer } \ \forall i, j, \tag{21}$$

where $V = \{1, 2, \ldots, n\}$. We assume throughout this paper that the variables $x_{ii}$ do not exist and thus we have a formulation of the TSP involving $n(n-1)$ zero-one variables and $O(2^n)$ constraints. The constraints (19) are referred to as subtour elimination constraints (SECs) and rule out cycles visiting a subset of nodes in $V$. They can be written equivalently in the cut form as

$$x(S : V - S) \geq 1.$$

We denote the convex hull of solutions to (17)-(21),

$$P^n = \text{conv}\{x \in \mathbb{R}^{n(n-1)} \mid x \text{ satisfies (17)-(21)}\},$$

the *travelling salesman polytope*, and the linear programming relaxation

$$P_S^n = \{x \in \mathbb{R}^{n(n-1)} \mid x \text{ satisfies (17)-(20)}\}, \tag{22}$$

the *subtour polytope.* The polytope $P_S^n$ is a *formulation* of the TSP, but an *ideal formulation* for the travelling salesman problem, i.e. a complete list of all linear inequalities that are needed to describe $P^n$, is unknown; see Grötschel and Padberg

(1985) for a survey of the partial results concerning the facial structure of $P^n$ obtained to date. It is well-known that $\dim P^n = \dim P_S^n = n(n-3)+1$ and that the constraints (19) and (20) define facets of $P^n$. We use the DFJ-formulation as the benchmark for our comparison.

### 3.2. *The Miller-Tucker-Zemlin (MTZ) formulation*

Miller, Tucker and Zemlin (1960) propose a formulation for a more general TSP on $V=\{1,\ldots,n\}$ nodes, which is known as the "clover-leaf" TSP with $t$ leaves and which goes as follows:

Denote city 1 the home city. The salesman is required to visit the other $n-1$ cities exactly once. During his travel he must return to the home city exactly $t$ times, including his final return, and he must visit no more than $p$ cities different from home in one tour. (A tour is a succession of visits to cities without stopping at city 1.) We require that $\lceil (n-1)/p \rceil \leq t \leq n-1$, where $\lceil a \rceil$ for any $a \in \mathbb{R}$ denotes the smallest integer greater than or equal to $a$, since otherwise there is no feasible tour and $p \geq 2$ since otherwise the problem is not interesting. The problem is written as the following mixed 0-1 linear program:

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{i=2}^{n} x_{i1} = t, \tag{23}$$

$$\sum_{i=2}^{n} x_{1i} = t, \tag{24}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 2, \ldots, n, \tag{25}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 2, \ldots, n, \tag{26}$$

$$u_i - u_j + p x_{ij} \leq p-1, \quad 2 \leq i \neq j \leq n, \tag{27}$$

$$u_i \geq 0, \quad 2 \leq i \leq n, \tag{28}$$

$$x_{ij} \geq 0 \quad \forall i, j, \tag{29}$$

$$x_{ij} \text{ integer } \forall i, j. \tag{30}$$

The constraint (24) is redundant, but we include it for convenience of analysis in Section 4.2. The restriction that the home city is visited exactly $t$ times is expressed in (23) and (24). The constraints (27) eliminate tours that visit more than $p$ cities. The formulation involves $O(n^2)$ constraints in $n^2-1$ variables. Furthermore, for $t=1$ and $p \geq n-1$ the MTZ-formulation models the standard model correctly.

### 3.2.1. The MTZ-formulation and a modified standard model

In order to compare the MTZ-formulation with the standard formulation we use the following modification of the DFJ-formulation to accommodate the more general form of the clover-leaf TSP:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{s.t.} \quad (23)\text{–}(26),\ (29),\ (30) \text{ and}$$

$$x(E(S)) \leqslant |S| - \lceil |S|/p \rceil \quad \forall S \subseteq V - \{1\},\ |S| \geqslant 2. \tag{31}$$

For this modification we define the polytope $P^n_{S,t}$ corresponding to the subtour polytope $P^n_S$ as follows:

$$P^n_{S,t} = \{x \in \mathbb{R}^{n(n-1)} \mid x \text{ satisfies } (23)\text{–}(26),\ (29) \text{ and } (31)\}.$$

This modified DFJ-formulation models the same problem as the MTZ-formulation and we can compare the two formulations by *projecting out* the $u$-variables of the latter formulation. To this end we associate with the MTZ-formulation the following sets:

$$Q = \{x \in \mathbb{R}^{n(n-1)} \mid x \text{ satisfies } (23)\text{–}(26)\},$$

$$\mathrm{UP_M} = \{(u, x) \in \mathbb{R}^{n^2-1} \mid (u, x) \text{ satisfies } (27)\text{–}(29) \text{ and } x \in Q\},$$

$$P_\mathrm{M} = \{x \in Q \mid \exists u \in \mathbb{R}^{n-1} \text{ such that } (u, x) \in \mathrm{UP_M}\}.$$

$\mathrm{UP_M}$ is the MTZ-polytope, i.e. the linear relaxation of the MTZ-formulation, and $P_\mathrm{M}$ its projection into $\mathbb{R}^{n(n-1)}$. The comparison of the two formulations is then reduced to a comparison of the two polytopes $P_\mathrm{M}$ and $P^n_{S,t}$.

To carry out this comparison *analytically* we use the results of Section 2 and determine a linear description of the polytope $P_\mathrm{M}$. Noting that the variables $x_e$ with $e \in \delta(1)$ do not appear in the constraints (27), we write these constraints as

$$A^\mathrm{T}u + Bx \leqslant b$$

where $A$ is the node-arc incidence matrix of a complete directed graph on nodes $\{2, \ldots, n\}$, $B = pI_{(n-1)(n-2)}$, $b = (p-1)e_{(n-1)(n-2)}$. In order to find the linear description of $P_\mathrm{M}$ we need to determine a generator system of the cone

$$C = \{v \in \mathbb{R}^{(n-1)(n-2)} \mid Av \geqslant 0,\ v \geqslant 0\}.$$

### 3.2.2. The arc cone of a complete digraph

We define the arc cone $C$ of a complete digraph with node-set $V = \{1, \ldots, n\}$ to be

$$C = \{v \in \mathbb{R}^{n(n-1)} \mid Av \geqslant 0,\ v \geqslant 0\}$$

where $A$ is the node-arc incidence matrix of the complete digraph $K_n$. Since $0 = e_nAv \geqslant 0$, it follows that $Av = 0$ for all $v \in C$. Thus $C$ can be written as

$$C = \{v \in \mathbb{R}^{n(n-1)} \mid Av = 0,\ v \geqslant 0\}. \tag{32}$$

Since $v \geqslant 0$ for all $v \in C$, it follows that $C$ is a pointed cone with apex at $\mathbf{0}$.

**Proposition 3.** *A vector $v \in C$ defines an extreme ray if and only if*

$$v_e = \begin{cases} 1 & \forall e \in \pi(S), \\ 0 & \textit{otherwise}, \end{cases} \tag{33}$$

*where $\pi(S) \subseteq E$ is the arc-set of a directed cycle on $S \subseteq V$.*

**Proof.** Let $v \in C$ be defined as in (33) and assume that $v = \lambda v^1 + (1-\lambda)v^2$ for some $v^1, v^2 \in C$ and $0 < \lambda < 1$. Then $\lambda v_e^1 + (1-\lambda)v_e^2 = 0$ for all $e \notin \pi(S)$ implies that $v_e^1 = v_e^2 = 0$ for all $e \notin \pi(S)$. If there exists $f \in \pi(S)$ such that $v_f^1 = 0$, we obtain $v_e^1 = 0$ for all $e \in \pi(S)$ by transitivity from the constraints $Av^1 = 0$ since $\pi(S)$ is a directed cycle. It follows that $v^2$ is a positive multiple of $v$. So suppose $v_e^1 > 0$ and $v_e^2 > 0$ for all $e \in \pi(S)$. Since $\pi(S)$ is a directed cycle we have by transitivity from $Av^i = 0$ that $v^1$ is a positive multiple of $v^2$. But then both $v^1$ and $v^2$ are positive multiples of $v$. Therefore $v$ defines an extreme ray.

For any extreme ray $v \in C$ let $S(v) = \{e \in E \mid v_e > 0\}$ be the support of $v$ and let $N \subseteq V$ be the set of nodes spanned by $S(v)$. If the partial subgraph $G^* = (N, S(v))$ is not connected and has $k \geqslant 2$ components, it follows that $v = \sum_{i=1}^k v^i$ and equivalently, $(v) = \sum_{i=1}^k (1/k)v^i$ where each point $v^i$ has nonzero entries in positions corresponding to a component of $G^*$ and zeroes elsewhere, which is a contradiction. Hence, $G^*$ is a connected graph. Let $i \in N$ and suppose that there do not exist $e_1 \in \delta^+(i)$ and $e_2 \in \delta^-(i)$ such that $v_{e_1} > 0$ and $v_{e_2} > 0$. It follows that $\sum_{e \in \delta^+(i)} v_e = 0$ or $\sum_{e \in \delta^-(i)} v_e = 0$. By the feasibility of $Av = 0$ we have $\sum_{e \in \delta^+(i)} v_e = \sum_{e \in \delta^-(i)} v_e = 0$ and thus $i \notin N$, which is a contradiction. Consequently, every node $i \in N$ has at least one arc $e \in \delta^+(i)$ and one arc $e \in \delta^-(i)$ in $G^*$ such that $v_e > 0$. By Lemma 1, $v$ satisfies $n(n-1)-1$ linearly independent rows of $\binom{A}{I}$ as equations since $C$ is a pointed cone. Let $q = |N|$ and $A^*$ be the node-arc incidence matrix of the complete digraph induced by the node-set $N$. Since the rank of $A^*$ equals $q-1$ it follows that $v$ satisfies *at most* $q-1$ linearly independent rows of $A$ as equations and thus, $v$ satisfies *at least* $n(n-1)-q$ equations of the form $v_e = 0$. It follows that $v$ has *exactly* $q$ positive entries of $v_e$ because each node $i$ in $G^*$ has an in-degree and an out-degree of at least one. Consequently, $G^*$ is a directed cycle and the proposition follows. □

*3.2.3. The projection of the MTZ-polytope*

Let $x^1$ be the vector with components $x_e$ for $e \in \delta(1)$ and $x^2$ be the vector with components $x_e$ for $e \in E - \delta(1)$. It follows from (16) that the projection of the MTZ-polytope $UP_M$ is given by

$$P_M = \{(x^1, x^2) \in Q \mid pvx^2 \leqslant (p-1)ve_{(n-1)(n-2)} \ \forall v \in C\}.$$

Since one needs to consider only extreme rays in the definition of $P_M$, it follows from Proposition 3 that it suffices to consider the inequalities

$$x(C) \leqslant |C| - |C|/p \quad \text{for all directed cycles } C \subseteq E - \delta(1). \tag{34}$$

Consequently, the following lemma gives the linear description of $P_M$:

**Lemma 2.** *The projection* $P_M$ *of the MTZ-polytope* $UP_M$ *is given by*

$$P_M = \{x \in \mathbb{R}^{n(n-1)} \mid x \text{ satisfies } (23)\text{-}(26),\ (29) \text{ and } (34)\}. \quad \square$$

Lemma 2 has been obtained independently by Egon Balas, see e.g. Balas, (1987) where it is stated without proof. The following theorem settles the comparison between the DFJ-formulation and the MTZ-formulation of the clover-leaf TSP:

**Theorem 3.** *The polytopes* $P_M$ *and* $P^n_{S,t}$ *satisfy*:

(i) $P_M = P^n_{S,t}$ *for* $t = n-1,\ n-2$ *and all* $p \geq 2,\ n \geq 3$.

(ii) $P^n_{S,t}$ *is a proper subset of* $P_M$ *for all* $\lceil (n-1)/p \rceil \leq t \leq n-3,\ p \geq 2$ *and* $n \geq 4$. $\square$

The proof of Theorem 3 is somewhat tedious and long and can be found in full in Sung, (1988).

### 3.3. The Fox-Gavish-Graves (FGG) formulation

Fox, Gavish and Graves (1980) formulate a "time-dependent" travelling salesman problem and thus a generalization of the standard TSP. Here the cost of travelling between city $i$ and city $j$ depends also on the position $t$ of the arc $(i, j)$ in the tour relative to a given "first" or "home" city that is indexed by 1. The time-dependent TSP was originally proposed as a formulation for the 1-machine $n$-job scheduling problem and studied by K. Fox [1973] in his dissertation; see Picard and Queyranne (1978) for further references. Of the various formulations proposed by Fox, Gavish and Graves we investigate here the most *compact* one involving $n$ linear constraints. The decision variables of this model are triple-indexed variables $z_{ijt}$ and $z_{ijt} = 1$ if the arc from $i$ to $j$ is assigned to the $t$th position in the tour, $z_{ijt} = 0$ otherwise.

The FGG-formulation of the time-dependent TSP goes as follows:

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{n} c_{ijt} z_{ijt}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{n} z_{ijt} = n, \tag{35}$$

$$\sum_{j=1}^{n} \sum_{t=2}^{n} t z_{ijt} - \sum_{j=1}^{n} \sum_{t=1}^{n} t z_{jit} = 1, \quad i = 2, \ldots, n, \tag{36}$$

$$z_{ijt} \geq 0 \quad \forall i, j, t, \tag{37}$$

$$z_{ijt} \text{ integer} \quad \forall i, j, t. \tag{38}$$

Constraint (35) is the "aggregation" of constraints of a 3-dimensional assignment problem which ensure that every city is visited exactly once and that every position of the tour has one arc assigned to it. The constraints (36) ensure that for each city other than the home city the position number of an arc leaving the city is exactly one more than the position number of an arc entering the city. For a city on a subtour of length $k \geq 2$ the position number of an arc leaving the city is $k$ units bigger than the position number of an arc entering the city and thus the constraint (36) cannot be satisfied unless the city is the home city. It follows that every feasible zero-one solution corresponds to a tour and in particular, that the constraints (36) rule out all subtours; see Garfinkel (1985) for a discussion of this formulation.

To simplify the exposition we note that the constraints (36) can be rewritten as follows:

$$-\sum_{j=1}^{n} z_{ji1} + n \sum_{j=1}^{n} (z_{ijn} - z_{jin}) + \sum_{t=2}^{n-1} t \sum_{j=1}^{n} (z_{ijt} - z_{jit}) = 1 \quad \text{for } i = 2, \ldots, n.$$

Observing that $z_{ij1} = z_{jin} = z_{1it} = z_{i1t} = 0$ for all $i \geq 2$, $j \geq 1$ and $2 \leq t \leq n-1$ in every feasible solution to (35)-(38) these constraints are simplified to

$$-z_{1i1} + nz_{i1n} + \sum_{t=2}^{n-1} t \sum_{j=2}^{n} (z_{ijt} - z_{jit}) = 1 \quad \text{for } i = 2, \ldots, n. \tag{39}$$

Furthermore, we do not need variables of the type $z_{iit}$ for all $i$ and $t$ and thus these variables as well as those that assume the value zero in every feasible solution to (35)-(38) are dropped from the model. If $c_{ijt} = c_{ij}$ for all $i$, $j$ and $t$, the FGG-formulation models the standard TSP correctly and involves $n$ linear constraints in $m = 2(n-1) + (n-1)(n-2)^2$ zero-one variables.

### 3.3.1. *The FGG-formulation and the standard model*

While the FGG-formulation permits more general cost functions than the standard DFJ-formulation, we can still investigate the "goodness" of the formulation in comparison to the DFJ-formulation by noting that the linear transformation $x = Lz$ given by

$$x_{ij} = \sum_{t} z_{ijt} \quad \forall i, j \in V, \tag{40}$$

maps the incidence vectors of tours of the FGG-formulation onto the incidence vectors of tours of the DFJ-formulation. To carry out this comparison we define two polytopes $\mathrm{TP_F}$ and $P_\mathrm{F}$ as follows:

$$\mathrm{TP_F} = \{z \in \mathbb{R}^m \mid z \text{ satisfies (35), (36) and (37)}\},$$

$$P_\mathrm{F} = \{x \in \mathbb{R}^{n(n-1)} \mid \exists z \in \mathrm{TP_F} \text{ such that } x = Lz\}.$$

We call $\mathrm{TP_F}$ the FGG-polytope and $P_\mathrm{F}$ its linear transformation. Since every nonnegative solution to (35) is clearly bounded, both $\mathrm{TP_F}$ and $P_\mathrm{F}$ are bounded polyhedra, i.e. they are indeed polytopes. Moreover, we have the following proposition which we state for completeness:

**Proposition 4.** *Every feasible solution to* (35)-(38) *satisfies* $z_{ijt} \leqslant 1$ *for all* $i, j, t$.

**Proof.** Let $z_{ijt}$ be a feasible solution to (35)-(38). Summing up (39) over all $i \neq 1$, we obtain

$$-\sum_{i=2}^{n} z_{1i1} + n \sum_{i=2}^{n} z_{i1n} = n-1.$$

If $\sum_{i=2}^{n} z_{i1n} \geqslant 2$, we have $\sum_{i=2}^{n} z_{1i1} \geqslant n+1$ which is a contradiction to (39). It follows that $\sum_{i=2}^{n} z_{1i1} = \sum_{i=2}^{n} z_{i1n} = 1$ by the integrality of $z$ and hence by (37) we obtain $z_{1i1} \leqslant 1$ and $z_{i1n} \leqslant 1$ for all $i \neq 1$. Furthermore, there exist $i^* \neq j^*$ such that $z_{1i1} = 1$ for $i = i^*$, $z_{1i1} = 0$ otherwise and $z_{j1n} = 1$ for $j = j^*$, $z_{j1n} = 0$ otherwise. Since none of the variables in (39) has a coefficient equal to 1, every nonnegative integer solution to (39) has at least two $z_{ijt}$ for all $i, j, t$ having positive values. Consequently, there are at least $2(n-1)$ positive $z_{ijt}$ appearing in the $n-1$ equations (39). Since $z_{1i^*1}$ and $z_{j^*1n}$ appear exactly once in (39) and every $z_{ijt}$ for $i \neq 1$ and $j \neq 1$ appears exactly twice in (39), we need to have *at least* $n-2$ $z_{ijt}$ for $i \neq 1$ and $j \neq 1$ having positive values. On the other hand, by (39) $z$ allows *at most* $n-2$ $z_{ijt}$ for $i \neq 1$ and $j \neq 1$ having positive values. It follows that $z_{ijt} \leqslant 1$ for all $i \neq 1$ and $j \neq 1$ and the proposition follows. □

Consequently, every integer extreme point of $\mathrm{TP_F}$ corresponds to a tour of the travelling salesman. Clearly, to every tour of the travelling salesman there corresponds a zero-one point in $P_{\mathrm{F}}$. To show that the converse holds we determine the linear description of the polytope $P_{\mathrm{F}}$ by applying the results of Section 2 to the linear transformation (40). In matrix form this transformation is written as $L = (L_1, L_2)$, where $L_1 = I_{n(n-1)}$ has columns corresponding to $z$-variables in the order

$$\{(1,2,1), \ldots, (1,n,1), (2,1,n), \ldots, (n,1,n)\} \cup \{(i,j,2) \mid 2 \leqslant i \neq j \leqslant n\}.$$

The matrix $L_2$ is a matrix having $n-2$ column blocks of $(0, I_{(n-1)(n-2)})^{\mathrm{T}}$ where the matrix 0 is a $2(n-1) \times (n-1)(n-2)$ matrix having all the components equal to 0. The $t$th block for $1 \leqslant t \leqslant n-2$ of $L_2$ has columns corresponding to $z$-variables indexed by $\{(i,j,t+2) \mid 2 \leqslant i \neq j \leqslant n\}$. Furthermore, the rows of zero matrices in $L_2$ correspond to all arcs with $e \in \delta(1)$.

We denote $Az = b$ the constraint system formed by the constraints (35) and (36). The vector $b$ is an $n \times 1$ vector having first component equal to $n$ and 1 elsewhere. The matrix $A$ can be partitioned according to $(L_1, L_2)$ as $(A_1, A_2)$ where $A_1$ and $A_2$ are written as follows:

$$A_1 = \begin{bmatrix} e_{n-1} & e_{n-1} & e_{(n-1)(n-2)} \\ -I_{n-1} & nI_{n-1} & 2M \end{bmatrix},$$

$$A_2 = \begin{bmatrix} e_{(n-1)(n-2)} & e_{(n-1)(n-2)} & \cdots & e_{(n-1)(n-2)} \\ 3M & 4M & \cdots & (n-1)M \end{bmatrix},$$

and $M$ is the node-arc incidence matrix of a complete digraph on $V=\{2,\ldots,n\}$. Following the definitions of $A_1$ and $A_2$, we obtain

$$A_2-A_1L_1^{-1}L_2=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ M & 2M & \cdots & (n-3)M \end{bmatrix},$$

since $L_1^{-1}=I_{n(n-1)}$. The cone corresponding to (5) is given by

$$C=\{(v,u,\xi,w)\in\mathbb{R}^{n^2} \mid kuM+w\geqslant 0,\ k=1,\ldots,n-3,\ \xi\geqslant 0,\ w\geqslant 0\} \tag{41}$$

where $v\in\mathbb{R}$, $u\in\mathbb{R}^{n-1}$, $\xi\in\mathbb{R}^{2(n-1)}$ and $w\in\mathbb{R}^{(n-1)(n-2)}$.

*3.3.2. Simplification of the cone C*

To make the analysis of the cone (41) easier we proceed as follows: The scalar $v$ gives rise to a lineality space that is generated by $(\pm 1,\mathbf{0},\mathbf{0},\mathbf{0})$ and consequently, in order to find extreme rays of $C$ we can get rid of it by intersecting $C$ with $v=0$. Furthermore, the inequalities $\xi\geqslant 0$ give rise to extreme rays of $C$ of the form $(0,\mathbf{0},\xi^i,\mathbf{0})$ where $\xi^i$ is the $i$th unit vector of $\mathbb{R}^{2(n-1)}$. So by the two preceding operations and the intersection property of cones, we can work with the smaller cone

$$C'=\{(u,w)\in\mathbb{R}^{(n-1)^2} \mid kuM+w\geqslant 0 \text{ for } k=1,\ldots,(n-3),\ w\geqslant 0\}.$$

Now consider

$$C''=\{(u,w)\in\mathbb{R}^{(n-1)^2} \mid (n-3)uM+w\geqslant 0,\ w\geqslant 0\}.$$

**Proposition 5.** $C'=C''$.

**Proof.** Since $(u,w)\in C'$ implies $(u,w)\in C''$, we have $C'\subseteq C''$. Suppose now that there exists $(u,w)\in C''$ such that $(u,w)\notin C'$. Then there exists a column $m^e$ of $M$ and an index $k$, $1\leqslant k<n-3$, such that

$$kum^e+w_e<0 \tag{42}$$

and thus from $w_e\geqslant 0$ we have

$$um^e<0 \tag{43}$$

since $k\geqslant 1$. On the other hand, since $(u,w)\in C''$ we have

$$(n-3)um^e+w_e\geqslant 0. \tag{44}$$

Adding the negative of (42) to (44) we find $(n-3-k)um^e>0$ and thus since $n-3-k>0$ we have $um^e>0$, which is a contradiction to (43). Consequently, we have $C'=C''$ and the proposition follows. $\square$

To get a full generator system of the cone $C'$ and the linear description of $P_{\mathrm{F}}$ corresponding to (6) it thus suffices to find a generator system for the cone $C''$.

*3.3.3. The node-arc cone of a complete digraph*

Let $C$ be a node-arc cone of a complete digraph $G$ on $V=\{1,\ldots,n\}$ and defined as follows:

$$C=\{(u,w)\in\mathbb{R}^{n^2} \mid puM+w\geqslant 0,\ w\geqslant 0\}$$

where $p$ is a positive integer and $M$ is the node-arc incidence matrix of $G$. Let

$$B=\begin{bmatrix} pM^{\mathrm{T}} & I \\ 0 & I \end{bmatrix}.$$

Since the rank of $M^{\mathrm{T}}$ is $n-1$, it follows that $B$ has a rank of $n^2-1$. Hence, the cone $C$ has a lineality space $L$ of dimension 1 and the basis of the lineality space is given by $u=\pm e_n$, $w=\mathbf{0}$.

To find the extreme rays of the cone $C$ we have to find solutions to all homogeneous equation systems corresponding to $n^2$ variables and $n^2-2$ linearly independent rows of $B$. The solution space to such a system is a family having one parameter and we derive a member *having minimal support* of this family.

**Proposition 6.** *Let $B^*$ be the $(n^2-2)\times n^2$ submatrix of $B$ that corresponds to the equation system of an extreme ray of $C$. Then every nonzero solution to*

$$B^*\begin{pmatrix} u \\ w \end{pmatrix}=0 \tag{45}$$

*of minimal support is a positive multiple of the vector $(u,w)$ given by* (i), (ii) *or* (iii) *where*

(i) $u_i=0\quad \forall i\in V,\qquad w_{ij}=\begin{cases} p & \text{for exactly one } i \text{ and } j\in V,\\ 0 & \text{otherwise,}\end{cases}$

(ii) $u_i=\begin{cases} 1 & \text{for all } i\in S,\\ 0 & \text{otherwise,}\end{cases}\qquad w_{ij}=\begin{cases} p & \text{for all } i\in \bar{S}, j\in S,\\ 0 & \text{otherwise,}\end{cases}$

(iii) $u_i=\begin{cases} -1 & \text{for all } i\in S,\\ 0 & \text{otherwise,}\end{cases}\qquad w_{ij}=\begin{cases} p & \text{for all } i\in S, j\in \bar{S},\\ 0 & \text{otherwise,}\end{cases}$

*and $S\subseteq V$, $1\leqslant |S|\leqslant n-1$.*

**Proof.** Since $B^*$ corresponds to an extreme ray of $C$ the system (45) admits a nonzero solution and we can partition $B^*=(B_1,f)$ where $B_1$ is an $(n^2-2)\times(n^2-1)$ submatrix of $B$ having a rank of $n^2-2$ and $f$ is some column of $B^*$. Suppose the column $f$ belongs to a $w$-variable $w_{ij}$, say. If $B^*$ contains the row corresponding to $w_{ij}\geqslant 0$, then $B_1$ has a rank of at most $n^2-3$ which is a contradiction. If $f$ is the zero column, then the vector defined in (i) is a solution of minimum support to (45) and we are done. In the remaining case $B^*$ contains the equation $pu_i-pu_j+w_{ij}=0$. Since no $(n-2)\times(n-2)$ submatrix of $B^*$ having rank $n-2$ contains all $n$ columns corresponding to the $u$-variables we can replace column $f$ by a column corresponding to some $u$-variable, i.e. we can assume without loss of generality that column $f$ belongs to a variable $u_k$, say, with $k\in V$. It follows that the matrix $B_1$ decomposes as follows:

$$B_1=\begin{pmatrix} A_0 & F & 0 \\ A_1 & 0 & I_1 \\ 0 & I_0 & 0 \end{pmatrix}$$

where $I_0$ and $I_1$ are identity matrices of size $|E_0|\times|E_0|$ and $|E_1|\times|E_1|$, respectively,

$$E_0=\{e\in E \mid w_e=0 \text{ in the extreme ray considered}\}$$

and $E_1=E-E_0$. Both $A_0$ and $A_1$ have $n-1$ columns and the matrices $A_0$ and $F$ have $n-2$ rows. $F$ is an $(n-2)\times|E_0|$ matrix having exactly one entry equal to 1 per row. Since $B_1$ has a rank of $n^2-2$ it follows that $A_0$ has a rank of $n-2$. Since $A_0$ is a submatrix of the arc-node incidence matrix of a directed graph it follows that the partial subgraph $H$ of $G$ induced by the $n-2$ rows and all columns of $A_0$ has at most $n-1$ nodes and that it is a forest consisting of at most one tree whose node-set $S$ satisfies $|S|>1$ and a possibly empty set of isolated nodes otherwise. $H$ has at least $n-2$ nodes since the rank of $A_0$ equals $n-2$. If $H$ has $n-2$ nodes, then $A_0$ has a zero column corresponding to some variable $u_j$ and $j\neq k$. From the rank of $A_0$ it follows that all $u_i=0$ for $i\neq j$ in every solution to (45) and $u_k=0$ in any solution of minimal support. Since (45) admits a nonzero solution it follows that $u_j\neq 0$. From $-pu_j+w_{ij}\geqslant 0$ and $pu_j+w_{ji}\geqslant 0$ for all $i\in V-j$ and the nonnegativity of all $w_{ij}$ it follows that every nonzero solution of minimal support to (45) is of the asserted form with $S=\{j\}$ since $E_0\cup E_1=E$. Suppose now that $H$ has $n-1$ nodes. Then there exists exactly one tree with node-set $S$ satisfying $|S|>1$. Since $k\notin S$ we have $|S|\leqslant n-1$. By transitivity we get $u_i=\lambda$ for all $i\in S$ in any nonzero solution to (45), $u_i=0$ for all $i\in\bar{S}$ in any solution of minimal support to (45). Moreover, from the partitioning of $B_1$ it follows that $w_{ij}=0$ for all $i,j\in S$ and that $w_{ij}=0$ for all $i,j\in\bar{S}$ in any solution of minimal support to (45). From $pu_i+w_{ij}\geqslant 0$ for all $i\in S, j\in\bar{S}$, $-pu_j+w_{ij}\geqslant 0$ for all $j\in S$, $i\in\bar{S}$ and the nonnegativity of all $w$-variables it follows that every nonzero solution of minimal support to (45) is of the form (ii) or (iii). The proposition follows. □

### 3.3.4. *The linear transformation of the FGG-polytope*

Since we are working on the linear transformation of the FGG-polytope, the cone $C$ is the one defined in (41). We use the generator system, which consists of a basis of the lineality space and a family of generators having minimal support, to derive the linear transformation of $P_{\mathrm{F}}$ of the FGG-polytope $\mathrm{TP}_{\mathrm{F}}$. By the intersection property of cones and Proposition 6 with $p=n-3$, the generator system that we get is summarized below:

(i) $v=\pm 1,\ u=\mathbf{0},\ \xi=\mathbf{0},\ w=\mathbf{0}$,

(ii) $v=0,\ u=\mathbf{0},\ \xi=\xi^i,\ w=\mathbf{0},\quad$ for $i=1,\ldots,2(n-1)$,

(iii) $v=0,\ u=\pm e_{n-1},\ \xi=\mathbf{0},\ w=\mathbf{0}$,

(iv) $v=0,\ u=\mathbf{0},\ \xi=\mathbf{0},\ w=(n-3)w^i\quad$ for $i=1,\ldots,(n-1)(n-2)$,

(v) $v=0,\ u=u^S,\ \xi=\mathbf{0},\ w=(n-3)w^{(\bar{S}-\{1\}:S)}$,

(vi) $v=0,\ u=-u^S,\ \xi=\mathbf{0},\ w=(n-3)w^{(S:\bar{S}-\{1\})}$, for all $S \subseteq V-\{1\}$, $1 \leqslant |S| \leqslant n-2$ where $v$ is a scalar, $u \in \mathbb{R}^{n-1}$, $\xi \in \mathbb{R}^{2(n-1)}$, $w \in \mathbb{R}^{(n-1)(n-2)}$, $\xi^i$ and $w^i$ are the $i$th unit vectors in $\mathbb{R}^{2(n-1)}$ and $\mathbb{R}^{(n-1)(n-2)}$ respectively, $u^S$ and $w^{E'}$ are the characteristic vectors of $S$ and $E'$ respectively.

By Corollary 1, $P_{\mathrm{F}}$ is given by

$$P_{\mathrm{F}} = \{x \in \mathbb{R}^{n(n-1)} \,|\, ((v, u)A_1 - (\xi, w))x \leqslant (v, u)b \;\; \forall (v, u, \xi, w) \in C\}.$$

It follows that (ii) and (iv) yield the nonnegativity constraints

$$x_e \geqslant 0 \quad \forall e \in E \tag{46}$$

and (i) and (iii) yield the following equations:

$$x(E) = n, \tag{47}$$

$$-x(\delta^+(1)) + nx(\delta^-(1)) = n-1. \tag{48}$$

The generators in (v) yield the following inequalities:

$$\begin{aligned} h(S) = -x(1:S) + nx(S:1) + 2x(S:\bar{S}-\{1\}) - (n-1)x(\bar{S}-\{1\}:S) \\ \leqslant |S| \quad \forall S \subseteq V-\{1\}, 1 \leqslant |S| \leqslant n-2. \end{aligned} \tag{49}$$

Similarly, the generators in (vi) define the following inequalities:

$$x(1:S) - nx(S:1) + 2x(\bar{S}-\{1\}:S) - (n-1)x(S:\bar{S}-\{1\}) \leqslant -|S| \tag{50}$$

for all $S \subseteq V-\{1\}$ and $1 \leqslant |S| \leqslant n-2$. Using (48) the constraint (50) can be rewritten as

$$\begin{aligned} &-(n-1) - x(1:\bar{S}-\{1\}) + nx(\bar{S}-\{1\}:1) + 2x(\bar{S}-\{1\}:S) \\ &\quad -(n-1)x(S:\bar{S}-\{1\}) \leqslant -|S| \end{aligned}$$

or equivalently,

$$\begin{aligned} &-x(1:\bar{S}-\{1\}) + nx(\bar{S}-\{1\}:1) + 2x(\bar{S}-\{1\}:S) \\ &\quad -(n-1)x(S:\bar{S}-\{1\}) \leqslant |\bar{S}| - 1, \end{aligned}$$

which is equivalent to the constraint (49) for the set $\bar{S}-\{1\}$ since $S = V-(\bar{S}-\{1\})-\{1\}$. Hence the constraints (50) define the same inequalities as (49) and we have the following lemma:

**Lemma 3.** *The linear transformation $P_{\mathrm{F}}$ of the FGG-polytope* $\mathrm{TP_F}$ *is given by*

$$P_{\mathrm{F}} = \{x \in \mathbb{R}^{n(n-1)} \,|\, x \textit{ satisfies } (46)\text{–}(49)\}.$$

*Moreover,* $\dim P_{\mathrm{F}} = n(n-1) - 2$ *for all* $n \geqslant 3$.

**Proof.** The first part of the lemma follows from the discussion preceding it. To prove the second part we observe that the point given by $x = 1/(n-1)e_{n(n-1)}$ is contained in $P_{\mathrm{F}}$ and satisfies all inequalities defining $P_{\mathrm{F}}$ strictly. Since the rank of the equation system defining $P_{\mathrm{F}}$ equals 2 the statement follows. $\square$

We show next that the relations (46)-(49) together with the integrality condition on all variables $x_{ij}$ constitute a valid formulation of the travelling salesman problem. Clearly, every incidence vector $x$ of a tour satisfies (46)-(49). On the other hand let $x$ be a integer solution to (46)-(49). Like in the proof of Proposition 4, equation (48) implies $x(\delta^+(1)) = x(\delta^-(1)) = 1$. Suppose that there exists a node $i \neq 1$ such that $x(\delta^+(i)) = 0$. Let $S = V - \{1, i\}$. Since $x(\delta^-(1)) = 1$ and $x(\delta^+(i)) = 0$ imply $x(S:1) = 1$, it follows that

$$h(S) = -x(1:S) + nx(S:1) + 2x(S:i) - (n-1)x(i:S)$$
$$= -x(1:S) + n + 2x(S:i) \geqslant n-1,$$

which is a contradiction. Hence, $x(\delta^+(i)) \geqslant 1$ for all $i \neq 1$. The constraint (49) for $S = \{i\}$ and $i \neq 1$ is written as

$$-x_{1i} + nx_{i1} + 2x(i:\bar{S}-\{1\}) - (n-1)x(\bar{S}-\{1\}:i)$$
$$= -(n-1)x(\delta^-(i)) + 2x(\delta^+(i)) + (n-2)(x_{i1} + x_{1i}) \leqslant 1.$$

It follows that $x(\delta^-(i)) \geqslant 1$ for all $i \neq 1$ because $x(\delta^+(i)) \geqslant 1$ for all $i \neq 1$. By (47), it follows that $x(\delta^-(i)) = x(\delta^+(i)) = 1$ for all $i$ and furthermore, the support of $x$ corresponds to $k \geqslant 1$ directed cycles. Suppose $k \geqslant 2$. Let $\pi(S')$ be a subset of support of $x$ that corresponds to a directed cycle on $S' \subseteq V$, $2 \leqslant |S'| \leqslant n-2$ and $1 \in S'$. Define $S = S' - \{1\}$. Because $2 \leqslant |S'| \leqslant n-2$ and $|S| = |S'| - 1$, it follows that $h(S) = -1 + n > |S|$, which is a contradiction. The statement follows.

Consequently, all integer extreme points of the polytope $P_F$ correspond to tours of the travelling salesman and the comparison between the FGG-formulation and the standard one is reduced to a comparison of the polytopes $P_F$ and $P_S^n$. Moreover, we note that if $c_{ijt} = c_{ij}$ for all $i$, $j$ and $t$ the sufficient condition of Proposition 2 for "comparability" is satisfied.

**Theorem 4.** *The subtour polytope $P_S^n$ is a proper subset of $P_F$ for all $n \geqslant 4$.*

**Proof.** The constraint (49) can be rewritten as

$$h(S) = -x(\bar{S}:S) + 2x(S:\bar{S}) + (n-2)(x(S:1) - x(\bar{S}-\{1\}:S)) \leqslant |S|.$$

Let $x$ be a vector in $P_S^n$. It follows that the equations (47) and (48) are satisfied. Furthermore, summing up the constraints (17) and (18) for all $i \in S$ we obtain $x(E(S)) + x(S:\bar{S}) = x(E(S)) + x(\bar{S}:S) = |S|$. It follows that $x(S:\bar{S}) = x(\bar{S}:S)$ and $x(S:\bar{S}) \leqslant |S|$. Since $x \in P_S^n$, we have $x(\bar{S}-\{1\}:S) + x(\bar{S}-\{1\}:1) \geqslant 1$ by connectivity and $x(\bar{S}-\{1\}:1) + x(S:1) = 1$ and hence, $x(\bar{S}-\{1\}:S) \geqslant x(S:1)$. It follows that

$$h(S) = x(S:\bar{S}) + (n-2)(x(S:1) - x(\bar{S}-\{1\}:S) \leqslant x(S:\bar{S}) \leqslant |S|$$

and thus the constraints (49) are satisfied. Consequently, $x \in P_S^n$ implies $x \in P_F$ and therefore $P_S^n \subseteq P_F$. To prove that $P_S^n$ is a proper subset, we construct a vector $x$ as follows:

$$x_e = \begin{cases} 1/n & \forall e \in \delta^-(1), \\ 0 & \forall e \in \delta^+(1), \\ (n^2-n+1)/n(n-1)(n-2) & \forall e \in E-\delta(1). \end{cases}$$

It follows that $x$ satisfies (46)-(48). Furthermore, $x(S:\bar{S}-\{1\}) = x(\bar{S}-\{1\}:S) > 0$ and thus

$$h(S) = 0 + |S| - (n-3)x(\bar{S}-\{1\}:S) \leqslant |S|$$

for all $S \subseteq V-\{1\}$ and $1 \leqslant |S| \leqslant n-2$. Hence, $x \in P_{\mathrm{F}}$. However $x$ does not satisfy constraints (17) and (18) of the DFJ-formulation and thus $x \notin P_{\mathrm{S}}^n$. The theorem follows. □

The point $x$ constructed in the proof of the theorem satisfies neither the degree constraints (17), (18) nor the subtour elimination constraints (19) of the DFJ-formulation. To prove the latter we observe that for $n \geqslant 4$ and $|S| = n-2$ we have

$$\begin{aligned} x(E(S)) &= (n-2)(n-3)(n^2-n+1)/n(n-1)(n-2) \\ &= (n-3)(n^2-n+1) > n-3 = |S|-1 \end{aligned}$$

and for $n=3$, $S=\{2,3\}$, we have $x(E(S)) = \frac{7}{3} > 1$, i.e. the subtour elimination constraint for $S=\{2,3\}$ is violated. Moreover, since $\dim P_{\mathrm{F}} > \dim P_{\mathrm{S}}^n$ it follows that the *affine hull* of $P_{\mathrm{F}}$ properly contains the affine hull of $P_{\mathrm{S}}^n$. The compact FGG-formulation is thus — from a linear programming point of view — a particularly "bad" formulation of the travelling salesman problem.

### *3.4. The Claus (C) formulation*

Claus (1984) proposes a different formulation of the standard TSP that uses network flow concepts involving multiple commodities. Denote $s$ the home city (the "source") and transform any hamiltonian cycle into a Hamiltonian path by duplicating the home city as a "sink" $t$. The TSP can be interpreted as the problem of finding a Hamiltonian path from $s$ to $t$ on an $(s,t)$-digraph $G=(V,E)$ where $V = V^1 \cup \{s,t\}$, $V^1 = \{1,\dots,n\}$ and $E = \{(i,j) \mid \forall i \neq j \in V^1\} \cup \{(s,i),(i,t) \mid \forall i \in V^1\}$. Furthermore, we interpret the variable $x_{ij}$ for $(i,j) \in E$ where

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is in the Hamiltonian path}, \\ 0 & \text{otherwise}, \end{cases}$$

as defining a capacity on arc $(i,j)$. The network flows involve $n+1$ commodities. The "$k$th commodity" is the commodity shipped from $s$ to vertex $k$ where $k = 1,\dots,n+1$ and $n+1$ is the index of the sink $t$. We define a variable $y_{ijk}$ as the flow of the $k$th commodity on arc $(i,j)$. Claus (1984) formulates the problem as the following 0–1 linear program:

$$\min \sum_{(i,j)\in E} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{j\in V} x_{ij} = \sum_{j\in V} x_{ji} = 1 \quad \forall i\in V^1, \tag{51}$$

$$\sum_{i\in V^1} x_{si} = \sum_{i\in V^1} x_{it} = 1, \tag{52}$$

$$\sum_{j\in V} y_{ijk} - \sum_{j\in V} y_{jik} = 0 \quad \forall i\in V^1,\ k\in V^*, \tag{53}$$

$$\sum_i -y_{ikk} = -1,\ \sum_i y_{sik} = 1,\ \sum_i y_{kik} = 0, \quad \forall k\in V^*, \tag{54}$$

$$y_{ijk} \leqslant x_{ij} \quad \forall i,j,k, \tag{55}$$

$$x_{ij} \geqslant 0 \quad \forall i,j, \tag{56}$$

$$y_{ijk} \geqslant 0 \quad \forall i,j,k, \tag{57}$$

$$x_{ij},\ y_{ijk} \text{ integer } \forall i,j,k, \tag{58}$$

where $V^* = V^1 \cup \{t\}$. When viewed in isolation the constraints (53), (54) and (57) assure that for every node $k\in V^*$ there exists a *path* from the source $s$ to the node $k$ along which one can push one unit of flow. The constraint (55) "couple" these requirements across all nodes by requiring additionally that for all feasible vectors $(x, y)$ such paths exist in the *support graph* corresponding to the positive values of $x$. It follows that in the support graph every cut has a value of at least one and thus all subtour elimination constraints are satisfied by the feasible values of $x$. Hence, in particular, all subtours are ruled out by the above formulation. For every feasible solution to (51)–(57) we have $y_{itk} = 0$ for all $i\in V$ and $k\in V^1$ because no commodity can be shipped out from the sink $t$ and $y_{kik} = 0$ for all $i, k\in V^*$ because $\sum_i y_{kik} = 0$ for all $k\in V^*$. Hence, these variables can be dropped from the formulation. The C-formulation involves a set of constraints of order $n^3$ and $n^3 + n^2 + 3n$ variables and models the standard problem of the DFJ-formulation correctly.

### *3.4.1. The C-formulation and the standard model*

To compare the C-formulation and the DFJ-formulation we have to *project out* all of the $y$-variables of the former formulation, i.e. the linear transformation that we have to analyze has the particular form $L = (I_{n^2+n}, 0)$ where 0 is an appropriately dimensioned matrix of zeros corresponding to the $y$-variables. Following the notation of Section 2 we define the following sets:

$$Q = \{x \in \mathbb{R}^{n^2+n} \mid x \text{ satisfies (51), (52) and (56)}\},$$

$$\mathrm{YP_C} = \{(y, x) \in \mathbb{R}^{n^3+n^2+3n} \mid (y, x) \text{ satisfies (51)–(57)}\},$$

$$P_C = \{x \in Q \mid \exists y \in \mathbb{R}^{n^3+2n} \text{ such that } (x, y) \in \mathrm{YP_C}\}.$$

We call $YP_C$ the C-polytope and $P_C$ its projection. Ignoring the (minor) conceptual change introduced by the modelling of Hamiltonian paths rather than Hamiltonian cycles, the comparison between the two formulations is thus reduced to comparing the polytopes $P_C$ and $P_S^n$. By the remarks following the C-formulation one is lead to expect that $P_C$ is a *smaller* polytope than $P_S^n$ and the question is simply whether or not this formulation provides any polyhedral information that is not already contained in the standard model. The answer to this question — as we shall see — is negative.

To analyze the C-formulation we write the constraints (53)–(55) in matrix form as follows:

$$Ay = b,$$

$$-Dx + y \leqslant 0.$$

Examining the formulation we find that $A$ is a block-diagonal matrix and consists of $n+1$ blocks $A_k$ for $k=1,\dots,n+1$. The decomposition of $A$ follows from the fact that the $n+1$ commodity flows do not interact with each other. For $k \in V^1$ the matrix $A_k$ is the node-arc incidence matrix of the complete $(s,k)$-digraph $G_k = (V^1 \cup \{s\}, E_k)$ having a source $s$, a sink $k$ and $E_k = E - (\delta^+(k) \cup \delta^-(t))$. $A_{n+1}$ is the node-arc incidence matrix of the entire network, i.e. $G_{n+1} = G$. The constraint $\sum_i y_{sik} = 1$ for the commodity $k$ defines the row of $A_k$ corresponding to the source $s$ and the constraint $\sum_i -y_{ikk} = -1$ defines the row of $A_k$ that corresponds to the sink $k$. Correspondingly, we partition $b$ as induced by $(A_1,\dots,A_{n+1})$ into $b = (b^1,\dots,b^{n+1})$ where $b^k$ for all $k$ has an entry of 1 in the row corresponding to the source $s$, $-1$ in the row corresponding to the sink $k$ and 0 elsewhere. Furthermore, we partition $D$ according to the partitioning of $A$ as $D = (D_1^T,\dots,D_{n+1}^T)^T$ where $D_k$ for $k \neq n+1$ is a matrix of size $(n^2-n+1)\times n^2$ and $D_{n+1} = I_{n^2+n}$. The columns of the matrix $D$ correspond to the variables $x_{ij}$ for all $(i,j) \in E$. Each row of $D_k$ corresponds to a variable $y_{ijk}$ and has an entry of 1 in the column corresponding to the variable $x_{ij}$ and 0 elsewhere. Let $y^k$ be a vector having components $y_{ijk}$ for all $(i,j) \in E_k$ and $x$ be a vector having components $x_{ij}$ for all $(i,j) \in E$. The C-formulation can then be written in matrix notation as follows:

$$\begin{aligned}
\min \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} \\
\text{s.t.} \quad & (51),\ (52) \text{ and} \\
& A_k y^k = b^k \ \text{ for } k=1,\dots,n+1, \\
& -D_k x + y^k \leqslant 0 \ \text{ for } k=1,\dots,n+1, \\
& x \geqslant 0,\ y^k \geqslant 0 \ \text{ for } k=1,\dots,n+1.
\end{aligned}$$

In order to carry out the comparison we have to find a generator system of the cone

$$C = \{(u, v, w) \mid uA + v \geqslant 0,\ v \geqslant 0,\ w \geqslant 0\}. \tag{59}$$

The inequalities $w \geqslant 0$ give rise to extreme rays of $C$ which are unit vectors of the form $(\mathbf{0}, \mathbf{0}, w^i)$ where $w^i$ is the $i$th unit vector of $\mathbb{R}^{n(n+1)}$. Furthermore, since $A$ decomposes, we can use the intersection property of cones and work on the smaller cones

$$C_k = \{(u^k, v^k) \mid u^k A_k + v^k \geqslant 0, v^k \geqslant 0\}$$

for $k = 1, \ldots, n+1$ to find a full system of generators for $C$ and thereby a linear description of the polytope $P_C$.

*3.4.2. The node-arc-cone of a (s, t)-digraph*

The node-arc cone $C$ of a $(s, t)$-digraph $(V, E)$ is defined as follows:

$$C = \{(u, v) \in \mathbb{R}^m \mid A^{\mathrm{T}} u + v \geqslant 0, v \geqslant 0\},$$

where $m = |V| + |E|$ and $A$ is the node-arc incidence matrix of the $(s, t)$-digraph having a source $s$ and a sink $t$. A full generator system for $C$ follows from the results of Proposition 6 by replacing $p$ with 1 because the arguments used in the proof of Proposition 6 do not make use of the completeness of the underlying digraph and thus apply to the sparse digraph considered here as well.

*3.4.3. The projection of the C-polytope*

Since in this section we are addressing the whole formulation, the cone $C$ is the cone defined in (59) and $V$ is the node-set of the original $(s, t)$-digraph, i.e. $V = \{s, 1, \ldots, n, t\}$. By Corollary 1, $P_C$ is defined by

$$P_C = \{x \in Q \mid (-vD - w)x \leqslant ub \ \forall (u, v, w) \in C\}.$$

A set of generators of the cone $C$ is given by the unit vectors $(\mathbf{0}, \mathbf{0}, w^i)$ which are generated by the inequalities $w \geqslant 0$. These generators yield the nonnegativity constraints

$$x_e \geqslant 0 \quad \forall e \in E. \tag{60}$$

In order to derive the remaining generators of the cone $C$, by Proposition 1 we embed the generators of $C_k$ for $k = 1, \ldots, n+1$ into the space of $C$ by adding zero entries elsewhere. However these zero entries in the generators do not affect the projection. We can still work with the generators of $C_k$ and correctly translate the results to the $x$-variables, i.e. we have to calculate $(-vD_k)x \leqslant ub^k$ for all generators $(u, v) \in C_k$ where $k = 1, \ldots, n+1$.

The vector $u^k = \pm e_{n+1}$, $v^k = \mathbf{0}$ defines a basis of the lineality space of $C_k$ for all $k$. This generator defines the trivial equality $0 = 0$ which is redundant. For all $k \neq n+1$ the remaining generators of $C_k$ are given by

(i) $u = u^S$, $v = v^{(\bar{S} - \{k\}:S)}$,

(ii) $u = -u^S$, $v = v^{(S - \{k\}:\bar{S} - \{t\})}$, where $S \subseteq V-\{t\}$ and $1 \leqslant |S| \leqslant n$. These generators yield the inequalities

$$-x(\bar{S}-\{k\}:S) \leqslant u_s - u_k \quad \text{for } u = u^S, \tag{61}$$

$$-x(S-\{k\}:\bar{S}-\{t\}) \leqslant u_s - u_k \quad \text{for } u = -u^S, \tag{62}$$

where $u_i = 1$ if $i \in S$, $u_i = 0$ otherwise. The remaining generators of $C_{n+1}$ are given by

(iii) $u = u^S,\ v = v^{(\bar{S}:S)}$,

(iv) $u = -u^S,\ v = v^{(S:\bar{S})}$,

where $S \subseteq V$ and $1 \leqslant |S| \leqslant n+1$. These generators produce the inequalities

$$-x(\bar{S}:S) \leqslant u_s - u_t \quad \text{for } u = u^S, \tag{63}$$

$$-x(S:\bar{S}) \leqslant u_s - u_t \quad \text{for } u = -u^S. \tag{64}$$

Moreover, the generators of $C_k$ given by Proposition 6, part (i), reproduce the nonnegativity constraints (60) when $k$ varies from 1 to $n+1$. If the right-hand side in (61) or (62) equals 0 or 1, then the corresponding inequality is implied by the nonnegativity conditions (60) and thus redundant. Therefore, we need to consider only the following inequalities:

$$-x(\bar{S}:S) \leqslant -1 \quad \forall S \subseteq V-\{t\}, 1 \leqslant |S| \leqslant n, s \notin S \text{ and } k \in S, \tag{65}$$

$$-x(S:\bar{S}-\{t\}) \leqslant -1 \quad \forall S \subseteq V-\{t\}, 1 \leqslant |S| \leqslant n, s \in S \text{ and } k \notin S, \tag{66}$$

where we have replaced $(\bar{S}-\{k\}:S)$ by $(\bar{S}:S)$ for $k \in S$ and $(S-\{k\}:\bar{S}-\{t\})$ by $(S:\bar{S}-\{t\})$ for $k \notin S$. Consequently, (66) defines the same set of inequalities as does (65). The left-hand sides of inequalities (63) and (64) except for $S=\{t\}$ and $S=V-\{t\}$, respectively, are less than or equal to the left-hand sides of (65) and (66), respectively, and hence, these inequalities are dominated by (65) and (66). For $S=\{t\}$ and $S=V-\{t\}$, respectively, the inequalities (63) and (64) define the same inequality $x(\delta^-(t)) \geqslant 1$ which is redundant.

Since $k$ varies from 1 to $n$, the inequalities (65) can be summarized as the following cut constraints:

$$x(\bar{S}:S) \geqslant 1 \quad \forall S \subseteq V-\{t\}, 1 \leqslant |S| \leqslant n \text{ and } s \in \bar{S}. \tag{67}$$

Consequently, we have the following lemma:

**Lemma 4.** *The projection $P_C$ of the C-polytope* $\mathrm{YP_C}$ *is given by*

$$P_C = \{x \in \mathbb{R}^{n^2+n} \mid x \text{ satisfies } (51), (52), (56), (60) \text{ and } (67)\}. \qquad \square$$

In order to compare the C-formulation with the standard formulation for TSP, we identify the source $s$ and the sink $t$ of the network $G$. It follows that constraints (67) can be written as

$$x(S:\bar{S}) \geqslant 1 \quad \forall S \subseteq V, 1 \leqslant |S| \leqslant n,$$

which are exactly the subtour elimination constraints in cut form. Hence, we conclude our comparison in the following theorem:

**Theorem 5.** *The projection $P_C$ of the C-polytope* $YP_C$ *is equivalent to the subtour polytope* $P_S^{n+1}$. □

## 4. Symmetrization of three TSP formulations

By "symmetrization" we mean the replacement of a single directed edge or of a pair of oppositely directed edges joining two nodes of a graph by an *undirected* edge of a graph on the same node set. As we are working in the space of variables corresponding to the edges of a directed graph this operation translates naturally into a linear transformation mapping the former space onto a space of variables corresponding to the edges of an undirected graph. We can thus use the results of Section 2 in order to derive analytically formulations or non-formulations of the *symmetric* travelling salesman problem from the formulations of the *asymmetric* travelling salesman problem that we have studied so far. We show that the symmetrization of the DFJ-formulation leads to the standard formulation of the symmetric travelling salesman problem also due to Dantzig, Fulkerson and Johnson (1954), a result that was known to Heller (1955); see also Grötschel and Padberg (1985) where several open problems that relate to this symmetrization technique are stated. We also "symmetrize" the MTZ-formulation and the FGG-formulation in this section.

### *4.1. Symmetrization of the DFJ-formulation*

We define a linear transformation $y = Lx$ by

$$y_{ij} = x_{ij} + x_{ji}, \quad 1 \leq i < j \leq n.$$

It is known that this transformation maps $P^n$ to the symmetric travelling salesman polytope $Q^n$, see Grötschel and Padberg (1985), which we can also define as

$$Q^n = \{y \in \mathbb{R}^{n(n-1)/2} \mid \exists x \in P^n \text{ such that } y = Lx\}.$$

In order to symmetrize the DFJ-formulation of Section 3.1 we derive the linear description of the polytope

$$Q_S^n = \{y \in \mathbb{R}^{n(n-1)/2} \mid \exists x \in P_S^n \text{ such that } y = Lx\},$$

where $P_S^n$ is the subtour polytope of the asymmetric TSP. It is clear that to every tour of the travelling salesman in the complete undirected graph there corresponds a zero-one point of $Q_S^n$ and we will show that the converse holds as well.

In matrix form the DFJ-formulation is written as $Ax = e_{2n}$, $Dx \leq d$, $x \geq 0$ and $x$ integer. The matrix $A$ is the node-edge incidence matrix of a complete bipartite graph; $D$ has an exponential number of rows and each row corresponds to the incidence vector $x^{E(S)}$ with a right-hand side of $|S|-1$ for $S\subseteq V$ and $2\leqslant |S|\leqslant n-1$. We write the above linear transformation in matrix form as $L=(L_1, L_2)=(I_m, I_m)$ where $L_1$ has columns corresponding to $x_{ij}$, $L_2$ has columns corresponding to $x_{ji}$ for all $1\leqslant i<j\leqslant n$ and $m=\frac{1}{2}n(n-1)$. We partition $A$ and $D$ according to $(L_1, L_2)$ as $(A_1, A_2)$ and $(D_1, D_2)$ and note that

$$A_1=\begin{bmatrix} H \\ K \end{bmatrix}, \qquad A_2=\begin{bmatrix} K \\ H \end{bmatrix},$$

where $H$ and $K$ are of the form

$$H=\begin{bmatrix} e_{n-1} & 0 & \cdots & 0 \\ 0 & e_{n-2} & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad K=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_{n-1} & \begin{pmatrix} 0 \\ I_{n-2} \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix}.$$

Then we have

$$A_2-A_1L_1^{-1}L_2=\begin{bmatrix} -M \\ M \end{bmatrix}, \qquad D_2-D_1L_1^{-1}L_2=0,$$

where $M=H-K$ is the node-arc incidence matrix of a complete acylic digraph $G=(V, E)$ with $|V|=n$ and $E=\{(i,j)\,|\,\text{for all } 1\leqslant i<j\leqslant n\}$.

The cone associated with this transformation is given by

$$C=\{(u, v, \xi, w)\,|\,-uM+vM+w\geqslant 0, \xi\geqslant 0, w\geqslant 0\},$$

and the linear description of $Q_S^n$ is given by

$$Q_S^n=\{y\in\mathbb{R}^{n(n-1)/2}\,|\,((u, v)A_1+\xi D_1-w)L_1^{-1}y\leqslant ue_n+ve_n+\xi d$$
$$\text{for all } (u, v, \xi, w)\in C\}.$$

The lineality space of $C$ is given by

(i) $u=v=\pm u^i$ for all $i\in V$, $\xi=\mathbf{0}$, $w=\mathbf{0}$,

(ii) $u=\pm e_n$, $v=\mathbf{0}$, $\xi=\mathbf{0}$, $w=\mathbf{0}$.

The generators $(\mathbf{0}, \mathbf{0}, \xi^S, \mathbf{0})$ and the basis of the lineality space give rise to the following inequalities:

$$y(E)=n, \tag{68}$$

$$y(\delta(i))=2 \quad \forall i\in V, \tag{69}$$

$$y(E(S))\leqslant |S|-1 \quad \forall S\subseteq V, 2\leqslant |S|\leqslant n-1, \tag{70}$$

where $\xi^S$ is the unit vector with entry 1 in the position corresponding to the constraint defined by $S$. Note that (69) implies equation (68) which is thus redundant.

Using the intersection property of cones we simplify the cone $C$ by dropping all $\xi$-variables. Intersecting the resulting cone with the lineality space given by (i) we can furthermore eliminate the variables $v$. Making the variable transformation $\tilde{w}=1/2w$, we can thus work on the smaller cone

$$C_1=\{(u,\tilde{w})\,|\,uM+\tilde{w}\geqslant 0,\ \tilde{w}\geqslant 0\}$$

to find the remaining generators of $C$ since they are in one-to-one correspondence. These generators yield the following inequalities of $Q_S^n$:

$$(uM-2\tilde{w})L_1^{-1}y\leqslant ue_n-ue_n=0 \quad \text{for all } (u,\tilde{w})\in C_1.$$

It follows from Proposition 6 that a generator system of $C_1$ is given by

(i) $u=\mathbf{0},\ \tilde{w}=w^i$,

(ii) $u=-u^S,\ \tilde{w}=w^{(\bar{S}:S)}$,

(iii) $u=u^S,\ \tilde{w}=w^{(S:\bar{S})}$,

for all $S\subseteq V-\{1\}$, $1\leqslant |S|\leqslant n-1$. These generators yield the following inequalities:

$$y_e\geqslant 0 \quad \forall e\in E, \tag{71}$$

$$-y(S:\bar{S})-y(\bar{S}:S)\leqslant 0 \quad \forall S\subseteq V, 1\leqslant |S|\leqslant n-1, \tag{72}$$

and (72) is redundant. Hence we have the following theorem:

**Theorem 6.** *The symmetrization $Q_S^n$ of the subtour polytope $P_S^n$ is given by*

$$Q_S^n=\{y\in\mathbb{R}^{n(n-1)/2}\,|\,y \text{ satisfies } (69)\text{–}(71)\}. \qquad \square$$

It follows that this symmetrization technique yields the standard formulation of the symmetric travelling salesman problem due to Dantzig, Fulkerson and Johnson (1954). Moreover, if $c_{ij}=c_{ji}$ for all $i$ and $j$, then the sufficient condition of Proposition 2 for comparability is satisfied and the linear programming relaxations of the asymmetric and the symmetric formulation, respectively, yield the same upper bound on the minimum-cost tour.

### 4.2. Symmetrization of the MTZ-formulation

We consider the linear transformation $\left[\begin{smallmatrix}y\\z\end{smallmatrix}\right]=L\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]$ given by

$$y_{ij}=x_{ij}+x_{ji}, \quad 1\leqslant i<j\leqslant n,$$

$$z_i=u_i, \quad 2\leqslant i\leqslant n,$$

and define the symmetric MTZ-polytope $\mathrm{SP_M}$ by

$$\mathrm{SP_M}=\left\{(y,z)\in\mathbb{R}^{(n-1)(n-2)/2}\,\middle|\,\exists(x,u)\in\mathrm{UP_M} \text{ such that } \begin{bmatrix}y\\z\end{bmatrix}=L\begin{bmatrix}x\\u\end{bmatrix}\right\},$$

where $UP_M$ is the MTZ-polytope defined in Section 3.2.1. It follows that to every *undirected* "permissible itinerary" (see Section 3.2) of the travelling salesman there corresponds a point of $SP_M$ satisfying $y_{1j} \in \{0, 1, 2\}$ and $y_{ij} \in \{0, 1\}$ for all $1 \leq i \leq j \leq n$. We derive the linear description of $SP_M$ using the results of Section 2 and show that the converse does not hold for any $p \geq 2$.

We decompose the vector $x$ of the MTZ-formulation as follows:

$$x^1 = (x_{12}, \ldots, x_{1n})^{\mathrm{T}},$$

$$x^2 = (x_{23}, \ldots, x_{2n}, \ldots, x_{n-1,n})^{\mathrm{T}},$$

$$x^3 = (x_{21}, \ldots, x_{n1})^{\mathrm{T}},$$

$$x^4 = (x_{32}, \ldots, x_{n2}, \ldots, x_{n,n-1})^{\mathrm{T}}.$$

Let $u = (u_2, \ldots, u_n)$, $A$ be the matrix defined by the constraints (23)-(26), $D$ be the matrix defined by the constraint (27), $b = (t, t, e_{n-1}, e_{n-1})^{\mathrm{T}}$ and $d = ((p-1)e_m, (p-1)e_m)^{\mathrm{T}}$ be the corresponding right-hand sides for $A$ and $D$ respectively, and $m = \frac{1}{2}(n-1)(n-2)$. We write the above linear transformation as

$$\begin{bmatrix} y \\ z \end{bmatrix} = L_1 \begin{bmatrix} x^1 \\ x^2 \\ u \end{bmatrix} + L_2 \begin{bmatrix} x^3 \\ x^4 \end{bmatrix}$$

where

$$L_1 = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix}, \qquad L_2 = \begin{bmatrix} I_{n-1} & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}.$$

We partition $A$ and $D$ according to $(L_1, L_2)$ as

$$A_1 = \begin{bmatrix} e_{n-1} & 0 & 0 \\ 0 & 0 & 0 \\ I_{n-1} & K & 0 \\ 0 & H & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ e_{n-1} & 0 \\ 0 & H \\ I_{n-1} & K \end{bmatrix},$$

and

$$D_1 = \begin{bmatrix} 0 & pI_m & M^{\mathrm{T}} \\ 0 & 0 & -M^{\mathrm{T}} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & pI_m \end{bmatrix},$$

where $H$ and $K$ are defined as in Section 4.1 and $M = H - K$ is the node-arc incidence matrix of a complete acyclic digraph $G = (V, E)$ with $|V| = n-1$ and $E = \{(i, j) \mid 2 \leq i < j \leq n\}$. It follows that

$$A_2 - A_1 L_1^{-1} L_2 = \begin{bmatrix} -e_{n-1} & 0 \\ e_{n-1} & 0 \\ -I_{n-1} & M \\ I_{n-1} & -M \end{bmatrix}, \qquad D_2 - D_1 L_1^{-1} L_2 = \begin{bmatrix} 0 & -pI_m \\ 0 & pI_m \end{bmatrix}.$$

The cone associated with this transformation is given by

$$C = \{(u, v, w) \mid -u_0 e_{n-1} + u_1 e_{n-1} - u^1 + u^2 + w^1 \geq 0,$$
$$u^1 M - u^2 M - pv^1 + pv^2 + w^2 \geq 0,\ v \geq 0,\ w \geq 0\}$$

where $u = (u_0, u_1, u^1, u^2)$, $u^1, u^2 \in \mathbb{R}^{n-1}$ are indexed from 2 to $n$, $v = (v^1, v^2)$ and $w = (w^1, w^2, w^3)$. (Note that $u^1$ and $u^2$ are vectors of *variables* and not unit-vectors.) The linear description of $\mathrm{SP_M}$ is given by

$$\begin{aligned}\mathrm{SP_M} = \{(y, z) \mid\ & (u_0 e_{n-1} + u^1 - w^1) y^1 + (u^1 K + u^2 H + pv^1 - w^2) y^2 \\ & + (v^1 M^{\mathrm{T}} - v^2 M^{\mathrm{T}} - w^3) z \\ & \leq (u_0 + u_1) t + e_{n-1}(u^1 + u^2) + (p-1)(v^1 + v^2) e_m \\ & \text{for all } (u, v, w) \in C\}\end{aligned}$$

where $y = (y^1, y^2)$, $y^1 = (y_{12}, \ldots, y_{1n})^{\mathrm{T}}$ and $y^2 = (y_{23}, \ldots, y_{2n}, \ldots, y_{n-1,n})^{\mathrm{T}}$.

*4.2.1. Simplification of the cone C*

The linealities of $C$ are given by

(i) $u_0 = u_1 = \pm 1,\ u^1 = \mathbf{0},\ u^2 = \mathbf{0},\ v = \mathbf{0},\ w = \mathbf{0},$

(ii) $u_i^1 = u_i^2 = \begin{cases} \pm 1 & i = i^*, \\ 0 & i \neq i^*, \end{cases} \quad u_0 = u_1 = 0,\ v = \mathbf{0},\ w = \mathbf{0}$ for $i^* = 2, \ldots, n,$

(iii) $u_0 = \pm 1,\ u^2 = \pm e_{n-1},\ u_1 = 0,\ u^1 = \mathbf{0},\ v = \mathbf{0},\ w = \mathbf{0},$

(iv) $u_1 = \pm 1,\ u^1 = \pm e_{n-1},\ u_0 = 0,\ u^2 = \mathbf{0},\ v = \mathbf{0},\ w = \mathbf{0}.$

The vectors (i), (ii) and (iii) are linearly independent and in the lineality space $L$ of $C$. We can thus intersect $C$ with the equations

$$u_0 + u_1 = 0, \quad u^1 + u^2 = 0, \quad u_0 + \sum_{i=2}^{n} u_i^2 = 0.$$

It follows that $u_1 = -u_0$, $u^1 = -u^2$ and

$$u_0 = -\frac{1}{n} \sum_{i=2}^{n} \tilde{u}_i, \quad u^2 = u_0 e_{n-1} + \tilde{u},$$

where $\tilde{u} = (\tilde{u}_2, \ldots, \tilde{u}_n)^{\mathrm{T}} \in \mathbb{R}^{n-1}$ is arbitrary. The cone $C$ is thus simplified to the following cone

$$\begin{array}{llllll} C_1: & 2\tilde{u} & & + w^1 & & \geq 0, \\ & -2\tilde{u} M - pv^1 + pv^2 & & & + w^2 & \geq 0, \\ & & v^1, \quad v^2, & w^1, & w^2, & w^3 \geq 0. \end{array}$$

Since $C_1$ is a pointed cone it follows that (i), (ii) and (iii) form a basis of $L$ and that $C_1 = C \cap L^{\perp}$. Using the intersection property of cone we can drop the variables $w^3$ and we note that

(v) all unit vectors associated with $w^1$, $w^2$ and $w^3$ define extreme rays of $C_1$ and thus of $C$.

Furthermore, for any extreme ray of $C_1$ it follows that either $w^1 = \mathbf{0}$ or $w^1$ is a unit vector if $\tilde{u} = \mathbf{0}$. If $\tilde{u} \neq 0$ then we have necessarily

$$w^1 = \max\{0, -2\tilde{u}\}$$

in any extreme ray of $C_1$. Consequently, it suffices to consider the cone

$$C_2: \qquad -\tilde{u}M - \tilde{v}^1 + \tilde{v}^2 + \tilde{w}^2 \geqslant 0, \ \tilde{v}^1 \geqslant 0, \ \tilde{v}^2 \geqslant 0, \ \tilde{w}^2 \geqslant 0,$$

where we set

$$\tilde{v}^1 = \tfrac{1}{2}pv^1, \quad \tilde{v}^2 = \tfrac{1}{2}pv^2, \quad \tilde{w}^2 = \tfrac{1}{2}w^2.$$

Different from $C_1$ the cone $C_2$ has a lineality that gives rise to the following two extreme rays of $C_1$:

(vi) $\tilde{u} = e_{n-1}, \ w^1 = \mathbf{0}, \ v^1 = v^2 = w^2 = \mathbf{0},$

(vii) $\tilde{u} = -e_{n-1}, \ w^1 = 2e_{n-1}, \ v^1 = v^2 = w^2 = \mathbf{0}.$

Let $\hat{v}^2 = \tilde{v}^2 + \tilde{w}^2$. Since $\tilde{v}_{ij}^2 \tilde{w}_{ij}^2 = 0$ for all $i$ and $j$ in every extreme ray of $C_2$, the cone $C_2$ is then reduced to

$$C_3: \qquad -\tilde{u}M - \tilde{v}^1 + \hat{v}^2 \geqslant 0, \ \tilde{v}^1 \geqslant 0, \ \hat{v}^2 \geqslant 0.$$

Although this variable transformation does not define a unique transformation, we can still derive all extreme rays of $C_2$ from those of the cone $C_3$ by replacing every positive $\hat{v}_{ij}^2$ in an extreme ray with $\tilde{v}_{ij}^2 = \hat{v}_{ij}^2$, $\tilde{w}_{ij}^2 = 0$ and $\tilde{v}_{ij}^2 = 0$, $\tilde{w}_{ij}^2 = \hat{v}_{ij}^2$. Thus to every extreme ray of $C_3$ having $k$ positive components $\hat{v}^2$ there correspond exactly $2^k$ extreme rays of $C_2$.

*4.2.2. A different node-arc cone of a directed graph*

Let $C$ be a node-arc cone of a digraph $G = (V, E)$ with $|V| = n$, $E = \{(i, j) \mid 1 \leqslant i < j \leqslant n\}$ and defined as

$$C = \{(u, v^1, v^2) \in \mathbb{R}^{n^2} \mid -uM - v^1 + v^2 \geqslant 0, v^1 \geqslant 0, v^2 \geqslant 0\}$$

where $M$ is the node-arc incidence of $G$. Let $B$ be the matrix defining all constraints of $C$. Then $B$ has a rank of $n^2 - 1$. Hence, $C$ has a lineality space of dimension 1 given by $(\pm e_n, \mathbf{0}, \mathbf{0})$. The remaining generators are given in the following proposition.

**Proposition 7.** *Let $B^*$ be the $(n^2-2) \times n^2$ submatrix of $B$ that corresponds to the equation system of an extreme ray of $C$. For any $S \subseteq V$ with $1 \leqslant |S| \leqslant n-1$, let $\mathscr{F}_1$ and $\mathscr{F}_2$ be the families of all subsets of $(S : \bar{S})$ and $(\bar{S} : S)$, respectively. Then every nonzero solution to*

$$B^* \begin{pmatrix} u \\ v^1 \\ v^2 \end{pmatrix} = 0 \tag{73}$$

*of minimal support is a positive multiple of the vector* $(u, v^1, v^2)$ *given by* (i), (ii), (iii) *or* (iv) *where*

(i) $u=\mathbf{0},\ v^1=\mathbf{0},\ v^2=u^i,$

(ii) $u=\mathbf{0},\ v^1=v^2=u^i,$

(iii) $u=-u^S,\ v^1=v^{F_1},\ v^2=v^{(\bar{S}:S)},$

(iv) $u=u^S,\ v^1=v^{F_2},\ v^2=v^{(S:\bar{S})},$

$u^i$ *is the ith unit vector in* $\mathbb{R}^{n(n-1)/2}$, $u^S$ *and* $v^{E_1}$ *are the characteristic vectors of S and* $E_1$, $F_1 \in \mathscr{F}_1$, $F_2 \in \mathscr{F}_2$ *and* $S \subseteq V$, $1 \leqslant |S| \leqslant n-1$.

**Proof.** This proof follows the proof of Proposition 6 and we let $B^*$, $B_1$ and $f$ be defined the same as in the proof of that proposition. Suppose the column $f$ belongs to a $v^k$-variable $v_{ij}^k$, say, with $i<j$ and $k=1$ or 2. If $B^*$ contains the row corresponding to $v_{ij}^k \geqslant 0$, then $B_1$ has a rank of at most $n^2-3$, which is a contradiction. Otherwise, if $k=2$ we prove like in Proposition 6 that we either get the vector defined by (i) or that we can assume without loss of generality that column $f$ belongs to some $u$-variable. If $k=1$ we either get the vector defined by (ii) or that $B^*$ contains the row $v_{ij}^2=0$. In the latter case we show again like in Proposition 6 that we can assume without loss of generality that column $f$ belongs to some $u$-variable. Moreover, for all solutions of minimal support to (73) not covered by (i) or (ii) it follows that:

*if* $B^*$ *contains the row* $-u_i+u_j-v_{ij}^1+v_{ij}^2=0$,
*then* $B^*$ *contains either the row* $v_{ij}^1=0$ *or* $v_{ij}^2=0$ *or both.*

By the preceding we can assume that the column $f$ belongs to variable $u_k$ with $k \in V$ and furthermore, that the matrix $B_1$ can be decomposed as follows:

$$B_1=\begin{bmatrix} A_0 & -I_0 & 0 & 0 & 0 & I_0 & 0 & 0 & 0\\ A_1 & 0 & -I_1 & 0 & 0 & 0 & I_1 & 0 & 0\\ A_2 & 0 & 0 & -I_2 & 0 & 0 & 0 & I_2 & 0\\ 0 & I_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I_1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \end{bmatrix}$$

where $I_0$, $I_1$, $I_2$ and $I_3$ are identity matrices of size $|E_0|$, $|E_1|$, $|E_2|$ and $|E_3|$, respectively,

$$E_0=\{e=(i,j)\in E \mid v_e^1=v_e^2=0 \text{ and } -u_i+u_j=0 \text{ in the extreme ray considered}\},$$

$$E_1=\{e\in E \mid v_e^1=0,\ v_e^2=1 \text{ in the extreme ray considered}\},$$

$$E_2=\{e\in E \mid v_e^1=1,\ v_e^2=0 \text{ in the extreme ray considered}\},$$

$$E_3=\{e=(i,j)\in E \mid v_e^1=v_e^2=0 \text{ and } -u_i+u_j>0 \text{ in the extreme ray considered}\}$$

and $E = E_0 \cup E_1 \cup E_2 \cup E_3$. The matrices $A_0$, $A_1$ and $A_2$ are submatrices of $-M^{\mathrm{T}}$ with $n-1$ columns, and $A_1, A_2$ can be vacuous. Since $B_1$ has a rank of $n^2-2$ it follows that $A_0$ has a rank of $n-2$. The rest of the proof goes exactly like the proof of Proposition 6 and hence the proposition follows. $\square$

*4.2.3. The symmetrization of the MTZ-polytope*

We can now derive the linear description of the polytope $\mathrm{SP_M}$ explicitly using the linealities and extreme rays of the cone $C$ of Section 4.2. The linealities of $C$ given by (i), (ii) and (iii) of Section 4.2.1 constitute a basis for the lineality space of $C$ and yield the following equations:

$$\sum_{i=2}^{n} y_{1i} = y(\delta(1)) = 2t, \tag{74}$$

$$y_{1i} + \sum_{j=2}^{i-1} y_{ji} + \sum_{j=i+1}^{n} y_{ij} = y(\delta(i)) = 2 \quad \text{for } i = 2, \ldots, n, \tag{75}$$

$$\sum_{i=2}^{n} y_{1i} + \sum_{i=2}^{n} \sum_{j=i+1}^{n} y_{ij} = y(E) = n + t - 1.$$

The last equation is implied by (74) and (75) and thus redundant. The extreme rays of $C$ given by (v) of Section 4.2.1 give rise to the nonnegativity constraints

$$z_j \geqslant 0, \;\; y_{ij} \geqslant 0, \quad \text{for } 1 \leqslant i < j \leqslant n, \tag{76}$$

while the extreme rays given by (vi) and (vii) of that section yield the inequality

$$-\sum_{j=2}^{n} y_{1j} \leqslant 0$$

which is implied by (76) and thus redundant. All of other extreme rays of $C$ are obtained from the extreme rays of $C_3$ of Section 4.2.1 which have been characterized in Proposition 7. Rather than reversing the variable transformations of Section 4.2.1 we note that in the notation of that section the additional linear inequalities of $\mathrm{SP_M}$ can be written as follows:

$$\begin{aligned}
&-p \sum_{j=2}^{n} (\tilde{u}_j + \max\{0, -2\tilde{u}_j\}) y_{1j} + p \sum_{2 \leqslant i < j \leqslant n} (\tilde{u}_i - \tilde{u}_j + 2\tilde{v}_{ij}^1 - 2\tilde{w}_{ij}^2) y_{ij} \\
&\quad + 2 \sum_{2 \leqslant i < j \leqslant n} (\tilde{v}_{ij}^1 (z_i - z_j) + \tilde{v}_{ij}^2 (-z_i + z_j)) \\
&= -p \sum_{j=2}^{n} |\tilde{u}_j| y_{1j} + p \sum_{2 \leqslant i < j \leqslant n} (\tilde{u}_i - \tilde{u}_j + 2\tilde{v}_{ij}^1 - 2\tilde{w}_{ij}^2) y_{ij} \\
&\quad + 2 \sum_{i=2}^{n} \left( \sum_{j=2}^{i-1} (-\tilde{v}_{ji}^1 + \tilde{v}_{ji}^2) + \sum_{j=i+1}^{n} (\tilde{v}_{ij}^1 - \tilde{v}_{ij}^2) \right) z_i \\
&\leqslant 2(p-1)(\tilde{v}^1 + \tilde{v}^2) e_m.
\end{aligned}$$

For convenience we write $\tilde{u}$ as $u$, $\tilde{v}^1$ as $v^1$, $\tilde{v}^2$ as $v^2$ and $\tilde{w}^2$ as $w^2$. Then a full system of generators for the conical part of $C_3$ is given by:

(i) $u=\mathbf{0},\ v^1=\mathbf{0},\ v^2=u^i,\ w^2=\mathbf{0},$

(ii) $u=\mathbf{0},\ v^1=v^2=\mathbf{0},\ w^2=u^i,$

(iii) $u=\mathbf{0},\ v^1=v^2=u^i,\ w^2=\mathbf{0},$

(iv) $u=\mathbf{0},\ v^2=\mathbf{0},\ v^1=w^2=u^i,$

(v) $u=-u^S,\ v^1=v^{E_1},\ v^2=v^{E_2},\ w^2=w^{(\bar{S}:S)-E_2},$

(vi) $u=u^S,\ v^1=v^{E_2},\ v^2=v^{E_1},\ w^2=w^{(S:\bar{S})-E_1},$

where $S\subseteq V-\{1\}$ with $1\leqslant|S|\leqslant n-2$, $E_1\subseteq(S:\bar{S})$ and $E_2\subseteq(\bar{S}:S)$ are any subsets of the respective sets and $\bar{S}=V-\{1\}-S$. The second set of generators is already contained in those of (v) of Section 4.2.1 while (i), (iii) and (iv) yield the following inequalities:

$$-z_i+z_j\leqslant p-1, \tag{77}$$

$$py_{ij}\leqslant 2(p-1), \tag{78}$$

$$z_i-z_j\leqslant p-1, \tag{79}$$

for all $2\leqslant i<j\leqslant n$. Noting that

$$u^SMy^2=-y(\bar{S}:S)+y(S:\bar{S}),$$

where $y^2$ and $M$ are defined above, we have the following inequalities from the generators (v):

$$\begin{aligned}&-py(1:S)-py(S:\bar{S})-py(\bar{S}:S)+2py(E_1)+2py(E_2)\\&\quad+2\sum_{(i,j)\in E_1}(z_i-z_j)+2\sum_{(i,j)\in E_2}(-z_i+z_j)\leqslant 2(p-1)(|E_1|+|E_2|)\end{aligned} \tag{80}$$

and the generators (vi) yield the inequalities

$$\begin{aligned}&-py(1:S)-py(S:\bar{S})-py(\bar{S}:S)+2py(E_1)+2py(E_2)\\&\quad+2\sum_{(i,j)\in E_2}(z_i-z_j)+2\sum_{(i,j)\in E_1}(-z_i+z_j)\leqslant 2(p-1)(|E_1|+|E_2|)\end{aligned} \tag{81}$$

with the above specification for the sets involved. Hence we have the following:

**Theorem 7.** *The symmetrization* $\mathrm{SP_M}$ *of the MTZ-polytope* $\mathrm{UP_M}$ *is given by*

$$\mathrm{SP_M}=\{(y,z)\in\mathbb{R}^{(n-1)(n+2)/2}\,|\,(y,z)\ \textit{satisfies}\ (74)\text{–}(81)\}. \qquad\square$$

From (75), (76) and (78) it follows that every integer-valued point in $SP_M$ satisfies $0 \leq y_{1j} \leq 2$ and $0 \leq y_{ij} \leq 1$ for all $1 \leq i < j \leq n$ and by construction $SP_M$ includes all the points with $y$-variables corresponding to a permissible itinerary. On the other hand, those points $(y, \mathbf{0})$ with $y$-variables corresponding to itineraries having subtours or tours visiting more than $p$ cities are also in $SP_M$. To prove this we note that the left-hand side of (80) and (81) is at most $p|E_1| + p|E_2|$ for all points $(y, \mathbf{0})$ having $y_{ij} \in \{0, 1\}$ for all $2 \leq i < j \leq n$ and that

$$p|E_1| + p|E_2| \leq 2(p-1)(|E_1| + |E_2|)$$

holds for all $p \geq 2$ and all sets $E_1$ and $E_2$. Consequently, the symmetrization of the MTZ-formulation does *not* provide a formulation of the symmetric TSP.

The explanation for this (unexpected) result is simply that the constraints (27) and (28) of the MTZ-formulation eliminate all zero-one solutions to (23)–(26) that correspond to infeasible itineraries, but that for every pair of *oppositely* directed infeasible itineraries e.g. the midpoint of the line joining the two corresponding zero-one points is contained in $UP_M$. Consequently, for every such pair the symmetrization produces an integer point in $SP_M$ that corresponds to the undirected infeasible itinerary. In addition, it follows that if $c_{ij} = c_{ji}$ for all $i$ and $j$ then the upper bound on the minimum-cost tour obtained from the MTZ-formulation equals the upper bound obtained from the "assignment problem" relaxation, i.e. the minimum of the linear form over the constraints (23)–(26) and (29). This shows that the MTZ-formulation is a very *weak* formulation of the asymmetric TSP.

Like in Section 4.1, one can symmetrize the modified DFJ-formulation of Section 3.2.1 to obtain a *better* formulation of the *symmetric* clover-leaf TSP that involves an exponential number of constraints corresponding to the constraints (31). A relative of the latter problem is studied in Araque (1988) from a polyhedral point of view.

### *4.3. Symmetrization of the FGG-formulation*

We define a linear transformation $y = Lz$ as follows:

$$y_{1j1} = z_{1j1}, \quad 2 \leq j \leq n,$$

$$y_{1jn} = z_{j1n}, \quad 2 \leq j \leq n,$$

$$y_{ijt} = z_{ijt} + z_{jit}, \quad 2 \leq i < j \leq n, \ 2 \leq t \leq n-1,$$

and define the symmetric FGG-polytope $SP_F$ as

$$SP_F = \{y \in \mathbb{R}^p \mid \exists z \in TP_F \text{ such that } y = Lz\}$$

where $TP_F$ is the FGG-polytope defined in Section 3.3 and $p = 2(n-1) + \frac{1}{2}(n-1) \times (n-2)^2$. It follows that to every *undirected* tour of the travelling salesman there corresponds a zero-one point of $SP_F$. To show that the converse holds we first derive the linear description of the polytope $SP_F$.

In matrix form, this linear transformation is written as $L=(L_1, L_2)$ where

$$L_1=\begin{bmatrix} I_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & I_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & I_m \end{bmatrix}, \qquad L_2=\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ I_m & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & I_m \end{bmatrix},$$

and $m=\frac{1}{2}(n-1)(n-2)$. The matrix $A$ and the vector $b$ defining the constraint system and corresponding right-hand side are the same as in Section 3.3. The matrix $A$ can be partitioned according to $(L_1, L_2)$ as

$$A_1=\begin{bmatrix} e_{n-1} & e_{n-1} & e_m & \cdots & e_m \\ -I_{n-1} & nI_n & 2M & \cdots & (n-1)M \end{bmatrix},$$

$$A_2=\begin{bmatrix} e_m & \cdots & e_m \\ -2M & \cdots & -(n-1)M \end{bmatrix}.$$

Then we have

$$A_2-A_1L_1^{-1}L_2=\begin{bmatrix} 0 & \cdots & 0 \\ -4M & \cdots & -2(n-1)M \end{bmatrix}.$$

It follows that the cone associated with this transformation is given by

$$C=\{(u_0, u, w)\,|\,-2tuM+w^t\geq 0 \text{ for } 2\leq t\leq n-1,\ w\geq 0\},$$

where $u_0\in\mathbb{R}$, $u\in\mathbb{R}^{n-1}$ and $w=(w^0, w^1, w^2, \ldots, w^{n-1})$ where $w^0, w^1\in\mathbb{R}^{n-1}$ and $w^t\in\mathbb{R}^m$ for $2\leq t\leq n-1$. Let $y=(y^0, y^1, y^2, \ldots, y^{n-1})$ where $y^0=(y_{121}, \ldots, y_{1n1})$, $y^1=(y_{12n}, \ldots, y_{1nn})$ and $y^t=(y_{23t}, \ldots, y_{2nt}, \ldots, y_{n-1,n,t})$ for $2\leq t\leq n-1$. Then the linear description of $SP_F$ is given by

$$SP_F=\left\{y\,\middle|\,(u_0e_{n-1}-u-w^0)y^0+(u_0e_{n-1}+nu-w^1)y^1 + \sum_{t=2}^{n-1}(u_0e_m+tuM-w^t)y^t\leq nu_0+ue_{n-1}\ \forall(u_0, u, w)\in C\right\}.$$

The lineality space of the cone is given by $(\pm 1, \mathbf{0}, \mathbf{0})$ and $(0, \pm e_{n-1}, \mathbf{0})$ and yields the following equations:

$$\sum_{t=1}^{n}\sum_{1\leq i<j\leq n} y_{ijt}=n, \tag{82}$$

$$-\sum_{j=2}^{n} y_{1j1}+n\sum_{j=2}^{n} y_{1jn}=n-1. \tag{83}$$

The unit vectors associated with $w^0, w^1$ are generators of $C$ and yield the non-negativity constraints

$$y_{1j1}\geq 0,\ \ y_{1jn}\geq 0, \quad \text{for } j=2, \ldots, n. \tag{84}$$

We intersect the cone with the linealities and the previous generators to find the remaining extreme rays of $C$. It suffices to analyze the cone

$$C_1 = \{(u, w) \mid -2tuM + w^t \geq 0,\ w^t \geq 0 \text{ for } t = 2, \ldots, n-1\}.$$

Making the variable substitution $\tilde{w}^t = 1/(2t)w^t$, the cone to be analyzed is given by

$$C_2 = \{(u, \tilde{w}) \mid -u_i + u_j + \tilde{w}_{ijt} \geq 0,\ \tilde{w}_{ijt} \geq 0 \text{ for } 2 \leq i < j \leq n,\ 2 \leq t \leq n-1\}.$$

Consequently, we have extreme rays corresponding to all unit vectors of the $(\tilde{w}^2, \ldots, \tilde{w}^{n-1})$-vector. These generators produce the nonnegativity constraints

$$y_{ijt} \geq 0, \quad 2 \leq i < j \leq n,\ 2 \leq t \leq n-1. \tag{85}$$

Moreover, every extreme ray of $C_2$ with $u \neq 0$ satisfies $\tilde{w}^2 = \cdots = \tilde{w}^{n-1}$ and thus, it suffices to determine all extreme rays of the cone

$$C_3 = \{(u, \tilde{w}) \mid -u_i + u_j + \tilde{w}_{ij} \geq 0,\ \tilde{w}_{ij} \geq 0,\ 2 \leq i < j \leq n\}.$$

For all generators of $C_3$ the linear inequalities of $\mathrm{SP_F}$ are given by

$$\sum_{j=2}^{n} u_j(-y_{1j1} + ny_{1jn}) + \sum_{t=2}^{n-1} t \sum_{2 \leq i < j \leq n} (u_i - u_j - 2\tilde{w}_{ij}) y_{ijt} \leq \sum_{j=2}^{n} u_j.$$

Using the results of Proposition 6, we obtain the generator system corresponding to the conical part of $C_3$. These generators give rise to the following inequalities:

$$\sum_{j \in S} y_{1j1} - n \sum_{j \in S} y_{1jn} - \sum_{t=2}^{n-1} t \left( \sum_{\substack{2 \leq i < j \leq n \\ i \in S, j \in \bar{S} - \{1\}}} y_{ijt} + \sum_{\substack{2 \leq i < j \leq n \\ i \in \bar{S} - \{1\}, j \in S}} y_{ijt} \right) \leq -|S|, \tag{86}$$

$$-\sum_{j \in S} y_{1j1} + n \sum_{j \in S} y_{1jn} - \sum_{t=2}^{n-1} t \left( \sum_{\substack{2 \leq i < j \leq n \\ i \in S, j \in \bar{S} - \{1\}}} y_{ijt} + \sum_{\substack{2 \leq i < j \leq n \\ i \in \bar{S} - \{1\}, j \in S}} y_{ijt} \right) \leq |S|, \tag{87}$$

for $S \subseteq V - \{1\}$, $1 \leq |S| \leq n-2$. Using (83) the inequality (86) for $S$ is equivalent to (87) for $\bar{S} - \{1\}$. Hence, we have the following theorem for the symmetrization of the FGG-formulation.

**Theorem 8.** *The symmetrization* $\mathrm{SP_F}$ *of the FGG-polytope* $\mathrm{TP_F}$ *is given by*

$$\mathrm{SP_F} = \{y \in \mathbb{R}^p \mid y \text{ satisfies } (82)\text{–}(86)\}. \quad \square$$

Moreover, we have the following result for the symmetrization of the FGG-formulation:

**Proposition 8.** *Every* 0–1 *point of* $\mathrm{SP_F}$ *corresponds to a tour.*

**Proof.** Let $y$ be a 0–1 point of $\mathrm{SP_F}$. We define the support of $y$, $S(y)$ as $\{(i, j) \mid y_{ijt} > 0\}$. Since the constraints (82) and (83) imply $\sum_{j=2}^{n} y_{1j1} = \sum_{j=2}^{n} y_{1jn} = 1$, node 1 has a degree of 2 in the support graph of $y$, $G(y) = (V, S(y))$. Furthermore, $G(y)$ does not have an isolated node otherwise (86) is violated for such a node. Suppose that there exists a node $v \neq 1$ having a degree of 1. If $y_{1vn} = 1$, the left-hand side of (86) for $S = \{v\}$ is $n$, which leads to a violation of (86) for $S = \{v\}$. If $y_{1vn} = 0$, the constraint (86) is violated for $S = V - \{1, v\}$. Consequently, every node $v \in V$ has a degree at least two and moreoever, $G(y)$ contains at least one cycle because of (82). Suppose that $G(y)$ is not connected, and let $C = (N, E)$ with $N \subset V$, $E \subset S(y)$ be a component of $G(y)$ with $1 \in N$. The constraint (86) is violated for the set $S = N - \{1\}$. Therefore, $G(y)$ is connected and the proposition follows. □

The symmetrization of the FGG-formulation thus yields a formulation for a symmetric "time-dependent" TSP. But Proposition 8 is less surprising than it may seem at first sight since our symmetrization leaves the index $t$ for the "position" number of an edge in the formulation. This index captures and "preserves" a certain asymmetry, a fact that is born out by the close resemblance of the linear description of $SP_F$ and of $P_F$ (see Lemma 3), respectively. In order to eliminate this residual "asymmetry" we could, of course, analyze the linear transformation given by

$$y_{ij} = \sum_t z_{ijt} + z_{jit}.$$

The results of Section 3.3 (Theorem 4) and of Section 4.1 (Theorem 6) prove, however, that we cannot get any new information about the facial structure of the symmetric TSP polytope $Q^n$ this way.

## 5. Conclusions

The transformation technique described in this paper is a generally applicable tool that permits one to compare different formulations of a given combinatorial optimization problem *analytically*. The comparison of different problem formulations was previously done by comparing *empirically* the performance of the respective formulations on some set of problem instances. As this paper shows it is not overly difficult to mathematically analyze different formulations of the travelling salesman problem.

The specific main findings of the work presented here are negative in the sense that not any one of the alternative formulations of the TSP considered here has yielded new insights about the facial structure of the (symmetric and asymmetric) polytopes associated with the Dantzig–Fulkerson–Johnson formulation. When we began this work, we hoped for positive results. From among the problems studied here it seems worthwhile to investigate the time-dependent travelling salesman problem in more detail from a polyhedral point of view because it permits more general objective functions than the standard formulation. The search for "more compact" formulations of this problem is, however, ill-directed if the envisaged solution method relies (directly or indirectly) relaxation of the formulation of the problem. Indeed, it is well-known that *any* integer program in bounded variables can be "reformulated" as a knapsack problem. If we apply e.g. Lemma 1 of Padberg

(1972) with $Q=n^3$ to the FGG-formulation (35)-(38) then we find that the zero-one solutions to the knapsack equation

$$\sum_{i=2}^{n}(1-n^{3(n+1-i)})z_{1i1}+\sum_{i=2}^{n}(1+n^{3(n+1-i)+1})z_{i1n}$$
$$+\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{t=2}^{n-1}tn^{3(n+1-i)}(z_{ijt}-z_{jit})+\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{t=2}^{n-1}z_{ijt}=n+\sum_{i=2}^{n}n^{3(n+1-i)}$$

are exactly the incidence vectors of tours. We note that $O(n^5)$ bits suffice to represent this equation on a digital computer. So — at least theoretically — this "super-compact" formulation of the TSP looks acceptable. Needless to say, we do not recommend its use in actual calculation.

A consistent fact that can be inferred from the results of Section 4 is the following: All of the asymmetric formulations involving polynomially many linear constraints result in constraint sets that are of exponential size when they are symmetrized. Evidently, the "breaking up" of an *undirected* edge into two oppositely *directed* edges permits one to capture information that results in the saving. While we did not symmetrize the Claus-formulation — it is clear by Theorem 5 that from a polyhedral point of view there is nothing to be gained from it — a symmetrization of that formulation will also result into a constraint set of exponential size. To prove this point we note that the minimum-cost $(s,t)$-flow problem on a network with node set $V$, a source $s$ and a sink $t$ is part of the C-formulation. So consider the problem

$$\text{(FP)}\qquad \min \sum_{i,j} c_{ij}x_{ij}$$
$$\text{s.t.}\quad \sum_{i\in V} x_{si}=\sum_{i\in V} x_{it}=1,$$
$$\sum_{j\in V\cup\{t\}} x_{ij}=\sum_{j\in V\cup\{s\}} x_{ji}\quad \forall i\in V,$$
$$x_{ij}\geqslant 0\quad \forall i,j.$$

By means of the linear transformation

$$y_{si}=x_{si}\quad \forall i\in V,$$
$$y_{it}=x_{it}\quad \forall i\in V,$$
$$y_{ij}=x_{ij}+x_{ji}\quad \forall i,j\in V \text{ and } i<j,$$

we find using Proposition 6 that the symmetrized problem becomes

$$\text{(SP)}\qquad \min \sum_{i<j} c_{ij}y_{ij}$$
$$\text{s.t.}\quad \sum_{i\in V} y_{si}=\sum_{i\in V} y_{it}=1,$$
$$y(S\cup\{s\}:(V-S)\cup\{t\})\geqslant 1\quad \forall S\subseteq V,\ 1\leqslant |S|\leqslant n-1,$$
$$y_{ij}\geqslant 0\quad \forall i,j,$$

where $(S \cup \{s\} : (V-S) \cup \{t\})$ is the cut-set defined by $S$ in the *undirected* graph. As expected the symmetrized minimum-cost $(s, t)$-flow problem has exponentially many constraints. More importantly, the symmetrized problem is — of course — the problem of finding a *shortest path* from node $s$ to node $t$ and we know by the results of Lehman (1963), see also Johnson (1974), that the basic feasible solutions to (SP) are all zero-one valued. This celebrated result now follows quite easily from the fact that — by the total unimodularity of the constraint set — all basic feasible solutions to (FP) are zero-one valued. Thus by Proposition 2, for all integer $c_{ij}$ satisfying $c_{ij} = c_{ji}$ for all $i$ and $j$ the objective function value of (SP) is an integer number and consequently, by a result of Hoffman (1974) all basic feasible solutions to (SP) are integer-valued. Likewise, one can symmetrize the *maximum capacitated* $(s, t)$*-flow* and *minimum weighted* $(s, t)$*-cut* problems and derive a symmetric version of the well-known max-flow-min-cut theorem. These and other applications of the transformation technique given in this paper are left for future work, see Padberg and Sung (1989).

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