Chapter 7

Valid Inequalities for Structured Integer Programs

In Chaps. 5 and 6 we have introduced several classes of valid inequalities that can be used to strengthen integer programming formulations in a cutting plane scheme. All these valid inequalities are "general purpose," in the sense that their derivation does not take into consideration the structure of the specific problem at hand. Many integer programs have an underlying combinatorial structure, which can be exploited to derive "strong" valid inequalities, where the term "strong" typically refers to the fact that the inequality is facet-defining for the convex hull of feasible solutions.

In this chapter we will present several examples. We will introduce the *cover* and *flow cover* inequalities, which are valid whenever the constraints exhibit certain combinatorial structures that often arise in integer programming. We will introduce lifting, which is a procedure for generating facet-defining inequalities starting from lower-dimensional faces, and a particularly attractive variant known as sequence-independent lifting. When applied to the above inequalities, we obtain lifted cover inequalities and lifted flow cover inequalities, which are standard features of current branch-and-cut solvers. We also discuss the traveling salesman problem, for which the polyhedral approach has produced spectacular results. Finally we present the equivalence between separation and optimization.

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7.1 Cover Inequalities for the 0,1 Knapsack Problem

Consider the 0,1 knapsack set

$$K := \left\{ x \in \{0,1\}^n \,:\, \sum_{j=1}^{n} a_j x_j \le b \right\}$$

where $b > 0$ and $a_j > 0$ for $j \in N := \{1, \dots, n\}$.

Recall from Example 3.19 that the dimension of $\text{conv}(K)$ is $n - |J|$ where $J = \{j \in N \,:\, a_j > b\}$. In the remainder, we assume that $a_j \le b$ for all $j \in N$, so that $\text{conv}(K)$ has dimension n.

In Sect. 2.2 we introduced the concept of minimal covers. Recall that a *cover* is a subset $C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and it is *minimal* if $\sum_{j \in C \setminus \{k\}} a_j \le b$ for all $k \in C$. For any cover C, the *cover inequality* associated with C is

$$\sum_{j \in C} x_j \le |C| - 1,$$

and it is valid for $\text{conv}(K)$.

Proposition 7.1. *Let C be a cover for K. The cover inequality associated with C is facet-defining for $P_C := \text{conv}(K) \cap \{x \in \mathbb{R}^n \,:\, x_j = 0,\ j \in N \setminus C\}$ if and only if C is a minimal cover.*

Proof. Note that $\dim(P_C) = |C|$. Assume C is a minimal cover. For all $j \in C$, let x^j be the point defined by $x^j_i = 1$ for all $i \in C \setminus \{j\}$ and $x^j_i = 0$ for all $i \in (N \setminus C) \cup \{j\}$. These are $|C|$ affinely independent points in P_C that satisfy the cover inequality associated with C at equality. This shows that the cover inequality associated with C is a facet of P_C.

Conversely, suppose that C is not a minimal cover, and let $C' \subset C$ be a cover contained in C. The cover inequality associated with C is the sum of the cover inequality associated with C' and the inequalities $x_j \le 1$, $j \in C \setminus C'$. Since these inequalities are valid for P_C, the cover inequality associated with C is not facet-defining for P_C. $\square$

Proposition 7.1 shows that minimal cover inequalities define facets of $\text{conv}(K) \cap \{x \in \mathbb{R}^n \,:\, x_j = 0,\, j \in N \setminus C\}$. In the next section we will discuss the following problem: given a minimal cover C, how can one compute coefficients α_j, $j \in N \setminus C$, so that the inequality $\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \le |C| - 1$ is facet-defining for $\text{conv}(K)$?

Separation

To use cover inequalities in a cutting plane scheme, one is faced with the *separation problem*, that is, given a vector $\bar{x} \in [0,1]^n$, find a cover inequality for K that is violated by $\bar{x}$, or show that none exists. Note that a cover inequality relative to C is violated by $\bar{x}$ if and only if $\sum_{j \in C}(1 - \bar{x}_j) < 1$. Thus, deciding whether a violated cover inequality exists reduces to solving the problem

$$\zeta = \min\{\sum_{j \in C}(1 - \bar{x}_j) \ : \ C \text{ is a cover for } K\}. \tag{7.1}$$

If $\zeta \geq 1$, then $\bar{x}$ satisfies all the cover inequalities for K. If $\zeta < 1$, then an optimal cover for (7.1) yields a violated cover inequality. Note that (7.1) always has an optimal solution that is a minimal cover.

Assuming that $a_1, \dots, a_n$ and b are integer, problem (7.1) can be formulated as the following integer program

$$\begin{array}{rl} \zeta = & \min \sum_{j=1}^n (1 - \bar{x}_j) z_j \\ & \sum_{j=1}^n a_j z_j \geq b + 1 \\ & z \in \{0,1\}^n. \end{array} \tag{7.2}$$

It is worth noting that the separation problem (7.1) is NP-hard in general [239]. In practice one is interested in fast heuristics to detect violated cover inequalities. A simple example is the following: find a basic optimal solution z^* of the linear programming relaxation of (7.2) (see Exercise 3.3); if the optimal objective value of the linear programming relaxation is ≥ 1, then also $\zeta \geq 1$ and there is no violated cover inequality. Otherwise (observing that z^* has at most one fractional coordinate) output the cover $C := \{j \in N \ : \ z_j^* > 0\}$. Note that this heuristic does not guarantee that the inequality associated with C cuts off the fractional point $\bar{x}$, even if there exists a cover inequality cutting off $\bar{x}$.

7.2 Lifting

Consider a mixed integer set $S := \{x \in \mathbb{Z}_+^n \times \mathbb{R}_+^p \ : \ Ax \leq b\}$. Given a subset C of $N := \{1, \dots, n+p\}$, and a valid inequality $\sum_{j \in C} \alpha_j x_j \leq \beta$ for $\operatorname{conv}(S) \cap \{x \in \mathbb{R}^{n+p} \ : \ x_j = 0, \ j \in N \setminus C\}$, an inequality $\sum_{j=1}^{n+p} \alpha_j x_j \leq \beta$ is called a *lifting* of $\sum_{j \in C} \alpha_j x_j \leq \beta$ if it is valid for $\operatorname{conv}(S)$. In the remainder of this section we will focus on the case where $S \subseteq \{0,1\}^n$.

Proposition 7.2. *Consider a set $S \subseteq \{0,1\}^n$ such that $S \cap \{x : x_n = 1\} \neq \emptyset$, and let $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ be a valid inequality for $S \cap \{x : x_n = 0\}$. Then*

$$\alpha_n := \beta - \max\left\{\sum_{i=1}^{n-1} \alpha_i x_i \ : \ x \in S,\ x_n = 1\right\} \tag{7.3}$$

is the largest coefficient such that $\sum_{i=1}^{n-1} \alpha_i x_i + \alpha_n x_n \leq \beta$ is valid for S.

Furthermore, if $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ defines a d-dimensional face of $\mathrm{conv}(S) \cap \{x_n = 0\}$, then $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ defines a face of $\mathrm{conv}(S)$ of dimension at least $d+1$.

Proof. The inequality $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ is valid for $S \cap \{x : x_n = 0\}$ by assumption, and it is valid for $S \cap \{x : x_n = 1\}$ by definition of α_n. Thus $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ is valid for S, and α_n is the largest coefficient with such property.

Consider $d+1$ affinely independent points of $\mathrm{conv}(S) \cap \{x_n = 0\}$ satisfying $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ at equality. These points also satisfy $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ at equality. Any point $\bar{x} \in S$ with $\bar{x}_n = 1$ achieving the maximum in (7.3) gives one more point satisfying $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ at equality, and it is affinely independent of the previous ones since it satisfies $x_n = 1$. Thus $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ defines a face of $\mathrm{conv}(S)$ of dimension at least $d+1$. $\square$

Sequential Lifting. Consider a set $S := \{x \in \{0,1\}^n : Ax \leq b\}$ of dimension n, where A is a nonnegative matrix. Proposition 7.2 suggests the following way of lifting a facet-defining inequality $\sum_{j \in C} \alpha_j x_j \leq \beta$ of $\mathrm{conv}(S) \cap \{x : x_j = 0,\ j \in N \setminus C\}$ into a facet-defining inequality $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ of $\mathrm{conv}(S)$.

> Choose an ordering $j_1, \ldots, j_\ell$ of the indices in $N \setminus C$. Let $C_0 = C$ and $C_h = C_{h-1} \cup \{j_h\}$ for $h = 1, \ldots, \ell$.
>
> For $h = 1$ up to $h = \ell$, compute
>
> $$\alpha_{j_h} := \beta - \max\left\{\sum_{j \in C_{h-1}} \alpha_j x_j \ : \ x \in S,\ \ x_j = 0,\ j \in N \setminus C_h,\ \ x_{j_h} = 1\right\}. \tag{7.4}$$

By Proposition 7.2 the inequality $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ obtained this way is facet-defining for $\mathrm{conv}(S)$.

The recursive procedure outlined above is called *sequential lifting*. Note that the assumption that $A \geq 0$ implies that, for every $\bar{x} \in S$, $\{x \in \{0,1\}^n :$

$x \leq \bar{x}\} \subseteq S$. This and the fact that $\dim(S) = n$ guarantee that (7.4) is feasible. We remark that different orderings of $N \setminus C$ may produce different lifted inequalities. Furthermore, not all possible liftings can be derived from the above procedure, as the next example illustrates.

Example 7.3. Consider the 0,1 knapsack set

$$\begin{aligned} & 8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 + 6x_6 + 6x_7 \leq 22 \\ & x_j \in \{0,1\} \quad \text{for } j = 1, \ldots, 7. \end{aligned}$$

The index set $C := \{1,2,3,4\}$ is a minimal cover. The corresponding minimal cover inequality is $x_1 + x_2 + x_3 + x_4 \leq 3$.

We perform sequential lifting according to the order $5, 6, 7$. According to Proposition 7.2, the largest lifting coefficient for x_5 is

$$\alpha_5 = 3 - \max\{x_1 + x_2 + x_3 + x_4 \,:\, 8x_1 + 7x_2 + 6x_3 + 4x_4 \leq 22 - 6,\, x_1, x_2, x_3, x_4 \in \{0,1\}\}.$$

It is easily verified that $\alpha_5 = 1$. The lifting coefficient of x_6 is

$$\alpha_6 = 3 - \max\{x_1 + x_2 + x_3 + x_4 + x_5 \,:\, 8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 \leq 16,\, x_1, \ldots, x_5 \in \{0,1\}\}.$$

It follows that $\alpha_6 = 0$. Similarly $\alpha_7 = 0$. This sequence yields the inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$. By symmetry, the sequences $6, 5, 7$ and $7, 5, 6$ yield the inequalities $x_1 + x_2 + x_3 + x_4 + x_6 \leq 3$ and $x_1 + x_2 + x_3 + x_4 + x_7 \leq 3$, respectively. By Propositions 7.1 and 7.2, all these inequalities are facet-defining.

Not all possible facet-defining lifted inequalities can be obtained sequentially. As an example, consider the following lifted inequality:

$$x_1 + x_2 + x_3 + x_4 + 0.5x_5 + 0.5x_6 + 0.5x_7 \leq 3.$$

We leave it to the reader to show that the inequality is valid and facet-defining for the knapsack set. However, it cannot be obtained by sequential lifting since its lifting coefficients are fractional. ■

7.2.1 Lifting Minimal Cover Inequalities

The following theorem was proved by Balas [25].

Theorem 7.4. *Let $K := \{x \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \le b\}$, where $b \ge a_j > 0$ for all $j \in N$. Let C be a minimal cover for K, and let*

$$\sum_{j\in C} x_j + \sum_{j\in N\setminus C} \alpha_j x_j \le |C| - 1 \tag{7.5}$$

be a lifting of the cover inequality associated with C. Up to permuting the indices, assume that $C = \{1,\dots,t\}$ and $a_1 \ge a_2 \ge \dots \ge a_t$. Let $\mu_0 := 0$ and $\mu_h := \sum_{\ell=1}^h a_\ell$ for $h = 1,\dots,t$. Let $\lambda := \mu_t - b$ (note that $\lambda > 0$).

If (7.5) defines a facet of $\operatorname{conv}(K)$, then the following hold for every $j \in N \setminus C$.

(i) If, for some h, $\mu_h \le a_j \le \mu_{h+1} - \lambda$, then $\alpha_j = h$.

(ii) If, for some h, $\mu_{h+1} - \lambda < a_j < \mu_{h+1}$, then $h \le \alpha_j \le h+1$.

Furthermore, for every $j \in N\setminus C$, if $\mu_{h+1} - \lambda < a_j < \mu_{h+1}$, then there exists a facet-defining inequality of the form (7.5) such that $\alpha_j = h+1$.

Proof. Assume that (7.5) is facet-defining for $\operatorname{conv}(K)$ and let $j \in N \setminus C$. Since $0 < a_j \le b < \mu_t$, there exists an index h, $0 \le h \le t-1$, such that $\mu_h \le a_j < \mu_{h+1}$.

By Proposition 7.2, $\alpha_j \le |C| - 1 - \theta$, where

$$\theta := \max\left\{\sum_{i=1}^t x_i \, : \, \sum_{i=1}^t a_i x_i \le b - a_j, \ x \in \{0,1\}^t\right\}.$$

Observe that, since $a_1 \ge a_2 \ge \dots \ge a_t$, $\theta = |C| - k + 1$, where k is the smallest index such that $\sum_{\ell=k}^t a_\ell \le b - a_j$. Therefore $\alpha_j \le k-2$.

Since $\sum_{\ell=k}^t a_\ell = \mu_t - \mu_{k-1} = b + \lambda - \mu_{k-1}$, it follows that k is the smallest index such that $a_j \le \mu_{k-1} - \lambda$. Therefore k is the index such that $\mu_{k-2} - \lambda < a_j \le \mu_{k-1} - \lambda$.

It follows that

$$\alpha_j \le k - 2 = \begin{cases} h & \text{when } \mu_h \le a_j \le \mu_{h+1} - \lambda \\ h+1 & \text{when } \mu_{h+1} - \lambda < a_j < \mu_{h+1}. \end{cases} \tag{7.6}$$

Next we show that $\alpha_j \ge h$. We apply Proposition 7.2 to the inequality

$$\sum_{i\in C} x_i + \sum_{i\in N\setminus(C\cup\{j\})} \alpha_i x_i \le |C| - 1. \tag{7.7}$$

Since (7.5) is facet-defining, it follows that

$$\alpha_j = |C| - 1 - \max\left\{\sum_{i\in C} x_i + \sum_{i\in N\setminus(C\cup\{j\})} \alpha_i x_i \, : \, \sum_{i\in N\setminus\{j\}} a_i x_i \le b - a_j, \ x \in \{0,1\}^{N\setminus\{j\}}\right\}. \tag{7.8}$$

Observe that, since $a_1 \geq a_2 \geq \ldots \geq a_t$, (7.8) admits an optimal solution $x^* \in \{0,1\}^{N\setminus\{j\}}$ such that $x_1^* \leq x_2^* \leq \ldots \leq x_t^*$. Since $\sum_{\ell=h+1}^{t} a_\ell = \mu_t - \mu_h \geq b + \lambda - a_j > b - a_j$, we have that $x_\ell^* = 0$ for some $\ell \in \{h+1,\ldots,t\}$. It follows that $x_1^* = \ldots = x_h^* = 0$. Let $\bar{x}$ be the vector in $\{0,1\}^n$ defined by $\bar{x}_j = 0$, $\bar{x}_i = 1$ for $i = 1,\ldots,h$, and $\bar{x}_i = x_i^*$ otherwise. We have that $\bar{x} \in K$ because $\sum_{i\in N} a_i\bar{x}_i = \sum_{i\in N\setminus\{j\}} a_i x_i^* + \mu_h \leq b - a_j + \mu_h \leq b$. Since (7.5) is valid for K, it follows that $\sum_{i\in C}\bar{x}_i + \sum_{i\in N\setminus C}\alpha_i\bar{x}_i \leq |C| - 1$. Therefore

$$\alpha_j = |C| - 1 - \left(\sum_{i\in C} x_i^* + \sum_{i\in N\setminus(C\cup\{j\})} \alpha_i x_i^*\right) \geq \sum_{i\in C}(\bar{x}_i - x_i^*) + \sum_{i\in N\setminus(C\cup\{j\})} \alpha_i(\bar{x}_i - x_i^*) = h.$$

This proves (i) and (ii). We prove the last statement of the theorem. Assume $a_j > \mu_{h+1} - \lambda$. If we do sequential lifting in which we lift x_j first, it follows from the proof of (7.6) that the coefficient of x_j in the resulting inequality is $h + 1$. By Proposition 7.2 this inequality is facet-defining for $\mathrm{conv}(K)$. □

Remark 7.5. *Let K and C be as in Theorem 7.4. For every $j \in N \setminus C$, let $h(j)$ be the index such that $\mu_{h(j)} \leq a_j < \mu_{h(j)+1}$. The inequality $\sum_{j\in C} x_j + \sum_{j\in N\setminus C} h(j)x_j \leq |C| - 1$ is a lifting of the minimal cover inequality associated with C. Furthermore, if $a_j \leq \mu_{h(j)+1} - \lambda$ for all $j \in N \setminus C$, then the above is the unique facet-defining lifting.*

Example 7.6. We illustrate the above theorem on the knapsack set

$$K := \{x \in \{0,1\}^5 \;:\; 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \leq 5\}.$$

The set $C := \{3,4,5\}$ is a minimal cover. We would like to lift the inequality $x_3 + x_4 + x_5 \leq 2$ into a facet of $\mathrm{conv}(K)$. We have $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 5$, $\mu_3 = 6$ and $\lambda = 1$. Therefore $\alpha_1 = 2$ since $\mu_2 \leq a_1 \leq \mu_3 - \lambda$. Similarly $\alpha_2 = 1$ since $\mu_1 \leq a_2 \leq \mu_2 - \lambda$. By Theorem 7.4, the inequality $2x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$ defines a facet of $\mathrm{conv}(K)$. Furthermore, by Remark 7.5, this is the unique facet-defining lifting. ■

7.2.2 Lifting Functions, Superadditivity, and Sequence Independent Lifting

Let $S := \{x \in \{0,1\}^n \;:\; Ax \leq b\}$, where we assume that $A \geq 0$ and $\dim(S) = n$. Therefore $b \geq 0$. Let $C \subset N := \{1,\ldots,n\}$, and let $\sum_{j\in C}\alpha_j x_j \leq \beta$ be a valid inequality for $S \cap \{x : x_j = 0 \text{ for } j \in N \setminus C\}$.

Consider any lifting of the above inequality,

$$\sum_{j=1}^{n} \alpha_j x_j \leq \beta. \tag{7.9}$$

Let a^j denote the jth column of A. By Proposition 7.2, for all $j \in N \setminus C$, inequality (7.9) must satisfy $\alpha_j \leq f(a^j)$ (because $A \geq 0$ and $\{x \in S : x_j = 0\} \neq \emptyset$), where $f : [0, b] \to \mathbb{R}$ is the function defined by

$$\begin{aligned} f(z) \quad := \quad & \beta - \max \sum_{i \in C} \alpha_i x_i \\ & \sum_{i \in C} a^i x_i \leq b - z \\ & x_i \in \{0,1\} \quad \text{for } i \in C. \end{aligned} \tag{7.10}$$

The function $f : [0, b] \to \mathbb{R}$ is the *lifting function* of the inequality $\sum_{j \in C} \alpha_j x_j \leq \beta$.

A function $g : U \to \mathbb{R}$ is *superadditive* if $g(u + v) \geq g(u) + g(v)$ for all $u, v \in U$ such that $u + v \in U$.

Theorem 7.7. *Let $g : [0, b] \to \mathbb{R}$ be a superadditive function such that $g \leq f$. Then $\sum_{j \in C} \alpha_j x_j + \sum_{j \in N \setminus C} g(a^j) x_j \leq \beta$ is a valid inequality for S. In particular, if f is superadditive, then the inequality $\sum_{j \in C} \alpha_j x_j + \sum_{j \in N \setminus C} f(a^j) x_j \leq \beta$ is the unique maximal lifting of $\sum_{j \in C} \alpha_j x_j \leq \beta$.*

Proof. For the first part of the statement, let $\alpha_j := g(a^j)$ for $j \in N \setminus C$. Let $t := n - |C|$. Given an ordering $j_1, \ldots, j_t$ of the indices in $N \setminus C$, let $C_0 := C$ and $C_i := C_{i-1} \cup \{j_i\}$, $i = 1, \ldots, t$, and define the function $f_i : [0, b] \to \mathbb{R}$ by

$$\begin{aligned} f_i(z) \quad := \quad & \beta - \max \sum_{j \in C_{i-1}} \alpha_j x_j \\ & \sum_{j \in C_{i-1}} a^j x_j \leq b - z \\ & x_j \in \{0,1\} \quad \text{for } j \in C_{i-1}. \end{aligned} \tag{7.11}$$

Note that $f_1 = f$ and, by definition, $f_1 \geq f_2 \geq \ldots \geq f_t$. By Proposition 7.2, the inequality $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ is valid for S if $\alpha_{j_i} \leq f_i(a^{j_i})$ for $i = 1, \ldots, t$. We will show that $g \leq f_i$ for $i = 1 \ldots, t$, implying that $\alpha_{j_i} = g(a^{j_i}) \leq f_i(a^{j_i})$.

The proof is by induction on i. By assumption $g \le f_1$. Consider $2 \le i \le t$, and assume by induction that $g \le f_{i-1}$. Given $z \in [0, b]$, we need to prove that $g(z) \le f_i(z)$. Let x^* be an optimal solution of (7.11), and define $u^* := a^{j_{i-1}} x^*_{j_{i-1}}$. It follows that

$$\begin{array}{rcll} f_i(z) & = & \beta - \sum_{j \in C_{i-2}} \alpha_j x_j^* - \alpha_{j_{i-1}} x^*_{j_{i-1}} & \\ & = & \beta - \max \left\{ \sum_{j \in C_{i-2}} \alpha_j x_j : \begin{array}{l} \sum_{j \in C_{i-2}} a^j x_j \le b - z - u^* \\ x_j \in \{0,1\} \text{ for } j \in C_{i-2} \end{array} \right\} - \alpha_{j_{i-1}} x^*_{j_{i-1}} & \\ & = & f_{i-1}(z + u^*) - g(a^{j_{i-1}}) x^*_{j_{i-1}} & \\ & \ge & g(z + u^*) - g(a^{j_{i-1}}) x^*_{j_{i-1}} & \text{(because } g \le f_{i-1}) \\ & \ge & g(z + u^*) - g(a^{j_{i-1}} x^*_{j_{i-1}}) & \text{(because } g \text{ is superadditive and } x^*_{j_{i-1}} \in \mathbb{Z}_+) \\ & = & g(z + u^*) - g(u^*) & \\ & \ge & g(z) & \text{(because } g \text{ is superadditive).} \end{array}$$

For the last part of the statement, assume that $f = f_1$ is superadditive. By the first part of the statement, $\sum_{j \in C} \alpha_j x_j + \sum_{j \in N \setminus C} f(a^j) x_j \le \beta$ is valid for S. It follows from the first part of the proof that $f_1 \le f_i$ for $i = 1, \ldots, t$. Since $f_1 \ge f_2 \ge \ldots \ge f_t$, we have $f_1 = f_2 = \ldots = f_t$. This shows that $\alpha_j \le f(a^j)$, $j \in N \setminus C$, for every lifting $\sum_{j=1}^n \alpha_j x_j \le \beta$ of $\sum_{j \in C} \alpha_j x_j \le \beta$. $\square$

Note that the inequality $\sum_{j \in C} \alpha_j x_j + \sum_{j \in N \setminus C} g(a^j) x_j \le \beta$ defined in the first part of the statement of Theorem 7.7 is valid even when $S \cap \{x : x_j = 1\} = \emptyset$ for some $j \in N \setminus C$ (If (7.10) is infeasible, we set $f(z) = +\infty$).

7.2.3 Sequence Independent Lifting for Minimal Cover Inequalities

Consider the 0,1 knapsack set $K := \{x \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \le b\}$ where $0 < a_j \le b$ for all $j = 1, \ldots, n$. Let C be a minimal cover. We present a sequence independent lifting of the cover inequality $\sum_{j \in C} x_j \le |C| - 1$.

The lifting function f defined in (7.10) becomes

$$\begin{array}{rcl} f(z) & = & |C| - 1 \; - \max \sum_{j \in C} x_j \\ & & \sum_{j \in C} a_j x_j \le b - z \\ & & x_j \in \{0,1\} \;\; \text{for } j \in C. \end{array}$$

We assume without loss of generality that $C = \{1, \ldots, t\}$ with $a_1 \ge \ldots \ge a_t$. Let $\mu_0 := 0$ and, for $h = 1, \ldots, t$, let $\mu_h := \sum_{\ell=1}^h a_\ell$. Let $\lambda := \mu_t - b > 0$.

The first part of the proof of Theorem 7.4 shows that

$$f(z) = \begin{cases} 0 & \text{if } 0 \le z \le \mu_1 - \lambda \\ h & \text{if } \mu_h - \lambda < z \le \mu_{h+1} - \lambda, \text{ for } h = 1, \dots, t-1. \end{cases}$$

The function f is not superadditive in general. Consider the function g defined by

$$g(z) := \begin{cases} 0 & \text{if } z = 0 \\ h & \text{if } \mu_h - \lambda + \rho_h < z \le \mu_{h+1} - \lambda, \text{ for } h = 0, \dots, t-1 \\ h - \frac{\mu_h - \lambda + \rho_h - z}{\rho_1} & \text{if } \mu_h - \lambda < z \le \mu_h - \lambda + \rho_h, \text{ for } h = 1, \dots, t-1 \end{cases} \tag{7.12}$$

where $\rho_h = \max\{0, a_{h+1} - (a_1 - \lambda)\}$ for $h = 0, \dots, r-1$. Note that $g \le f$. It can be shown that the function g is superadditive (see [192]). Hence by Theorem 7.7 the inequality

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} g(a^j) x_j \le |C| - 1.$$

is a lifting of the minimal cover inequality.

Example 7.8. Consider the 0,1 knapsack set from Example 7.3

$$8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 + 6x_6 + 6x_7 \le 22$$
$$x_j \in \{0,1\} \quad \text{for } j = 1, \dots, 7.$$

We consider the minimal cover $C := \{1,2,3,4\}$ of Example 7.3 and the corresponding minimal cover inequality is $x_1 + x_2 + x_3 + x_4 \le 3$. We lift it with the superadditive function g defined in (7.12). Figure 7.1 plots the function. The lifted minimal cover inequality is $x_1 + x_2 + x_3 + x_4 + 0.5x_5 + 0.5x_6 + 0.5x_7 \le 3$. ■

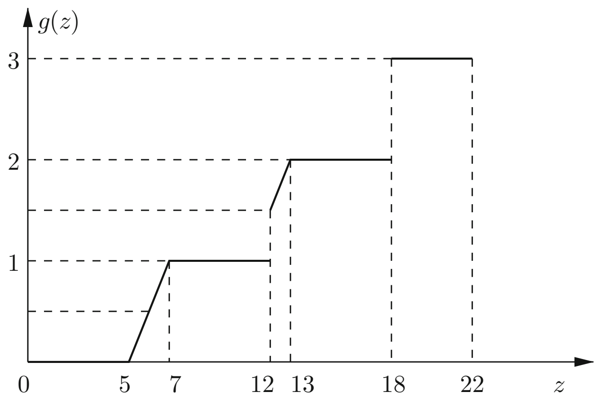


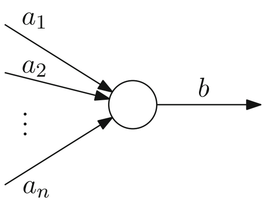
Figure 7.1: A sequence independent lifting function

7.3 Flow Cover Inequalities

The *single-node flow set* is the mixed integer linear set defined as follows

$$T := \left\{ (x,y) \in \{0,1\}^n \times \mathbb{R}^n_+ : \begin{array}{ll} \sum_{j=1}^{n} y_j \le b & \\ y_j \le a_j x_j & \text{for } j = 1,\dots,n \end{array} \right\} \tag{7.13}$$

where $0 < a_j \le b$ for all $j = 1,\dots,n$. This structure appears in many integer programming formulations that model fixed charges (Sect. 2.10). The elements of the set T can be interpreted in terms of a network consisting of n arcs with capacities $a_1,\dots,a_n$ entering the same node, and one arc of capacity b going out. The variable x_j indicates whether arc j is open, while y_j is the flow through arc j, $j = 1,\dots,n$. Note that $\dim(T) = 2n$.



Let $N := \{1,\dots,n\}$. A set $C \subseteq N$ is a *flow cover* of T if $\sum_{j\in C} a_j > b$. Let $\lambda := \sum_{j\in C} a_j - b$. The inequality

$$\sum_{j\in C} y_j + \sum_{j\in C}(a_j - \lambda)^+(1 - x_j) \le b \tag{7.14}$$

is the *flow cover inequality* defined by C.

Theorem 7.9 (Padberg et al. [303])**.** *Let C be a flow cover for the single-node flow set T, and let $\lambda := \sum_{j\in C} a_j - b$. The flow cover inequality defined by C is valid for T. Furthermore, it defines a facet of* $\operatorname{conv}(T)$ *if $\lambda < \max_{j\in C} a_j$.*

Proof. The flow cover inequality defined by C is valid for T since

$$\begin{aligned} \sum_{j\in C} y_j &\le \min\{b, \sum_{j\in C} a_j x_j\} = b - (b - \sum_{j\in C} a_j x_j)^+ = b - (\sum_{j\in C} a_j(1-x_j) - \lambda)^+ \\ &\le b - \sum_{j\in C}(a_j - \lambda)^+(1 - x_j), \end{aligned}$$

where the last inequality holds because x is a 0,1 vector.

Let F be the face of $\operatorname{conv}(T)$ defined by (7.14). Note that F is a proper face, since the point defined by $x_j = 1$, $y_j = 0$ for $j = 1, \dots, n$ is in $T \setminus F$.

Assume that $\lambda < \max_{j \in C} a_j$. We will show that F is a facet. It suffices to provide a set $X \subseteq F$ of points such that $\dim(X) = 2n - 1$.

Without loss of generality, assume that $C = \{1, \dots, k\}$, and $a_1 \geq \dots \geq a_t \geq \lambda$, $a_{t+1}, \dots, a_k < \lambda$ where $1 \leq t \leq k$.

Define the point $\tilde{x} \in \{0,1\}^n$ by $\tilde{x}_j = 1$ for $j \in C$, $\tilde{x}_j = 0$ for $j \in N \setminus C$. For $i \in C$, let $x^i := \tilde{x} - e^i$, where e^i denotes the ith unit vector. For $i \in N \setminus C$, let $x^i := \tilde{x} - e^1 + e^i$.

For $i = 1, \dots, t$, define the points $y^i, \tilde{y}^i \in \mathbb{R}^n$ by

$$y^i_j := \begin{cases} a_j & j \in C \setminus \{i\} \\ 0 & j \in (N \setminus C) \cup \{i\} \end{cases}, \quad \tilde{y}^i_j := \begin{cases} y^i_j & j \in N \setminus \{i\} \\ a_i - \lambda & j = i. \end{cases}$$

For $i = t+1, \dots, k$, define the point $y^i \in \mathbb{R}^n$ by

$$y^i_j := \begin{cases} a_1 + a_i - \lambda & j = 1 \\ a_j & j \in C \setminus \{1, i\} \\ 0 & j \in (N \setminus C) \cup \{i\}. \end{cases}$$

Finally, for $i \in N \setminus C$, let $y^i \in \mathbb{R}^n$ be defined by

$$y^i_j := \begin{cases} y^1_j & j \in N \setminus \{i\} \\ \min\{a_i, a_1 - \lambda\} & j = i. \end{cases}$$

Let X be the following set of $2n$ points in $\{0,1\}^n \times \mathbb{R}^n$: (x^i, y^i) for $i \in N$; (x^i, y^1) for $i \in N \setminus C$; $(\tilde{x}, \tilde{y}^i)$ for $i = 1, \dots, t$; $(\tilde{x}, y^i)$ for $i = t+1, \dots, k$.

One can verify that $X \subseteq T \cap F$. We will conclude by showing that $\dim(X) = 2n - 1$. It suffices to show that the system

$$\alpha x + \beta y = \gamma \quad \text{for all } (x, y) \in X, \tag{7.15}$$

in the variables $(\alpha, \beta, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, has a unique nonzero solution up to scalar multiplication. Consider such a nonzero solution (α, β, γ).

Let $i \in N \setminus C$. Then $\alpha_i = (\alpha x^i + \beta y^1) - (\alpha x^1 + \beta y^1) = \gamma - \gamma = 0$. Similarly, $\min\{a_i, a_1 - \lambda\}\beta_i = (\alpha x^i + \beta y^i) - (\alpha x^i + \beta y^1) = 0$, implying $\beta_i = 0$. This shows $\alpha_i = \beta_i = 0$ for all $i \in N \setminus C$.

For $i = 2, \dots, t$, $\lambda(\beta_1 - \beta_i) = (\alpha \tilde{x} + \beta \tilde{y}^i) - (\alpha \tilde{x} + \beta \tilde{y}^1) = 0$. For $i = t+1, \dots, k$, $a_i(\beta_1 - \beta_i) = (\alpha \tilde{x} + \beta y^i) - (\alpha \tilde{x} + \beta \tilde{y}^1) = 0$. This shows that $\beta_i = \beta_1$ for all $i \in C$.

For $i = t+1, \dots, k$, $\alpha_i = (\alpha\tilde{x} + \beta y^i) - (\alpha x^i + \beta y^i) = 0$. For $i = 1, \dots, t$, $\alpha_i + \beta_i(a_i - \lambda) = (\alpha\tilde{x} + \beta\tilde{y}^i) - (\alpha x^i + \beta y^i) = 0$, thus $\alpha_i = -\beta_1(a_i - \lambda)$. Since (α, β) is not the zero vector, it follows that $\beta_1 \neq 0$, and up to rescaling we may assume that $\beta_1 = 1$.

Finally, substituting $(\tilde{x}^1, \tilde{y}^1)$ into (7.15) gives $\gamma = b - \sum_{j \in C}(a_j - \lambda)^+$.

Therefore the points in F defined above generate the affine space

$$\sum_{j \in C} y_j + \sum_{j \in C}(a_j - \lambda)^+(1 - x_j) = b.$$

This proves that (7.14) defines a facet of $\mathrm{conv}(T)$. □

Note that when the inclusion $C \subset N$ is strict, the condition $\lambda < \max_{j \in C} a_j$ is also necessary for the flow cover inequality (7.14) to define a facet of $\mathrm{conv}(T)$ (Exercise 7.14).

Example 7.10. (Minimal Knapsack Covers Are a Special Case of Flow Cover Inequalities) Consider the knapsack set $K := \{x \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \leq b\}$. Note that $\mathrm{conv}(K)$ is isomorphic to the face of the single-node flow set $\mathrm{conv}(T)$ defined in (7.13), namely the face $\mathrm{conv}(T) \cap \{(x, y) : y_j = a_j x_j, j = 1, \dots, n\}$.

Let C be a minimal cover for K. Then C is a flow cover for T. Substituting $a_j x_j$ for y_j, for all $j = 1, \dots, n$, in the expression (7.14) of the flow cover inequality relative to C, we obtain the following valid inequality for K

$$\sum_{j \in C} a_j x_j + \sum_{j \in C}(a_j - \lambda)^+(1 - x_j) \leq b.$$

Note that, since C is a minimal cover, $a_j > \lambda$ for $j = 1, \dots, n$, thus $(a_j - \lambda)^+ = a_j - \lambda$. Rearranging the terms in the expression above, we obtain

$$\lambda \sum_{j \in C} x_j \leq b - \sum_{j \in C} a_j + |C|\lambda = (|C| - 1)\lambda.$$

The above is the knapsack cover inequality relative to C multiplied by λ. ■

Example 7.11. (Application to Facility Location) Consider the facility location problem described in Sect. 2.10.1. The problem can be written in the form

$$\begin{array}{rrcll}
\min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}y_{ij} & + & \displaystyle\sum_{j=1}^{n} f_j x_j & \\
& \displaystyle\sum_{j=1}^{n} y_{ij} & = & d_i & i=1,\dots,m \\
& \displaystyle\sum_{i=1}^{m} y_{ij} & \leq & u_j x_j & j=1,\dots,n \\
& y & \geq & 0 & \\
& x & \in & \{0,1\}^n. &
\end{array}$$

Note that the above formulation differs slightly from the one in Sect. 2.10.1, in that here y_{ij} represents the amount of goods transported from facility i to client j, whereas in Sect. 2.10.1 y_{ij} represented the fraction of demand of customer i satisfied by facility j. Nonetheless the two formulations are obviously equivalent. Let us introduce the variables z_j, $j = 1, \dots, n$, where

$$z_j = \sum_{i=1}^{m} y_{ij}.$$

If we define $b := \sum_{i=1}^{m} d_i$, then the points $(x, z) \in \{0,1\} \times \mathbb{R}^n$ corresponding to feasible solutions must satisfy the constraints

$$\begin{array}{rcll}
\displaystyle\sum_{j=1}^{n} z_j & \leq & b & \\
z_j & \leq & u_j x_j & j=1,\dots,n \\
z_j & \geq & 0 & j=1,\dots,n \\
x_j & \in & \{0,1\} & j=1,\dots,n.
\end{array}$$

This defines a single-node flow set. Any known family of valid inequalities for the single-node flow set, such as the flow cover inequalities, can therefore be adopted to strengthen the formulation of the facility location problem. ■

Theorem 7.9 shows that, whenever $\lambda < \max_{j \in C} a_j$, the inequality $\sum_{j \in C} y_j + \sum_{j \in C}(a_j - \lambda)^+(1 - x_j) \leq b$ can be lifted into a facet of $\mathrm{conv}(T)$ by simply setting to 0 the coefficients of the variables x_j, y_j, for $j \in N \setminus C$. The next section provides other ways of lifting the coefficients of x_j, y_j, for $j \in N \setminus C$.

Lifted Flow Cover Inequalities

Let C be a flow cover for the single-node flow set T defined in (7.13), where $0 < a_j \leq b$ for all $j = 1, \ldots, n$. Let $\lambda := \sum_{j \in C} a_j - b$. Throughout this section, we assume that $\lambda < \max_{j \in C} a_j$.

By Theorem 7.9, the flow cover inequality defined by C is facet-defining for $\mathrm{conv}(T)$. We intend to characterize the pairs of coefficients (α_j, β_j), $j \in N \setminus C$, such that the inequality

$$\sum_{j \in C} y_j + \sum_{j \in C} (a_j - \lambda)^+ (1 - x_j) + \sum_{j \in N \setminus C} (\alpha_j y_j + \beta_j x_j) \leq b \tag{7.16}$$

is facet-defining for $\mathrm{conv}(T)$.

Let $C := \{j_1, \ldots, j_t\}$ and assume $a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_t}$. Let $\mu_0 := 0$ and $\mu_h := \sum_{\ell=1}^{h} a_{j_\ell}$, $h = 1, \ldots, t$. Assume also that $N \setminus C = \{1, \ldots, n-t\}$ and let $T^i := T \cap \{(x, y) \in \mathbb{R}^{2n} : x_j = y_j = 0,\ j = i+1, \ldots, n-t\}$, $i = 0, \ldots, n-t$.

Suppose we want to sequentially lift the pairs of variables (x_i, y_i) starting from $i = 1$ up to $i = n - t$. That is, once we have determined pairs of coefficients $(\alpha_1, \beta_1), \ldots, (\alpha_{i-1}, \beta_{i-1})$ so that

$$\sum_{j \in C} y_j + \sum_{j \in C} (a_j - \lambda)^+ (1 - x_j) + \sum_{j=1}^{i-1} (\alpha_j y_j + \beta_j x_j) \leq b \tag{7.17}$$

is facet-defining for $\mathrm{conv}(T^{i-1})$, we want to find coefficients (α_i, β_i) such that

$$\sum_{j \in C} y_j + \sum_{j \in C} (a_j - \lambda)^+ (1 - x_j) + \sum_{j=1}^{i} (\alpha_j y_j + \beta_j x_j) \leq b \tag{7.18}$$

is facet-defining for $\mathrm{conv}(T^i)$.

Let $f_i : [0, b] \to \mathbb{R}$ be the function defined by

$$\begin{aligned} f_i(z) := \quad & b - \max \sum_{j \in C} y_j + \sum_{j \in C} (a_j - \lambda)^+ (1 - x_j) + \sum_{j=1}^{i-1} (\alpha_j y_j + \beta_j x_j) \\ & \sum_{j \in C} y_j + \sum_{j=1}^{i-1} y_j \leq b - z \\ & 0 \leq y_j \leq a_j x_j,\ x_j \in \{0, 1\} \quad j \in C \cup \{1, \ldots, i-1\}. \end{aligned} \tag{7.19}$$

Note that, since (7.17) is valid for $\mathrm{conv}(T^{i-1})$, $f_i(z) \geq 0$ for $z \in [0, b]$. It follows from the definition that $f_1 \geq f_2 \geq \ldots, f_{n-t}$. The function $f := f_1$ is called the *lifting function* for C.

Lemma 7.12. *Assume that* (7.17) *is valid for* $\mathrm{conv}(T^{i-1})$. *Then* (7.18) *is valid for* $\mathrm{conv}(T^i)$ *if and only if* (α_i, β_i) *satisfies*

$$\alpha_i y_i + \beta_i \leq f_i(y_i) \text{ for all } y_i \in [0, a_i].$$

Furthermore, if (7.17) *defines a facet of* $\mathrm{conv}(T^{i-1})$, *then* (7.18) *is a facet of* $\mathrm{conv}(T^i)$ *if and only if it is valid for* $\mathrm{conv}(T^i)$ *and there exist* $y'_i, y''_i \in [0, a_i]$, $y'_i \neq y''_i$, *such that* $\alpha_i y'_i + \beta_i = f_i(y'_i)$ *and* $\alpha_i y''_i + \beta_i = f_i(y''_i)$.

Proof. By definition of the function f_i, (7.18) is valid for $\mathrm{conv}(T^i)$ if and only if (α_i, β_i) satisfies $\alpha_i y_i + \beta_i x_i \leq f_i(y_i)$ for all $(x_i, y_i) \in \{0,1\} \times \mathbb{R}_+$ such that $y_i \leq a_i x_i$. Since $f_i \geq 0$, such condition is verified if and only if $\alpha_i y_i + \beta_i \leq f_i(y_i)$ for all $y_i \in [0, a_i]$.

For the second part of the lemma, assume that (7.17) defines a facet of $\mathrm{conv}(T^{i-1})$. So, in particular, there exists a set $X \subseteq T^{i-1}$ of points satisfying (7.17) to equality such that $\dim(X) = \dim(T^{i-1}) - 1$. Suppose there exist $y'_i, y''_i \in [0, a_i]$, $y'_i \neq y''_i$, such that $\alpha_i y'_i + \beta_i = f_i(y'_i)$ and $\alpha_i y''_i + \beta_i = f_i(y''_i)$. Then there exist points $(\bar{x}', \bar{y}')$ and $(\bar{x}'', \bar{y}'')$ in T^i that are optimal solutions to (7.19) for $z = y'_i$ and $z = y''_i$, respectively, and where $(\bar{x}'_i, \bar{y}'_i) = (1, y'_i)$ and $(\bar{x}''_i, \bar{y}''_i) = (1, y''_i)$. Then the points in $X \cup \{(\bar{x}', \bar{y}'), (\bar{x}'', \bar{y}'')\} \subseteq T^i$ satisfy (7.18) at equality and $\dim(X \cup \{(\bar{x}', \bar{y}'), (\bar{x}'', \bar{y}'')\}) = \dim(X) + 2 = \dim(T^i) - 1$.

Conversely, assume that (7.18) defines a facet of $\mathrm{conv}(T^i)$. Then there exist two linearly independent points (x', y'), (x'', y'') in T_i satisfying (7.18) at equality such that $(x'_i, y'_i) \neq (0, 0)$ and $(x''_i, y''_i) \neq (0, 0)$. If follows that $x'_i = x''_i = 1$, $y'_i \neq y''_i$, $\alpha_i y'_i + \beta_i = f_i(y'_i)$ and $\alpha_i y''_i + \beta_i = f_i(y''_i)$. □

Lemma 7.13. *Let* $r := \max\{i \in C : a_{j_i} > \lambda\}$. *For* $z \in [0, b]$, *the lifting function for* C *evaluated at* z *is*

$$f(z) = \begin{cases} h\lambda, & \text{if } \mu_h \leq z < \mu_{h+1} - \lambda,\ h = 0, \dots, r-1 \\ z - \mu_h + h\lambda, & \text{if } \mu_h - \lambda \leq z < \mu_h,\ h = 1, \dots, r-1, \\ z - \mu_r + r\lambda, & \text{if } \mu_r - \lambda \leq z \leq b. \end{cases}$$

Proof. Recall that

$$f(z) := b - \max\{\sum_{j \in C} (y_j + (a_j - \lambda)^+ (1 - x_j)) : \sum_{j \in C} y_j \leq b - z,\ y_j \leq a_j x_j,\ x_j \in \{0,1\},\ j \in C\}.$$

Consider a point (x, y) achieving the maximum in the above equation. For $i = r+1, \dots, t$, we can assume that $x_{j_i} = 1$, since $(a_{j_i} - \lambda)^+ = 0$.

Assume that $\sum_{i=r+1}^{t} a_{j_i} \geq b - z$, which is the case if and only if $\mu_r - \lambda \leq z$. Then the maximum is achieved by setting $x_{j_i} = 0$ for $i = 1, \dots, r$ and setting the value of y_{j_i}, $i = r+1, \ldots, t$, so that $\sum_{i=r+1}^{t} y_{j_i} = b - z$. The value of the objective function is then $b - (b - z + \sum_{i=1}^{r}(a_{j_i} - \lambda)) = z - \mu_r + r\lambda$.

Assume next that $z < \mu_r - \lambda$. Then $\mu_h - \lambda \leq z < \mu_{h+1} - \lambda$ for some h, $0 \leq h \leq r-1$. Observe that we can assume that $x_{j_1} \leq x_{j_2} \leq \ldots \leq x_{j_r}$. Indeed, given $i < \ell \leq r$, if $x_{j_i} = 1$ and $x_{j_\ell} = 0$, then the solution (x', y') obtained from (x, y) by setting $x'_{j_i} = 0$, $x'_{j_\ell} = 1$, $y'_{j_i} = 0$, $y'_{j_\ell} = (y_{j_\ell} - a_{j_i} + a_{j_\ell})^+$ has value greater than or equal to that of (x, y), and it is feasible because $a_{j_i} \geq a_{j_\ell}$.

Note that it is optimal to set $x_{j_\ell} = 1$, $y_{j_\ell} = a_{j_\ell}$ for $\ell = h+2, \ldots, t$ because $\sum_{\ell=h+2}^{t} a_{j_\ell} = b + \lambda - \mu_{h+1} < b - z$; and it is optimal to set $x_{j_\ell} = 0$, $y_{j_\ell} = 0$ for $\ell = 1, \ldots, h$ because $\sum_{\ell=h+1}^{t} a_{j_\ell} = b + \lambda - \mu_h \geq b - z$. It remains to determine optimal values for $x_{j_{h+1}}$ and $y_{j_{h+1}}$.

If $z \geq \mu_h$, then $b - z - \sum_{\ell=h+2}^{t} a_{j_\ell} \leq a_{j_{h+1}} - \lambda$, so an optimal solution is

$$x_{j_i} = \begin{cases} 0 & i = 1, \ldots, h+1, \\ 1 & i = h+2, \ldots, t, \end{cases} \qquad y_{j_i} = \begin{cases} 0 & i = 1, \ldots, h+1 \\ a_{j_i} & i = h+2, \ldots, t. \end{cases}$$

Thus $f(z) = b - \sum_{i=h+2}^{t} a_{j_i} - \sum_{i=1}^{h+1}(a_{j_i} - \lambda) = h\lambda$.

If $z < \mu_h$, then $b - z - \sum_{\ell=h+2}^{t} a_{j_\ell} > a_{j_{h+1}} - \lambda$, so an optimal solution is

$$x_{j_i} = \begin{cases} 0 & i = 1, \ldots, h, \\ 1 & i = h+1, \ldots, t, \end{cases} \qquad y_{j_i} = \begin{cases} 0 & i = 1, \ldots, h \\ b - z - \sum_{\ell=h+2}^{t} a_{j_\ell} & i = h+1 \\ a_{j_i} & i = h+2, \ldots, t. \end{cases}$$

Thus $f(z) = b - (b - z - \sum_{\ell=h+2}^{t} a_{j_\ell}) - \sum_{i=h+2}^{t} a_{j_i} - \sum_{i=1}^{h}(a_{j_i} - \lambda) = z - \mu_h + h\lambda$. $\square$

Lemma 7.14. *The function f is superadditive in the interval $[0, b]$.*

The proof of the above lemma can be found in [192]. Lemma 7.14 implies that the lifting of flow cover inequalities is always sequence independent, as explained in the next lemma, which closely resembles Theorem 7.7.

Lemma 7.15. *Let C be a flow cover of T. For $i = 1, \ldots, n-t$, the function f_i defined in (7.19) coincides with the lifting function f.*

Proof. Let $i \geq 2$ and assume by induction that $f = f_1 = \cdots = f_{i-1}$. Let $z \in [0, b]$ and let (x^*, y^*) be an optimal solution for (7.19). It follows from the definition of $f_i(z)$ that

$$0 \leq f_i(0) \leq f_{i-1}(y^*_{i-1}) - (\alpha_{i-1} y^*_{i-1} + \beta_{i-1} x^*_{i-1}),$$

thus $\alpha_{i-1}y^*_{i-1}+\beta_{i-1}x^*_{i-1} \le f(y^*_{i-1})$. By the choice of (x^*, y^*), it follows that $f_i(z) = f_{i-1}(z+y^*_{i-1})-(\alpha_{i-1}y^*_{i-1}+\beta_{i-1}x^*_{i-1}) \ge f(z+y^*_{i-1})-f(y^*_{i-1}) \ge f(z)$, where the last inequality follows by the superadditivity of the function f. Since the definition of f_i implies $f \ge f_i$, it follows that $f_i = f$. □

Lemma 7.15 shows that each pair (α_i, β_i), $i \in N \setminus C$, can be lifted independently of the others.

Theorem 7.16 (Gu et al. [191])**.** *Let C be a flow cover for T such that $\lambda < \max_{i\in C} a_j$. Let $r := \max\{i \in C : a_{j_i} > \lambda\}$. The inequality (7.16) is facet-defining for T if and only if, for each $i \in N \setminus C$, one of the following holds*

(i) $\alpha_i = 0$, $\beta_i = 0$;

(ii) $\alpha_i = \frac{\lambda}{a_{j_h}}$, $\beta_i = \lambda(h-1-\frac{\mu_h-\lambda}{a_{j_h}})$ *for some* $h \in \{2,\dots,r\}$ *such that* $\mu_h - \lambda \le a_i$;

(iii) $\alpha_i = 1$, $\beta_i = \ell\lambda - \mu_\ell$ *where* $a_i > \mu_\ell - \lambda$ *and either* $\ell = r$ *or* $\ell < r$ *and* $a_i \le \mu_\ell$;

(iv) $\alpha_i = \frac{\lambda}{a_i+\lambda-\mu_\ell}$, $\beta_i = \ell\lambda - \frac{\lambda a_i}{a_i+\lambda-\mu_\ell}$ *where* ℓ *is such that* $\mu_\ell < a_i \le \mu_{\ell+1}-\lambda$ *and* $\ell < r$.

Proof. By Lemmas 7.12 and 7.15, the inequality (7.16) is facet-defining for $\operatorname{conv}(T)$ if and only if, for every $i \in N \setminus C$, the line of equation $v = \alpha_i u + \beta_i$ lies below the graph of the function f in the interval $[0, a_i]$ (i.e., $\{(u, v) \in [0, a_i] \times \mathbb{R} : v = f(u)\}$), and it intersects such graph in at least two points in $[0, a_i]$. Since $a_{j_1} \ge a_{j_2} \ge \ldots \ge a_{j_t}$, then all possible such lines are a) the line passing through $(0, 0)$ and $(\mu_1 - \lambda, 0)$, b) the line passing through $(\mu_{h-1} - \lambda, f(\mu_{h-1} - \lambda))$ and $(\mu_h - \lambda, f(\mu_h - \lambda))$, if $\mu_h - \lambda \le a_i$ and $h \le r$, c) the line passing through $(\mu_\ell - \lambda, f(\mu_\ell - \lambda))$ and $(a_i, f(a_i))$ where ℓ is the largest index such that $0 \le \ell \le r$ and $a_i > \mu_\ell - \lambda$. The line of equation $v = \alpha_i u + \beta_i$ satisfies a) or b) if (α_i, β_i) satisfy (i) or (ii), respectively. If $v = \alpha_i u + \beta_i$ satisfies c), then (α_i, β_i) satisfy (iii) if $\ell = r$ or $\ell < r$ and $\mu_\ell - \lambda < a_i \le \mu_\ell$ and (iv) if $\ell < r$ and $\mu_\ell < a_i \le \mu_{\ell+1} - \lambda$. □

Example 7.17. Consider the single-node flow set

$$T := \left\{ \begin{array}{lll} (x, y) \in \{0,1\}^6 \times \mathbb{R}^6_+ : & y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \le 20 \\ & y_1 \le 17x_1,\ y_2 \le 9x_2,\ y_3 \le 8x_3 \\ & y_4 \le 6x_4,\ \ y_5 \le 5x_5,\ y_6 \le 4x_6 \end{array} \right\}$$

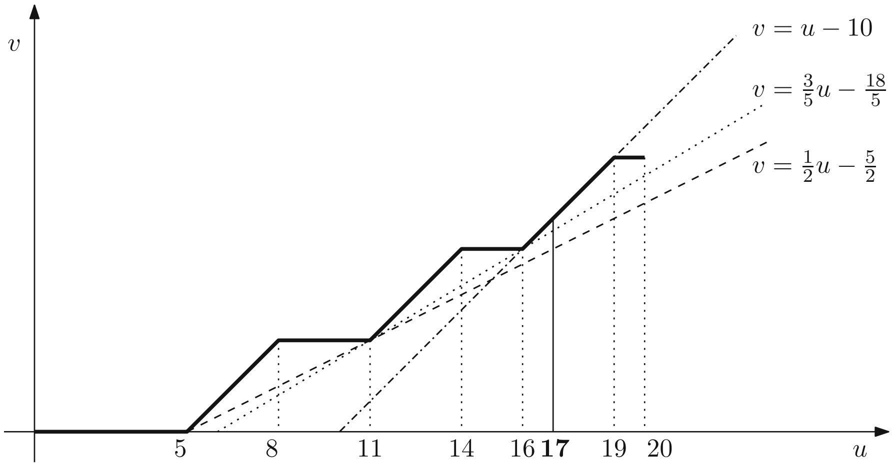


Figure 7.2: Lifting function f and possible lifting coefficients for (y_1, x_1)

Consider the flow cover $C := \{3, 4, 5, 6\}$. Note that $\mu_1 = 8$, $\mu_2 = 14$, $\mu_3 = 19$, $\mu_4 = 23$, $\lambda = 3$ and $r = 4$. For $a_1 = 17$, Case (ii) of the theorem holds for $h = 2$ and $h = 3$, and Case (iii) holds for $\ell = 3$. For $a_2 = 9$, Case (iv) holds for $\ell = 1$. Therefore it follows from Theorem 7.16 that the lifted flow cover inequality

$$\alpha_1 y_1 + \beta_1 x_1 + \alpha_2 y_2 + \beta_2 x_2 + y_3 + y_4 + y_5 + y_6 - 5x_3 - 3x_4 - 2x_5 - x_6 \leq 9$$

is facet-defining for $\text{conv}(T)$ if and only if $(\alpha_1, \beta_1) \in \{(0,0), (\frac{1}{2}, -\frac{5}{2}), (\frac{3}{5}, -\frac{18}{5}), (1, -10)\}$ and $(\alpha_2, \beta_2) \in \{(0,0), (\frac{3}{4}, -\frac{15}{4})\}$ (see Fig. 7.2).

■

7.4 Faces of the Symmetric Traveling Salesman Polytope

In this section we consider the symmetric traveling salesman problem, introduced in Sect. 2.7. Among the formulations we presented, the most successful in practice has been the Dantzig–Fulkerson–Johnson formulation (2.15), which we restate here. Let $G = (V, E)$ be the complete graph on n nodes, where $V := \{1, \ldots, n\}$.

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
& \sum_{e \in \delta(i)} x_e = 2 && \text{for } i \in V \\
& \sum_{e \in \delta(S)} x_e \geq 2 && \text{for } S \subset V \text{ s.t. } 2 \leq |S| \leq n-2 \\
& x_e \in \{0,1\} && \text{for } e \in E.
\end{aligned} \tag{7.20}$$

The convex hull of feasible solutions to (7.20) is the *traveling salesman polytope*, which will be denoted by P_{tsp}. The constraints $\sum_{e\in\delta(i)} x_e = 2$ are the degree constraints, while the constraints $\sum_{e\in\delta(S)} x_e \geq 2$ are the *subtour elimination constraints.*

Theorem 7.18. *The affine hull of the traveling salesman polytope on $n \geq 3$ nodes is $\{x \in \mathbb{R}^{\binom{n}{2}} : \sum_{e\in\delta(i)} x_e = 2\}$. Furthermore, $\dim(P_{\text{tsp}}) = \binom{n}{2} - n$.*

Proof. Note that every point in P_{tsp} must satisfy the n degree constraints $\sum_{e\in\delta(i)} x_e = 2$ for $i \in V$. We first note that such constraints are linearly independent. Indeed, let $Ax = \mathbf{2}$ be the system formed by the n degree constraints. Let A' be the $n \times n$ submatrix of A obtained by the columns corresponding to edges $1j$, $j = 2, \dots, n$ and edge 23. It is routine to show that $\det(A') = \pm 2$. Therefore $\dim(P_{\text{tsp}}) \leq \binom{n}{2} - n$. To show equality, consider the Hamiltonian-path polytope P of the complete graph on nodes $\{1, \dots, n-1\}$. We showed in Example 3.21 that $\dim(P) = \binom{n-1}{2} - 1 = \binom{n}{2} - n$, thus there exists a family $\mathcal{Q}$ of $\binom{n}{2} - n + 1$ Hamiltonian paths on $n-1$ nodes whose incidence vectors are affinely independent. Let $\mathcal{T}$ be the family of $\binom{n}{2} - n + 1$ Hamiltonian tours on nodes $\{1, \dots, n\}$ obtained by completing each Hamiltonian path $Q \in \mathcal{Q}$ to a Hamiltonian tour by adding the two edges between node n and the two endnodes of Q. Since the incidence vectors of elements in $\mathcal{Q}$ are affinely independent, the incidence vectors of the elements of $\mathcal{T}$ are $\binom{n}{2} - n + 1$ affinely independent points in P_{tsp}. □

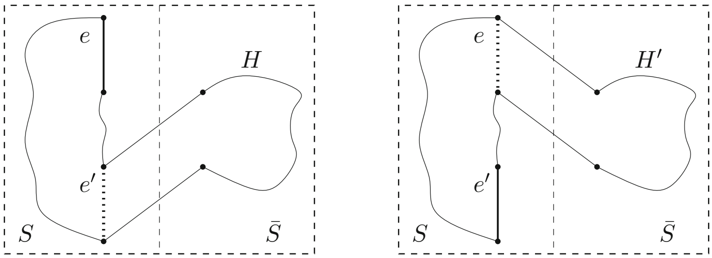
Theorem 7.19. *For $S \subset V$ with $2 \leq |S| \leq n-2$, the subtour elimination constraint $\sum_{e\in\delta(S)} x_e \geq 2$ defines a facet of the traveling salesman polytope on $n \geq 4$ nodes.*

Proof. Given $S \subset V$, $2 \leq |S| \leq n-2$, let F be the face defined by $\sum_{e\in\delta(S)} x_e \geq 2$. Then there exists some valid inequality $\alpha x \leq \beta$ for P_{tsp} which defines a facet $\bar{F}$ such that $F \subseteq \bar{F}$. We want to show that $F = \bar{F}$.

We first show that, up to linear combinations with the degree constraints, we may assume that $\alpha_e = 0$ for all $e \in \delta(S)$. Indeed, assume w.l.o.g. $1 \in S$.

By subtracting from $\alpha x \leq \beta$ the constraint $\sum_{i\in\bar{S}} \alpha_{1i} \sum_{e\in\delta(i)} x_e = 2\sum_{i\in\bar{S}} \alpha_{1i}$, we may assume that $\alpha_{1i} = 0$ for all $i \in \bar{S}$. Let $k \in S \setminus \{1\}$, and let $i, j \in \bar{S}$, $i \neq j$. Let H be a tour containing edges $1i$ and kj, and no other edge in $\delta(S)$. Note that $H' := H \cup \{1j, ki\} \setminus \{1i, kj\}$ is also a Hamiltonian tour. If $\bar{x}$ and $\bar{x}'$ are the incidence vectors of H and H', then $\bar{x}, \bar{x}' \in F \subseteq \bar{F}$, thus $\alpha\bar{x} = \alpha\bar{x}' = \beta$. It follows that $\alpha_{1i} + \alpha_{kj} = \alpha_{1j} + \alpha_{ki}$, thus $\alpha_{kj} = \alpha_{ki}$ for all $i, j \in \bar{S}$. This shows that, for all $k \in S$, there exists λ_k such that $\alpha_{ki} = \lambda_k$ for all $i \in \bar{S}$. Subtracting from $\alpha x \leq \beta$ the constraint $\sum_{k\in S} \lambda_k \sum_{e\in\delta(k)} x_e = 2\sum_{k\in S} \lambda_k$ we may assume that $\alpha_e = 0$ for all $e \in \delta(S)$.

Next, we show that there exist constants λ and $\bar{\lambda}$ such that $\alpha_e = \lambda$ for all $e \in E[S]$ and $\alpha_e = \bar{\lambda}$ for all $e \in E[\bar{S}]$. Indeed, given distinct edges $e, e' \in E[S]$, there exist Hamiltonian tours H and H' such that $|H \cap \delta(S)| = |H' \cap \delta(S)| = 2$, and $(H \triangle H') \setminus \delta(S) = \{e, e'\}$.



Let $\bar{x}$ and $\bar{x}'$ be the incidence vectors of H and H' respectively. Since $\bar{x}, \bar{x}' \in F \subseteq \bar{F}$, it follows that $\alpha\bar{x} = \alpha\bar{x}' = \beta$. Thus $\alpha_e = \alpha_{e'}$, because $\alpha_e = 0$ for all $e \in \delta(S)$ and $(H \triangle H') \setminus \delta(S) = \{e, e'\}$.

Since every tour H such that $|H \cap \delta(S)| = 2$ satisfies $|H \cap E[S]| = |S| - 1$ and $|H \cap E[\bar{S}]| = |\bar{S}| - 1$, and since $F \subseteq \bar{F}$, it follows that the equation $\alpha x = \beta$ is equivalent to $\lambda \sum_{e\in E[S]} x_e + \bar{\lambda} \sum_{e\in E[\bar{S}]} x_e = \lambda(|S| - 1) + \bar{\lambda}(|\bar{S}| - 1)$. Since the inequalities $\sum_{e\in E[S]} x_e \leq |S| - 1$ and $\sum_{e\in E[\bar{S}]} x_e \leq |\bar{S}| - 1$ both define the face F, it follows that $F = \bar{F}$. $\square$

Because there are exponentially many subtour elimination constraints, solving the linear programming relaxation of (7.20) is itself a challenge.

$$
\begin{array}{lll}
\min & \sum_{e\in E} c_e x_e & \\
& \sum_{e\in\delta(i)} x_e = 2 & \text{for } i \in V \\
& \sum_{e\in\delta(S)} x_e \geq 2 & \text{for } S \subset V \text{ s.t. } 2 \leq |S| \leq n-2 \\
& 0 \leq x_e \leq 1 & \text{for } e \in E.
\end{array} \tag{7.21}
$$

The feasible set of (7.21) is called the *subtour elimination polytope*. It is impossible to input all the subtour elimination constraints in a solver for medium or large instances (say $n \geq 30$); they must be generated as needed. One starts by solving the following linear programming relaxation.

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ & \sum_{e\in\delta(i)} x_e = 2 & \text{for } i \in V \\ & 0 \leq x_e \leq 1 & \text{for } e \in E. \end{array} \tag{7.22}$$

One then adds inequalities that are valid for the subtour elimination polytope but violated by the current linear programming solution $\bar{x}$. The linear program is strengthened iteratively until an optimal solution of (7.21) is found (we will explain how to do this shortly). But solving (7.21) is usually not enough. The formulation is strengthened further by generating additional inequalities that are valid for the traveling salesman polytope but violated by the current linear programming solution $\bar{x}$. This idea was pioneered by Dantzig et al. [103], who solved a 49-city instance in 1954. It was improved in the 1980s by Grötschel [184] and Padberg and Rinaldi [301] who solved instances with hundreds of cities, and refined by Applegate et al. [13] in the 2000s, who managed to solve instances with thousands and even tens of thousands of cities. The formulation strengthening approach mentioned above is typically combined with some amount of enumeration performed within the context of a branch-and-cut algorithm. However the generation of cutting planes is absolutely crucial. This involves solving the separation problem: given a points $\bar{x} \in \mathbb{R}^E$, find a valid inequality for the traveling salesman polytope that is violated by $\bar{x}$, or show that no such inequality exists.

7.4.1 Separation of Subtour Elimination Constraints

Assume that we have a solution $\bar{x}$ of the linear program (7.22) or of some strengthened linear program. The separation problem for subtour elimination inequalities is the following: Prove that $\bar{x}$ is in the subtour elimination polytope, or find one or more subtour elimination constraints that are violated by $\bar{x}$. Note that $\sum_{e\in\delta(S)} \bar{x}_e$ is the weight of the cut $\delta(S)$ in the graph $G = (V, E)$ with edge weights $\bar{x}_e$, $e \in E$. There are efficient polynomial-time algorithms for finding a minimum weight cut in a graph (see Sect. 4.11). If the algorithm finds that the minimum weight of a cut is 2 or more, then all subtour elimination constraints are satisfied, i.e., $\bar{x}$ is in the subtour elimination polytope. On the other hand, if the algorithm finds a cut $\delta(S^*)$ of weight strictly less than 2, the corresponding subtour elimination constraint

$\sum_{e\in\delta(S^*)} x_e \geq 2$ is violated by $\bar{x}$. One then adds $\sum_{e\in\delta(S^*)} x_e \geq 2$ to the linear programming formulation, finds an improved solution $\bar{x}$, and repeats the process.

In order to make the separation of subtour elimination inequalities more efficient, fast procedures are typically applied first before resorting to the more expensive minimum weight cut algorithm. For example, let $\bar{E} := \{e \in E : \bar{x}_e > 0\}$. If the graph $(V, \bar{E})$ has at least two connected components, any node set S^* that induces a connected component provides a violated subtour elimination constraint $\sum_{e\in\delta(S^*)} x_e \geq 2$. Identifying the connected components of a graph can be done extremely fast [335].

7.4.2 Comb Inequalities

A solution $\bar{x}$ in the subtour elimination polytope is not necessarily in the traveling salesman polytope as shown by the following example with $n = 6$ nodes. The cost between each pair of nodes is defined as follows. For the edges represented in Fig. 7.3 the costs are shown on the graph (left figure), and the cost of any edge ij not represented in the figure is the cost of a shortest path between i and j in the graph. It is easy to verify that every tour has cost at least 4, but the fractional solution $\bar{x}$ shown on the right figure has cost 3 (the value $\bar{x}_e$ on any edge not represented in Fig. 7.3 is 0). One can check directly that $\bar{x}$ satisfies all the subtour elimination constraints. We will describe a valid inequality for the traveling salesman polytope that separates $\bar{x}$.

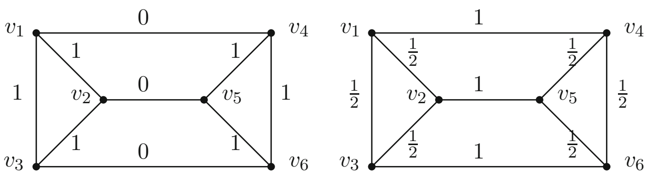


Figure 7.3: Traveling salesman problem on 6 nodes, and a fractional vertex of the subtour elimination polytope

For $k \geq 3$ odd, let $S_0, S_1, \ldots, S_k \subseteq V$ be such that $S_1, \ldots, S_k$ are pairwise disjoint, and for each $i = 1, \ldots, k$, $S_i \cap S_0 \neq \emptyset$ and $S_i \setminus S_0 \neq \emptyset$. The inequality

$$\sum_{i=0}^{k} \sum_{e\in E[S_i]} x_e \leq \sum_{i=0}^{k} |S_i| - \frac{3k+1}{2} \tag{7.23}$$

is called a *comb inequality*.

Proposition 7.20. *The comb inequality (7.23) is valid for the traveling salesman polytope.*

Proof. We show that (7.23) is a Chvátal inequality for the subtour elimination polytope. Consider the following inequalities, valid for the subtour elimination polytope.

$$\begin{array}{rcll} \sum_{e\in\delta(v)} x_e & = & 2 & v \in S_0; \\ -x_e & \leq & 0 & e \in \delta(S_0) \setminus \cup_{i=1}^k E[S_i]; \\ \sum_{e\in E[S_i]} x_e & \leq & |S_i| - 1 & i = 1,\dots,k; \\ \sum_{e\in E[S_i\setminus S_0]} x_e & \leq & |S_i \setminus S_0| - 1 & i = 1,\dots,k; \\ \sum_{e\in E[S_i\cap S_0]} x_e & \leq & |S_i \cap S_0| - 1 & i = 1,\dots,k. \end{array}$$

Summing the above inequalities multiplied by $\frac{1}{2}$, one obtains the inequality

$$\sum_{i=0}^{k} \sum_{e\in E[S_i]} x_e \leq \sum_{i=0}^{k} |S_i| - \frac{3k}{2}.$$

Observe that, since k is odd, $\lfloor -\frac{3k}{2} \rfloor = -\frac{3k+1}{2}$, therefore rounding down the right-hand side of the previous inequality one obtains (7.23). $\square$

Grötschel and Padberg [189] showed that the comb inequalities define facets of the traveling salesman polytope for $n \geq 6$.

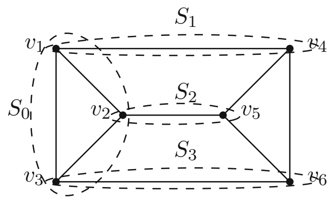


Figure 7.4: A comb

In the example of Fig. 7.3, let $S_0 = \{v_1, v_2, v_3\}$, and $S_1 = \{v_1, v_4\}$, $S_2 = \{v_2, v_5\}$, $S_3 = \{v_3, v_6\}$ (see Fig. 7.4). The corresponding comb inequality is $x_{12} + x_{13} + x_{23} + x_{14} + x_{25} + x_{36} \leq 4$. However $\bar{x}_{12} + \bar{x}_{13} + \bar{x}_{23} + \bar{x}_{14} + \bar{x}_{25} + \bar{x}_{36} = 4.5$, showing that the above comb inequality cuts off $\bar{x}$. Unlike for the subtour elimination inequalities, no polynomial algorithm is known for separating comb inequalities in general. In the special case where $|S_i| = 2$ for $i = 1,\dots,k$, comb inequalities are known as *blossom inequalities*, and there is a polynomial separation algorithm for this class (Padberg and Rao [300]). In addition to the separation of blossom inequalities, state-of-the-art software for the traveling salesman problem have sophisticated heuristics to separate more general comb inequalities. Note that, even if all comb inequalities could be separated, we would still not be done in general since the traveling salesman polytope has many other types of facets. In fact, Billera and Sarangarajan [53] showed that any 0,1 polytope is affinely equivalent to a face of an asymmetric traveling salesman polytope of sufficiently large dimension. We are discussing the symmetric traveling salesman polytope in this section, but the Billera–Sarangarajan result is a good indication of how complicated the traveling salesman polytope is. The following idea tries to bypass understanding its structure.

7.4.3 Local Cuts

In their solver for the symmetric traveling salesman problem, Applegate et al. [13] separate subtour elimination constraints and comb inequalities. But then, instead of going on separating other classes of inequalities with known structure, they introduce an interesting approach, the separation of *local cuts*. To get a sense of the contribution of each of these three steps, they considered an Euclidean traveling salesman problem in the plane with 100,000 cities (the cities were generated randomly in a square, the costs were the Euclidean distance between cities up to a small rounding to avoid irrationals), and they constructed a good feasible solution using a heuristic. The lower bound obtained using subtour elimination constraints was already less than 1 % from the heuristic solution. After adding comb inequalities, the gap was reduced to less than 0.2 %, and after adding local cuts, the gap was reduced to below 0.1 %. We now discuss the generation of local cuts.

Let $\mathcal{S} \subset \{0,1\}^E$ denote the set of incidence vectors of tours, and let $\bar{x} \in \mathbb{R}^E$ be a fractional solution that we would like to separate from $\mathcal{S}$. The idea is to map the space $\mathbb{R}^E$ to a space of much lower dimension by a linear mapping Φ and then, using general-purpose methods, to look for linear inequalities $ay \leq b$ that are satisfied by all points $y \in \Phi(\mathcal{S})$ and violated by $\bar{y} := \Phi(\bar{x})$. Every such inequality yields a cut $a\Phi(x) \leq b$ separating $\bar{x}$ from $\mathcal{S}$. For the traveling salesman problem, Applegate, Bixby, Chvátal, and Cook chose Φ as follows. Partition V into pairwise disjoint nonempty sets $V_1, \ldots, V_k$, let $H = (U, F)$ be the graph obtained from G by shrinking each set V_i into a single node u_i, and let $y = \Phi(x) \in \{0,1\}^{|F|}$ be defined by $y_{ij} = \sum_{v \in V_i} \sum_{w \in V_j} x_{vw}$ for all $ij \in F$. This mapping transforms a tour x into a vector y with the following properties.

- $y_e \in \mathbb{Z}_+$ for all $e \in F$,
- $\sum_{e\in\delta(i)} y_e$ is even for all $i \in U$,
- the subgraph of H induced by the edge set $\{e \in F : y_e > 0\}$ is connected.

The convex hull of such vectors is known as the *graphical traveling salesman polyhedron.* Let us denote it by $GTSP^k$ for a graph on k nodes. The goal is to find an inequality that separates $\bar{y}$ from the graphical traveling salesman polyhedron $GTSP^k$, or prove that $\bar{y} \in GTSP^k$. Because k is chosen to be relatively small, this separation can be done by brute force. To simplify the exposition, let us intersect $GTSP^k$ with $\sum_{e\in F} y_e \leq n$ (every $y := \Phi(x)$ satisfies this inequality since $\sum_{e\in E} x_e = n$ for $x \in \mathcal{S}$). Let $GTSP^{k,n}$ denote this polytope. We want to solve the following separation problem: Find an inequality that separates $\bar{y}$ from the polytope $GTSP^{k,n}$, or prove that $\bar{y} \in GTSP^{k,n}$. More generally, we want to solve the following separation problem.

Let $\mathcal{Y}$ be a finite set of points in $\mathbb{R}^t$. Given a point $\bar{y} \in \mathbb{R}^t$, either find an inequality that separates $\bar{y}$ from the polytope $\operatorname{conv}(\mathcal{Y})$, or prove that $\bar{y} \in \operatorname{conv}(\mathcal{Y})$.

This can be done by *delayed column generation.*

At a general iteration i, we have a set S_i of points in $\mathcal{Y}$.

At the first iteration, we initialize $S_1 := \{y^1\}$ where y^1 is an arbitrary point in $\mathcal{Y}$.

At iteration i, we check whether $\bar{y} \in \operatorname{conv}(S_i)$ (this amounts to checking the existence of a vector $u \geq 0$ satisfying $\sum_{h=1}^{i} u_h = 1$ and $\bar{y} = \sum_{h=1}^{i} y^h u_h$, which can be done by linear programming). If this is the case we have proved that $\bar{y} \in \operatorname{conv}(\mathcal{Y})$. Otherwise we find a linear inequality $ay \leq b$ separating $\bar{y}$ from $\operatorname{conv}(S_i)$ (see Proposition 7.21 below). We then solve $\max\{ay : y \in \mathcal{Y}\}$ (this is where brute force may be needed). If the solution y^{i+1} found satisfies $ay^{i+1} \leq b$, then the inequality $ay \leq b$ separates $\bar{y}$ from $\operatorname{conv}(\mathcal{Y})$. Otherwise we set $S^{i+1} := S^i \cup \{y^{i+1}\}$ and we perform the next iteration.

Proposition 7.21. *If $\bar{y} \notin \operatorname{conv}(S_i)$, an inequality $ay \leq b$ separating $\bar{y}$ from $\operatorname{conv}(S_i)$ can be found by solving a linear program.*

Proof. If $\bar{y} \notin \operatorname{conv}(S_i)$, the linear program

$$\begin{array}{lrl} \min & & 0 \\ & \sum_{h=1}^{i} y^h u_h & = \bar{y} \\ & \sum_{h=1}^{i} u_h & = 1 \\ & u & \geq 0 \end{array}$$

has no solution. Therefore its dual

$$\begin{array}{lll} \max & a\bar{y} - b & \\ & ay^h - b & \leq 0 \quad h = 1, \dots, i \end{array}$$

has an unbounded solution (a, b). □

Applegate, Bixby, Chvátal and Cook call *local cuts* the inequalities generated by this procedure. In their implementation, they refined the procedure so that it only generates facets of the graphical traveling salesman polyhedron. Different choices of the shrunk node sets $V_1, \dots, V_k$ are used to try to generate several inequalities cutting off the current fractional solution $\bar{x}$. The interested reader is referred to [13] for details.

7.5 Equivalence Between Optimization and Separation

By Meyer's theorem (Theorem 4.30), solving an integer program is equivalent to solving a linear program with a potentially very large number of constraints. In fact, several integer programming formulations, such as the subtour elimination formulation of the traveling salesman polytope or the single-node flow set formulation given by all flow cover inequalities, already have a number of constraints that is exponential in the data size of the problem, so solving the corresponding linear programming relaxations is not straightforward. We would like to solve these linear programs without generating explicitly all the constraints. A fundamental result of Grötschel et al. [186] establishes the equivalence of *optimization* and *separation*: solving a linear programming problem is as hard as finding a constraint cutting off a given point, or deciding that none exists.

Optimization Problem. *Given a polyhedron $P \subset \mathbb{R}^n$ and an objective $c \in \mathbb{R}^n$, find $x^* \in P$ such that $cx^* = \max\{cx : x \in P\}$, or show $P = \emptyset$, or find a direction z in P for which cz is unbounded.*

Separation Problem. *Given a polyhedron $P \subset \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$, either show that $\bar{x} \in P$ or give a valid inequality $\alpha x \leq \alpha_0$ for P such that $\alpha\bar{x} > \alpha_0$.*

We are particularly interested in solving the above separation problem when the inequalities defining P are not given explicitly. This is typically the case in integer programming, where P is given as the convex hull of a mixed integer set $\{(x, y) \in \mathbb{Z}_+^p \times \mathbb{R}_+^q : Ax + Gy \leq b\}$ with data A, G, b.

An important theorem of Grötschel et al. [186] states that the optimization problem can be solved in polynomial time if and only if the separation problem can be solved in polynomial time. Similar results were obtained by Padberg and Rao [299] and Karp and Papadimitriou [233]. Of course, P needs to be described in a reasonable fashion for the polynomiality statement to make sense. We will return to this issue later. First, we introduce the main tool needed for proving the equivalence, namely the *ellipsoid algorithm.* We only give a brief outline here. The reader is referred to [188] for a detailed treatment.

Ellipsoid Algorithm

Input. A matrix $A \in \mathbb{Q}^{m\times n}$ and a vector $b \in \mathbb{Q}^m$.
Output. A point of $P := \{x \in \mathbb{R}^n : Ax \le b\}$ or a proof that P is not full dimensional.

Initialize with a large enough integer t^* and an ellipsoid E_0 that is guaranteed to contain P. Set $t = 0$.

Iteration t**.** If the center x^t of E^t is in P, stop. Otherwise find a constraint $a^i x \le b_i$ from $Ax \le b$ such that $a^i x^t > b_i$. Find the smallest ellipsoid E_{t+1} containing $E_t \cap \{a^i x \le b_i\}$. Increment t by 1. If $t < t^*$, perform the next iteration. If $t = t^*$, stop: P is not full-dimensional.

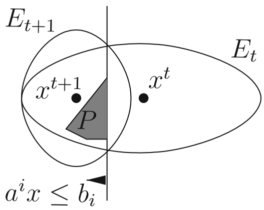


Figure 7.5: Illustration of the ellipsoid algorithm

Figure 7.5 illustrates an iteration of the ellipsoid algorithm. Khachiyan [235] showed that the ellipsoid algorithm can be made to run in polynomial time.

Theorem 7.22. *The ellipsoid algorithm terminates with a correct output if* E_0 *and* t^* *are chosen large enough. Furthermore this choice can be made so that the number of iterations is polynomial.*

The following observations about volumes are key to proving that only a polynomial number of iterations are required. We state them without proof.

- The smallest ellipsoid E_{t+1} containing $E_t \cap \{a^i x \leq b_i\}$ can be computed in closed form.

- $\text{Vol}(E_{t+1}) \leq \rho \text{Vol}(E_t)$, where $\rho < 1$ is a constant that depends only on n.

- There exists $\epsilon > 0$, whose encoding size is polynomial in n and in the size of the coefficients of (A, b), such that either P has no interior, or $\text{Vol}(P) \geq \epsilon$.

- $\text{Vol}(E_0) \leq \Delta$, where the encoding size of Δ is polynomial in n and in the size of the coefficients of (A, b).

Since $\text{Vol}(E_t) \leq \rho^t \text{Vol}(E_0)$, the ellipsoid algorithm requires at most $t^* = \log \frac{\Delta}{\epsilon}$ iterations before one can conclude that P has an empty interior. Thus the number of iterations is polynomial. To turn the ellipsoid algorithm into a polynomial algorithm, one needs to keep a polynomial description of the ellipsoids used in the algorithm. This can be achieved by working with slightly larger ellipsoids, instead of the family E_t defined above. We skip the details.

The ellipsoid algorithm returns a point in P whenever P is full dimensional. Dealing with non-full dimensional polyhedra is tricky. Grötschel et al. [187] describe a polynomial-time algorithm that, upon termination of the ellipsoid algorithm with the outcome that P is not full-dimensional, determines an equation $\alpha x = \beta$ satisfied by all $x \in P$. Once such equation is known, one can reduce the dimension of the problem by one, and iterate. A detailed description can be found in [188].

Another issue is the optimization of a linear function cx over P, instead of just finding a feasible point, as described in the above algorithm. This can be done in polynomial time by using binary search on the objective value, or a "sliding objective." Again, we refer to [188] for a description of these techniques.

Finally, we note a beautiful aspect of the ellipsoid algorithm: It does not require an explicit description of P as $\{x \in \mathbb{R}^n : Ax \leq b\}$, but instead it can rely on a separation algorithm that, given the point x^t, either shows that this point is in P, or produces a valid inequality $a^i x \leq b_i$ for P such that $a^i x^t > b_i$.

As a consequence, if we have a separation algorithm at our disposal, the ellipsoid algorithm with a sliding objective solves the optimization problem.

Example 7.23. Consider the traveling salesman problem in an undirected graph $G = (V, E)$. As observed in Sect. 7.4.1, the separation problem for the subtour elimination polytope can be solved in polynomial time (as it amounts to finding a minimum cut in G). Therefore, by applying the ellipsoid algorithm, one can optimize over the subtour elimination polytope in polynomial time. ■

The complexity of the separation algorithm depends on how P is given to us. We will need P to be "well-described" in the following sense.

Definition 7.24. *A polyhedron $P \subset \mathbb{R}^n$ belongs to a* well-described family *if the length L of the input needed to describe P satisfies $n \leq L$, and there exists a rational matrix (A, b) such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and the encoding size of each of the entries in the matrix (A, b) is polynomially bounded by L.*

Examples of well-described polyhedra are

- $P := \{x \in \mathbb{R}^n : Ax \leq b\}$, where A, b have rational entries and are given as input.
- $P := \text{conv}\{x \in \mathbb{Z}^n : Ax \leq b\}$, where A, b have rational entries and are given as input.
- P is the subtour elimination polytope of a graph G, where G is given as input.

On the other hand, the subtour elimination polytope of a complete graph on n nodes in not well-described if the given input is just the positive integer n in binary encoding, because in this case the length of the input is $\lceil \log_2(n+1) \rceil$, which is smaller than n for $n \geq 3$.

Remark 7.25. *It follows from Theorems 3.38 and 3.39 that P belongs to a well-described family of polyhedra if and only if there exist rational vectors $x^1, \dots, x^k$, $r^1, \dots, r^t$ each of which has an encoding size that is polynomially bounded by the length L of the input used to describe P, and such that $P = \text{conv}\{x^1, \dots, x^k\} + \text{cone}\{r^1, \dots, r^t\}$.*

Theorem 7.26. *For well-described polyhedra, the separation problem is solvable in polynomial time if and only if the optimization problem is.*

Proof. We give the proof for the case when P is full-dimensional and bounded. The proof is more complicated when P is not full-dimensional, and we refer the reader to [188] in that case.
"Polynomial Separation ⇒ Polynomial Optimization." This follows from the ellipsoid algorithm.

"Polynomial Optimization ⇒ Polynomial Separation."

Claim 1: If Optimization can be solved in polynomial time on P, then an interior point of P can be found in polynomial time.

Indeed, find a first point x^0 by maximizing any objective function over P. Assume affinely independent points $x^0, \dots, x^i$ have been found. Choose c orthogonal to the affine hull of $x^0, \dots, x^i$. Solve $\max cx$ and $\max -cx$ over P, respectively. Al least one of these programs gives an optimal solution x^{i+1} that is affinely independent of $x^0, \dots, x^i$. Repeat until $i = n$. Now $\bar{x} = \frac{1}{n+1} \sum_{i=0}^{n} x^i$ is an interior point of P. This proves Claim 1.

Translate P so that the origin is in the interior of P. By Claim 1, this can be done in polynomial time; indeed, if $\bar{x}$ is an interior point of P, $P - \bar{x}$ contains the origin in the interior.

Claim 2: If Optimization can be solved in polynomial time on P, then Separation can be solved in polynomial time on its polar P^*.

Given $\pi^* \in \mathbb{R}^n$, let x^* be an optimal solution to $\max\{\pi^* x : x \in P\}$. If $\pi^* x^* \leq 1$, then $\pi^* \in P^*$. If $\pi^* x^* > 1$, then $\pi x^* \leq 1$ is a valid inequality for P^* which cuts off π^*. Its description is polynomial in the input size of the separation problem on P^* (the input is the description of P (by Remark 7.25 P^* is well-described by the same input) and π^*). This proves the claim.

By Claim 2 and by the first part of the proof (Polynomial Separation ⇒ Polynomial Optimization), it follows that Optimization can be solved in polynomial time on P^*. Applying Claim 2 to P^*, we get that Separation can be solved in polynomial time on P^{**}. Since P contain the origin in its interior, it follows from Corollary 3.50 that $P^{**} = P$. □

7.6 Further Readings

The solution of an instance of the traveling salesman problem on 49 cities, detailed by Dantzig et al. [103] in 1954, laid out the foundations of the cutting plane method, and has served as a template for tackling hard combinatorial problems using integer programming (see for example Grötschel

[183, 184], Crowder and Padberg [98], Grötschel et al. [185]). We refer the reader to Cook [85] for an account of the history of combinatorial integer programming, to the monograph by Applegate et al. [13] for a history of traveling salesman computations, and to Cook's book [86] for an expository introduction to the traveling salesman problem. On the theory side, the key insight of Dantzig, Fulkerson and Johnson that one can solve integer programs by introducing inequalities as needed, culminated in the proof of equivalence of separation and optimization by Grötschel et al. [186].

Several early works on valid inequalities for structured problems focused on packing problems, see for example Padberg [297], Nemhauser and Trotter [283, 284], and Wolsey [350]. Padberg [297] introduced the notion of sequential lifting in the context of odd-holes inequalities and generalized it in [298], where he described the sequential lifting procedure discussed in Sect. 7.2. The effectiveness of polyhedral methods in solving general $0, 1$ problems was illustrated in the 1983 paper of Crowder et al. [99], where they reported successfully solving 10 pure 0,1 linear programs with up to 2750 variables, employing a variety of tools including lifted cover inequalities. Van Roy and Wolsey [340] reported computational experience in solving a variety of mixed 0,1 programming problems using strong valid inequalities. The paper formalized the *automatic reformulation* approach, that has since become a staple in integer programming: identify a suitable "structured relaxation" R of the feasible region (such as, for example, a single-node flow set), find a family of "strong" valid inequalities for R, and devise an efficient separation algorithm for the inequalities in the family.

The results of Sect. 7.2.2 on superadditive liftings were proved by Wolsey [351], and generalized to mixed 0,1 linear problems by Gu et al. [192]. The sequence independent liftings of cover and flow cover inequalities (Sects. 7.2.3 and 7.3) are given in [192]. Gu et al. [191] report on a successful application of lifted flow cover inequalities to solving mixed 0,1 linear problems. See Louveaux and Wolsey [258] for a survey on sequence-independent liftings.

Wolsey [352] showed that the subtour formulation of the symmetric traveling salesman problem has an integrality gap of $3/2$ whenever the distances define a metric. Goemans [169] computed the worst-case improvement resulting from the addition of many of the known classes of inequalities for the graphical traveling salesman polyhedron, showing for example that the comb inequalities cannot improve the subtour bound by a factor greater than $10/9$.

Equivalence of Separation and Optimization for Convex Sets

The ellipsoid method was first introduced by Yudin and Nemirovski [356] and Shor [332] for convex nonlinear programming, and was used by Khachiyan [235] in a seminal paper in 1979 to give the first polynomial-time algorithm for linear programming. Several researchers realized, soon after Khachiyan's breakthrough, that the method could be modified to run in polynomial time even if the polyhedron is implicitly described by a separation oracle. The strongest version of this result is given by Grötschel et al. [186] (see also [188]), and it can be extended to general convex sets, but similar results have also been discovered by Karp and Papadimitriou [233] and Padberg and Rao [299].

As mentioned above, the equivalence of linear optimization and separation holds also for general convex sets. However, given a convex set $K \subset \mathbb{R}^n$ and $c \in \mathbb{Q}^n$, it may very well be that the optimal solutions of $\max\{cx : x \in K\}$ have irrational components. Analogously, given $y \notin K$, there is no guarantee that a rational hyperplane separating y from K exists, in general. Therefore optimization and separation over K can only be solved in an approximate sense. Formally, given a convex set $K \subseteq \mathbb{R}^n$ and a number $\varepsilon > 0$, let $S(K, \varepsilon) := \{x \in \mathbb{R}^n : \|x - y\| \leq \varepsilon \text{ for some } y \in K\}$ and $S(K, -\varepsilon) := \{x \in K : S(\{x\}, \varepsilon) \subseteq K\}$.

The *weak optimization problem* is the following: given a vector $c \in \mathbb{Q}^n$, and a rational number $\varepsilon > 0$, either determine that $S(K, -\varepsilon)$ is empty, or find $y \in S(K, \varepsilon) \cap \mathbb{Q}^n$ such that $cx \leq cy + \varepsilon$ for all $x \in S(K, -\varepsilon)$.

The *weak separation problem* is the following: given a point $y \in \mathbb{Q}^n$, and a rational number $\delta > 0$, either determine that $y \in S(K, \delta)$, or find $c \in \mathbb{Q}^n$, $\|c\|_\infty = 1$, such that $cx \leq cy + \delta$ for all $x \in K$.

In order to state the equivalence of the two problems, one needs to specify how K is described. Furthermore, the equivalence holds under some restrictive assumptions. Namely, we say that a convex set K is *circumscribed* if the following information is given as part of the input: a positive integer n such that $K \subset \mathbb{R}^n$, and a rational positive number R such that K is contained in the ball of radius R centered at 0. A circumscribed convex set K is denoted by $(K; n, R)$.

We say that a circumscribed convex set $(K; n, R)$ is given by a *weak separation oracle* if we have access to an oracle that provides a solution c to the weak separation problem for every choice of y and δ, where the encoding size of c is polynomially bounded by n and the encoding sizes of R, y, and δ.

We say that a circumscribed convex set $(K; n, R)$ is given by a *weak optimization oracle* if we have access to an oracle providing a solution y

to the weak optimization problem for every choice of c and ε, where the encoding size of y is polynomially bounded by n and the encoding sizes of R, c and ε.

If $(K; n, R)$ is expressed by a weak separation or a weak optimization oracle, an algorithm involving K is said to be *oracle-polynomial time* if the total number of operations, including calls to the oracle, is bounded by a polynomial in n and the encoding sizes of R and of other input data (such as objective function c and tolerance ε).

Theorem 7.27 (Grötschel et al. [186]). *There exists an oracle-polynomial time algorithm that solves the weak optimization problem for every circumscribed convex set $(K; n, R)$ given by a weak separation oracle and every choice of $c \in \mathbb{Q}^n$ and $\varepsilon > 0$.*
There exists an oracle-polynomial time algorithm that solves the weak separation problem for every circumscribed convex set $(K; n, R)$ given by a weak optimization oracle and every choice of $y \in \mathbb{Q}^n$ and $\delta > 0$.

The equivalence hinges on an approximate version of the ellipsoid method. Below we give a high-level description of the method.

Input. A rational number $\varepsilon > 0$ and a circumscribed closed convex set $(K; n, R)$ given by a separation oracle.
Output. Either a rational point $y \in S(K, \varepsilon)$, or an ellipsoid E such that $K \subseteq E$ and $\text{vol}(E) < \varepsilon$.

Initialize with a large enough integer t^* and a small enough $\delta < \varepsilon$. Set $t = 0$, and let E^0 be the ball of radius R centered at 0.

Iteration t**.** Let x^t be the center of the current ellipsoid E^t containing K. Make a call to the separation oracle with $y = x^t$. If the oracle concludes that x^t is in $S(K, \delta)$, then $x^t \in S(K, \varepsilon)$, stop. If the oracle returns $c \in \mathbb{Q}^n$ such that $cx \le cx^t + \delta$ for all $x \in K$, then find an ellipsoid E^{t+1} that is an appropriate approximation of the smallest ellipsoid containing $E^t \cap \{cx \le cx^t + \delta\}$. Increment t by 1. If $t < t^*$, perform the next iteration. If $t = t^*$, stop: $\text{vol}(E^t) < \varepsilon$.

The algorithm described above is oracle-polynomial time because it can be shown that t^* and δ can be chosen so that their encoding size is polynomial in n and in the encoding sizes of R and ε. Furthermore, the ellipsoid E^{t+1} can be computed by a closed form formula.

7.7 Exercises

Exercise 7.1. Consider the 0,1 knapsack set $K := \{x \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \leq b\}$ where $0 < a_j \leq b$ for all $j = 1, \ldots, n$.

(i) Show that $x_j \geq 0$ defines a facet of $\text{conv}(K)$.

(ii) Give conditions for the inequality $x_j \leq 1$ to define a facet of $\text{conv}(K)$.

Exercise 7.2. Consider the graph C_5 with five vertices v_i for $i = 1, \ldots, 5$ and five edges $v_1v_2, \ldots, v_4v_5, v_5v_1$. Let $STAB(C_5)$ denote the stable set polytope of C_5, namely the convex hull of its stable sets.

(i) Show that $x_j \geq 0$ is a facet of $STAB(C_5)$.

(ii) Show that $x_j + x_k \leq 1$ is a facet of $STAB(C_5)$ whenever v_jv_k is an edge of C_5.

(iii) Show that $\sum_{j=1}^5 x_j \leq 2$ is a facet of $STAB(C_5)$.

(iv) Let W_5 be the graph obtained from C_5 by adding a new vertex w adjacent to every v_j, $j = 1, \ldots, 5$. Show how each facet in (i), (ii) and (iii) is lifted to a facet of $STAB(W_5)$.

Exercise 7.3. A *wheel* W_n is the graph with $n+1$ vertices $v_0, v_1, \ldots, v_n$, and $2n$ edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$ and v_0v_i for all $i = 1, \ldots n$. A Hamiltonian cycle is one that goes though each vertex exactly once. We represent each Hamiltonian cycle by a 0,1 vector in the edge space of the graph, namely $\mathbb{R}^{2n}$. Define $Hamilton(W_n)$ to be the convex hull of the 0,1 vectors representing Hamiltonian cycles of W_n.

(i) What is the dimension of $Hamilton(W_n)$? How many vertices does $Hamilton(W_n)$ have? How many facets?

(ii) Show that the inequalities $x_e \leq 1$ define facets of $Hamilton(W_n)$ for $e = v_1v_2, \ldots, v_{n-1}v_n, v_nv_1$.

(iii) Give a minimal description of $Hamilton(W_n)$.

Exercise 7.4. Let $G = (V, E)$ be a graph.

1. Show that the blossom inequalities (4.17) for the matching polytope are Chvátal inequalities for the system $\sum_{e \in \delta(v)} x_e \leq 1$, $v \in V$, $x \geq 0$.

2. Show that, if G is not bipartite, then there is at least one blossom inequality that is facet-defining for the matching polytope of G.

In particular, the matching polytope has Chvátal rank zero or one, and the rank is one if and only if G is not bipartite.

Exercise 7.5. Consider $S \subseteq \{0,1\}^n$. Suppose $S \cap \{x_n = 0\} \neq \emptyset$ and $S \cap \{x_n = 1\} \neq \emptyset$. Let $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ be a valid inequality for $S \cap \{x_n = 1\}$. State and prove a result similar to Proposition 7.2 that lifts this inequality into a valid inequality for $\text{conv}(S)$.

Exercise 7.6. Consider $S \subseteq \{0,1\}^n$. Suppose that $\text{conv}(S) \cap \{x : x_k = 0$ for all $k = p+1, \dots, n\}$ has dimension p, and that $\sum_{j=1}^{p} \alpha_j x_j \leq \beta$ defines one of its faces of dimension $p-2$ or smaller. Construct an example showing that a lifting may still produce a facet of $\text{conv}(S)$.

Exercise 7.7. Consider the sequential lifting procedure. Prove that the largest possible value of the lifting coefficient α_j is obtained when x_j is lifted first in the sequence. Prove that the smallest value is obtained when x_j is lifted last.

Exercise 7.8. Consider the 0,1 knapsack set $K := \mathbb{Z}^n \cap P$ where $P := \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x \leq \mathbf{1}\}$. Let C be a minimal cover, and let $h \in C$ such that $a_h = \max_{j \in C} a_j$. Show that the inequality

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \lfloor \frac{a_j}{a_h} \rfloor x_j \leq |C| - 1$$

is a Chvátal inequality for P.

Exercise 7.9. Let K be a knapsack set where $b \geq a_1, \dots, \geq a_n > 0$, Let $C = \{j_1, \dots, j_t\}$ be a minimal cover of K. The *extension* of C is the set $E(C) := C \cup \{k \in N \setminus C : a_k \geq a_j$ for all $j \in C\}$. Let ℓ be the smallest index in $\{1, \dots, n\} \setminus E(C)$ (if the latter is nonempty).

(i) Prove that, if $\sum_{j \in C \cup \{1\} \setminus \{j_1, j_2\}} a_j \leq b$ and $\sum_{j \in C \cup \{\ell\} \setminus \{j_1\}} a_j \leq b$, then the *extended cover inequality* $\sum_{j \in E(C)} x_j \leq |C| - 1$ defines a facet of $\text{conv}(K)$.

(ii) Prove that extended cover inequalities are Chvátal inequalities.

Exercise 7.10. Consider the knapsack set $\{x \in \{0,1\}^4 : 8x_1 + 5x_2 + 3x_3 + 12x_4 \leq 14\}$. Given the minimal cover $C = \{1,2,3\}$, compute the best possible lifting coefficient of variable x_4 using Theorem 7.4. Is the corresponding lifted cover inequality a Chvátal inequality?

Exercise 7.11. Let set S and the inequality $\sum_{j\in C} \alpha_j x_j \le \beta$ be defined as in Sect. 7.2.2. Suppose that the lifting function defined in (7.10) is superadditive. Prove that $\sum_{j\in C} \alpha_j x_j + \sum_{j\in N\setminus C} f(a^j)x_j \le \beta$ is valid for S and, for every valid inequality $\sum_{j\in N} \alpha_j x_j \le \beta$, $\alpha_j \le f(a^j)$ for all $j \in N \setminus C$.

Exercise 7.12. Show that the function g defined in (7.12) is superadditive.

Exercise 7.13. Show that the lifted minimal cover inequality of Example 7.8 induces a facet.

Exercise 7.14. Prove that, when the inclusion $C \subset N$ is strict, the condition $\lambda < \max\limits_{j\in C} a_j$ is necessary for the flow cover inequality (7.14) to define a facet of $\mathrm{conv}(T)$.

Exercise 7.15. Consider the following mixed integer linear set.

$$T := \{x \in \{0,1\}^n, y \in \mathbb{R}^n_+ : \begin{array}{l} \sum_{j=1}^k y_j - \sum_{j=k+1}^n y_j \le b \\ y_j \le a_j x_j \text{ for all } j = 1, \dots, n\} \end{array}$$

where $b > 0$ and $a_j > 0$ for all $j = 1, \dots, n$. Consider $C \subseteq \{1, \dots, k\}$ such that $\sum_{j\in C} a_j > b$. Let $\lambda := \sum_{j\in C} a_j - b$. Consider $L \subseteq \{k+1, \dots, n\}$ and let $\bar{L} := \{k+1, \dots, n\} \setminus L$. Prove that if $\max_{j\in C} a_j > \lambda$ and $a_j > \lambda$ for all $j \in L$, then

$$\sum_{j\in C} y_j - \sum_{j\in \bar{L}} y_j + \sum_{j\in C} (a_j - \lambda)^+ (1 - x_j) - \sum_{j\in L} \lambda x_j \le b$$

defines a facet of $\mathrm{conv}(T)$.

Exercise 7.16. Prove that the function f defined in Lemma 7.13 is superadditive in the interval $[0, b]$.

Exercise 7.17. Prove that flow cover inequalities (7.14) are Gomory mixed integer inequalities.

Exercise 7.18. Show that the comb inequality (7.23) can be written in the following equivalent form $\sum_{i=0}^k \sum_{e\in\delta(S_i)} x_e \ge 3k + 1$.

Exercise 7.19. Let $G = (V, E)$ be an undirected graph. Recall from Sect. 2.4.2 that $\mathrm{stab}(G)$ is the set of the incidence vectors of all the stable sets of G. The *stable set polytope of* G is $\mathrm{STAB}(G) = \mathrm{conv}(\mathrm{stab}(G))$. Let

$Q(G) := \{x \in \mathbb{R}^V : x_i + x_j \leq 1, ij \in E\}$ and $K(G) := \{x \in \mathbb{R}^V : \sum_{i \in K} \leq 1, K \text{ clique of } G\}$. Recall that $\mathrm{stab}(G) = Q(G) \cap \mathbb{Z}^V = K(G) \cap \mathbb{Z}^V$.

(i) Prove that, given a clique K of G, the clique inequality $\sum_{v \in K} x_v \leq 1$ is facet-defining for $\mathrm{STAB}(G)$ if and only if K is a maximal clique.

(ii) Given an odd cycle C of G, the *odd cycle inequality* is $\sum_{v \in V(C)} x_v \leq (|C| - 1)/2$. The cycle C is *chordless* if and only if $E \setminus C$ has no edge with both endnodes in $V(C)$.

- Show that the odd cycle inequality is a Chvátal inequality for $Q(G)$.
- Show that the odd cycle inequality is facet-defining for $\mathrm{STAB}(G) \cap \{x : x_i = 0, i \in V \setminus V(C)\}$ if and only if C is chordless.

(iii) A graph $H = (V(H), E(H))$ is an *antihole* if the nodes of H can be labeled $v_1, \dots, v_h$ so that v_i is adjacent to v_j, $j \neq i$, if and only if both $i - j \pmod h \geq 2$ and $j - i \pmod h \geq 2$. The inequality $\sum_{i \in V(H)} x_i \leq 2$ is the *antihole inequality* relative to H. Let H be an antihole contained in G such that $|V(H)|$ is odd.

- Show that the antihole inequality relative to H is a Chvátal inequality for $K(G)$.
- Show that, if $E \setminus E(H)$ has no edge with both endnodes in $V(H)$, then the antihole inequality relative to H is facet-defining for $\mathrm{STAB}(G) \cap \{x : x_i = 0, i \in V \setminus V(H)\}$.

(iv) Given positive integers n, k, $n \geq 2k+1$, a graph $W_n^k = (V(W_n^k), E(W_n^k))$ is a *web* if the nodes of W_n^k can be labeled $v_1, \dots, v_n$ so that v_i is adjacent to v_j, $j \neq i$, if and only if $i - j \pmod h \leq k$ or $j - i \pmod h \leq k$. Show that, if W_n^k is a web contained in G and n is not divisible by $k+1$, then the *web inequality* $\sum_{i \in V(W_n^k)} x_i \leq \lfloor n/(k+1) \rfloor$ is a Chvátal inequality for $K(G)$.

Exercise 7.20. Given an undirected graph $G = (V, E)$, the *stability number* $\alpha(G)$ of G is the size of the largest stable set in G. An edge $e \in E$ is α-*critical* if $\alpha(G \setminus e) = \alpha(G) + 1$. Let $\tilde{E} \subseteq E$ be the set of α-critical edges in G. Show that, if the graph $\tilde{G} = (V, \tilde{E})$ is connected, then the inequality $\sum_{i \in V} x_i \leq \alpha(G)$ is facet-defining for $\mathrm{STAB}(G)$.

Exercise 7.21. Show that the mixing inequalities (4.29) and (4.30) are facet-defining for P^{mix} (defined in Sect. 4.8.1).

Exercise 7.22. Show that the separation problem for the mixing inequalities (4.29) can be reduced to a shortest path problem in a graph with $O(n)$ nodes.
Show that the separation problem for the mixing inequalities (4.30) can be reduced to finding a negative cost cycle in a graph with $O(n)$ nodes.