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Facilities, Locations, Customers: Building Blocks of Location Models. A Survey.

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Abstract

As evidenced by the remarkable diversity of real world applications which have been modeled and solved as location problems, the field studying the optimal location of facilities is a very interdisciplinary and broad research area. The purpose of this paper is to fit the large variety of location models within a general unified framework, which arises from the description of the three buildings blocks of location problems, namely: facilities, customers, and locations. We provide evidences of how a particular problem specification can be stated mathematically as an optimization problem by opportunely combining into a compact and workable model the main features that characterize and relate these three elements.

1 Introduction

The analysis of facility location problems has represented an attractive field of research since the beginning of the century. The very first location model, due to Alfred Weber [104], appeared in 1909 and dominated the literature for many years hence. However, a unified field of study called facility location did not emerge until the 1960s. The seminal paper by Hakimi [43], published in 1964, established important results in location theory and sparked new theoretical interest among researchers, who from then on developed very wide-ranging models, challenging combinatorial problems and many creative ideas for designing location methodologies.

The remarkable diversity of facility location models arising from real world applications suggests the need for a unifying classification scheme in order to provide a scientific method for unambiguously and precisely describing the model class at hand, and to sketch a guiding theme running through the large variance of papers published in the field. An extensive bibliography devoted to facility location analysis has been provided by Travor Hale [44]. It contains a listing of over 2900 location science, facility location, and related references. The first proposal of classification schemes for location models dates back to 1979 and was due to Handler and Mirchandani [51].

Since then, many authors publishing in the area have designed classification schemes in an attempt at covering a large variety of location models (see for example [9, 18, 33, 49, 61]).

Location problems in the most general form can be stated as follows. A set of customers spatially distributed in a geographical area originates demands for some kind of goods or services. Customers demand must be supplied by one or more facilities, which can operate in a cooperative or competitive framework, depending on the type of good or service being required. The decision process must establish where to locate the facilities in the territorial space taking into account users requirements and possible geographical restrictions. Each particular choice of facility site implies some set up cost for establishing the facility, and some operational costs for serving the customers. Issues like cost reduction, demand capture, equitable service supply, fast response time and so on, drive the selection of facility placement.

Not all location problems, however, are inherently geographical in nature. More abstract settings can originate location decisions for less obvious objects, such as electric components, industrial products, concentrators and so on.

The generalized statement of the problem we provided above emphasizes the presence of three main elements which play an essential role in all location problems, namely: facilities, customers, and locations. The characterizing features of these elements and the different combinations of them within a model are sufficient to define the structure underlying a wide range of applications and allow the characterization of a representative sample of problems.

In this context, more than providing a symbolic scheme to compose a specific instance of the model classification, we will focus on a verbal description of the basic elements involved in any location problem. Variations, extensions and combinations of the characteristics of these elements reflect the particulars of the model under examination.

This survey of location models is organized as follows. In Section 2, we identify the basic elements shared by all location problems and describe their main features. For each feature we provide a comprehensive list of possible options reflecting the peculiarities of the specific problem to be modeled. Section 3 illustrates the interdependence among elements and Section 4 shows how the element properties and interaction aspects can be combined into objectives and constraints to generate location models. In Section 5, a survey of major location models is provided, including some standard problems, such as the median, center, set covering, and maximal covering problems, as well as less traditional problems which have emerged in recent years. Finally, the last section is devoted to some concluding remarks on the future research avenues in locational analysis.

2 Basic elements in location models

Within the realm of location models, we can almost always identify three essential components: facilities, i.e. the objects to be located to provide a service or good; locations, i.e. the set of candidate points for facility sites; and customers, i.e. the users of the facilities who demand certain services or goods. In this section we discuss each of these basic elements in turn, describing the main properties they can exhibit in the sizable collection of location problems.

2.1 Facilities

The generic term *facility* is used to denote a large variety of objects for which we must determine a spatial position in order to optimize their interaction with other preexisting objects. The most classical examples of the use of the word facility in location theory refer to objects such as warehouses, plants, schools, hospitals, retail outlets and many other industrial, commercial or public structures. A less obvious use include electronic components, warning sirens, exploratory oil wells, radar beams or even new products to be positioned in a market.

The main properties characterizing the facilities are: their *number*, their *type* and the *cost* associated with them.

In several location models, the *number* of new facilities is given. In the simplest case, only one facility is to be located relative to a number of existing facilities. This kind of problem is referred to as *single-facility problem.* In the general case, models involve the simultaneous location of many facilities and are referred to as *multi-facility models.* Multi-facility problems include both cases of fixed and prespecified numbers of facilities to be located, and variable numbers of facilities to be determined during the decision process. In the latter case, the decision involves a trade-off among the improved accessibility of the customers to the facilities obtained by opening a larger number of centers, and the increased costs for establishing and operating the facilities. Problems in this category might even take some additional restrictions on the number of facility to be opened, forcing it to be greater and/or smaller than a given value.

Another important property of the facilities is represented by their *type*, which involves capacity, service, and structure considerations. In the simplest case, location problems require the placement of identical facilities with respect to both size and the kind of service they can supply. However, in many applications it is often necessary to locate simultaneously facilities which differ from each other. A typical example is given by multi-level distribution systems where both plants and warehouses must be located in order to produce and distribute goods in an efficient way. The number of levels at which different facilities operate distinguishes location models between *single-echelon* and *multi-echelon*. Models can also be differentiated according to *single-service* and *multi-service*, based on whether the facilities can provide only one or many kinds of service respectively. Moreover, some location problems admit facilities that supply an infinite demand, whereas other applications look for the best placement of facilities with limited production or supply capacity. In this respect, problems are denoted as *capacitated* or *uncapacitated*. The capacity might even depend on the particular site where the facility is established. Finally, facilities can have different configurations: they can be considered to occupy either point locations or area locations, or even take a specialized shape, like a graph or a tree. As an example, path-shaped facility location models arise in many transportation planning problems concerning the location of new highways, railroad lines and subway lines, or the design of airlines routes. Another illustration of path-shaped facilities arises in telecommunication systems for the location of fiber optic cables to be lashed to telephone poles or existing electrical cables.

The *costs* associated with facilities concern the fixed expenses incurred for their opening and the variable charges related to the service delivery. While the first type of costs are usually connected to the specific location where the facilities are established, the second is usually some function of the distance from the users of the service. A more in-depth analysis of facility costs will be presented in Sections 3.1 and 3.4.

2.2 Locations

The second essential element for setting the stage for arbitrary location models is the physical place where the facilities can be positioned, i.e. the set of *locations*. With respect to the set of eligible points (usually referred as the solution space), three spatial representations are possible, namely: continuous, discrete, and network.

In the *discrete case*, the decision maker can specify a list of plausible sites for facility locations. This kind of solution space proves to be very flexible because it makes it possible to incorporate a number of geographical and economic features into the model. Further, the discrete space results the most natural one for designing problems when land availability, zoning regulations or the presence of pre-existing structures require that new facilities be opened only at some prespecified points within the area under consideration.

On the other hand, many location applications are based on the assumption that the underlying space both for facilities sites and preexisting points is a continuous one, where all points are determined by way of one or more coordinates which may vary continuously. *Continuous location problems* are usually considered in the Euclidean space ($\Re^2$) or, more generally, in a n-dimensional space ($\Re^n$). Two-dimensional problems are the most popular for evident reasons of geographical nature, but more abstract settings may give rise to even more dimensions. For example, in marketing and design research, product positioning is cast as a location problem in an n-dimensional attribute space, where demand points correspond to the desired attribute coordinates of representative consumer groups [41, 76].

Continuous space models are referred to by [71] as *site-generation* models since no a-priori knowledge of particular candidate sites is assumed, and the generation of appropriate sites is left to the model at hand. Conversely, the discrete case is termed the *site-selection* model by the same authors.

Discrete facility location has been extensively studied in OR since Hakimi [43]. The study of the continuous location problem also dates back many years (Weber's problem [104]), but its development was relatively slow due to the analytical difficulties in handling geometrical computation in a continuous plane. Recently, this difficulty has been overcome to a great extent by the progress in computational geometry [82].

The third type of location models we can identify with respect to the solution space is the *network-based model.* For many applications in both public and private service systems, the graph-theoretic approach lends itself in an excellent way to an intuitive representation of the problem. Some natural examples are: the set up of plants in a transportation system to reduce production and shipment costs; the placement of emergency services in rural areas so to guarantee fast intervention to population centers; the design of computer communication networks involving the connection of remote terminal sites to a central site to optimize transmission costs, and so on. The advantage of using graph-theoretic approaches to model location problems is that algorithms motivated by graph-theoretic considerations are generally much more efficient than the more traditional mathematical programming algorithms, and hence can solve problems of much larger size.

Problems defined on networks can be seen as both continuous or discrete, depending on whether links are considered as a continuous set of candidate points for facility location, or only the nodes are eligible for the placement of new facilities. When the model is defined on a network, the underlying graph can have different structures (such as undirected graphs, directed graphs, or trees). A structure that has been particularly exploited is that of a tree. Tree-like networks can be encountered in sparsely occupied regions or when having cycles is very expensive, as with portions of interstate highway systems. Further, simple distribution systems with a single distributor as the "hub" can often be modeled as star-like trees.

Many other factors can be involved in the eligibility of location points. For example, the selection space available may be restricted by the presence of forbidden zones, i.e. areas in which facilities may not locate. Other restrictions on acceptable points might include requirements such as selecting at least one location out of a given subset of candidate sites in order to guarantee fast service to a particular area or, conversely, conditions forcing the choice of at most one site in a given region. Also, additional constraints might be needed to account for compatibility considerations. For example, when locating an hospital the solution space should be defined so as to exclude the possibility of placing it in the vicinity of a facility producing hazardous materials or pollutants.

2.3 Customers

Facility location problems arise from the need to locate supply centers in order to optimally satisfy the demand of a set of customers. The generic word *customer* can be used in its most traditional meaning to denote the person requiring accessibility to a service or supply of a good, or more abstractly to indicate any object which must interact with some new facility. Examples of abstract customers include: elements of a wiring circuit to be connected to new electric components, remote terminals in communication networks, irrigation systems or livestock barns to be served by new wells, and communities in rural areas requiring public services.

When dealing with the presence of customers in location analysis, it is essential to know their *distribution*, their *demand*, and their *behavior*.

As for *distribution*, it may be assumed that customers are either spread uniformly over a given set or that they are located at specific points in space or vertices in a network.

As for the actual demand, each customer is assigned a weight which expresses the amount of service he/she requires, i.e. its *demand*. When the customer is a single user, the associated weight can be a unit weight, or a fixed weight representing the effective demand of the user for the good or service. When the demand point is a symbolic representation of an area destination for the service (such as a community or a city), the weight is often used to account for the total demand arising in that area (for instance it might be a function of the population size).

In both cases of single users and demand areas, the demand may not necessarily be known with certainty. If facilities provide essential services (such as issuing driver's licenses, or providing polling places on election day), consumer demand may be deterministic and known a priori. However, for facilities that provide non-essential services (for example fast food restaurants, retail stores or ATM machines), consumer demand may be a function of the total cost of receiving service. Many models handle variable demands [65, 80]. For instance, production models with price-sensitive demands[101]; models with demand weights satisfying a specified distribution [8]; models with elastic demands with respect to both distance and price [60, 68], and so on.

Analogous to the possibility of facilities providing multiple services, we note the possibility that customers may require different services or goods. Problems involving demand for different kinds of service are referred to by many authors as *multi-commodity location problems*. Further, customer demand need not be always static: it can show temporal variability, meaning that it shifts at intervals over the planning horizon.

The last characterization about customers concerns their *behavior*. In some applications, customers are free to choose from which facility to be served, in which case the question is whether they will always patronize the closest facility or use some other criterion which reflects their preferences. In this respect, they can behave individually or as a group, meaning that when choosing a facility they might consider the convenience of all the other members of the group. Conversely, location problems exist where the assignment of customers to specific facilities is compulsory, as in the case of schools location in some districts.

Other aspects involving customers behavior will be further explored in Section 3.5 while analyzing the interrelation among customers.

3 Interrelationships among basic elements

Facilities, locations and customers constitute the building blocks of any location problem. However, a complete model specification can only be achieved through the description of the interactions among the three basic elements. The main purpose of this section is to illustrate how the components previously described are related to each other, and how their interrelation reveals essential features characterizing both generic and specialized models.

3.1 Facility-Location Relationships

The interaction between facilities and locations mainly concerns the variability of investment costs and capacity restrictions due to the assignment of supply centers to particular sites. In fact, specific geographical features, territorial or administrative regulations, or the presence of existing infrastructures, might cause the construction or remodeling of a facility to be more costly at some positions rather than at others, or might set bounds to the center size and production capacity. Usually, restricting the number of supply centers to a fixed number p amounts to an investment constraint with the assumption that all supply centers require the same level of investment. However, in many models this assumption is relaxed and it is assumed that a fixed investment expense is incurred if a facility is established at a particular location. Thus, the number of facilities is determined by budget constraints plus tradeoffs of higher investment costs versus lower customer service costs. The same considerations hold for facility capacities, which can vary depending on the specific position that the facility will occupy.

3.2 Facility-Facility Relationships

Even if many location models assume that the facilities to be located are not directly dependent on each other, there may be cases in which there is some kind of synergistic or competitive interaction among the new centers. This can happen under two different circumstances. First, when the model allows a flow of services or goods directly among new facilities. This is often the case in layout location problems, where for each pair of facilities to be assigned to locations, there is a weight associated with the activity between them. In the second case, there is no direct activity among new facilities, but the facilities operate in a competitive environment, where the centers relative positions strongly affect the performance of the whole system. This aspect characterizes many economic location problems, hence commonly referred to as *competitive location problems*. A typical example is provided by the location of retail outlets and other commercial facilities.

The facility-facility relationship also includes cooperative aspects among facilities, such as in the case of post office locations. A close proximity among offices might cause a large variance in the number of users choosing among them, thus increasing the probability of queueing delays and, hence, worsening the overall service quality.

Finally, the interdependence among facilities naturally arises when the problem requires a covering of the population with respect to some service delivery. In this case, the spatial distribution of facilities is crucial to guarantee that all users receive the service from at least one facility within some time or distance limit.

3.3 Customer-Facility Relationships

The relationship between customers and facilities mostly focuses on the way the users demand is satisfied by the supply centers.

In some location problems, solutions must simply indicate the best location for the new facilities, under the assumption that the flows of goods or services between the facilities and the demand points are given. However, in situations where there is more than one new facility to be located, determining the flows is often part of the problem. When the location of several new facilities is to be determined simultaneously with the allocation of customer demand to the supply centers, the problem is referred to as a *location-allocation* problem. Such models are particularly suitable when planning for the addition of new facilities to an existing structure or rearranging an existing layout. Many problems in locational analysis belong to this category.

Another element defining the relationship among customers and facilities is the number of centers that customers can use to satisfy their demand. Some applications might require each customer to be served by a single facility (in this case we talk about *single-source problems*); others allow customers to split their demand among several centers (*multi-source problems*).

Customer-facility relationships can include even more complex issues, such as the ones often arising in problems which deal with the location of emergency facilities. In these problems, a criterion for judging the efficiency of the service provided by the facility to the users is the speed at which the system reacts to an emergency call, which is critically affected by the availability of service. Anytime a server is not immediately available to provide service to the customer, the phenomenon of congestion can arise, economic loss occurs, and the model needs to include some additional tools to handle those features. Congestion and delay aspects also involve the interaction among customers and will be further analyzed in Section 3.5.

Finally, customers and facilities are related through customers preferences for supply center location. In this respect, users can have four different orientations depending on the facility to be located. Namely, they can find it desirable, meaning that its closeness is an attractive feature and they would like to have the center as close as possible; they can consider it undesirable, in the sense that its closeness provides a disutility to the individuals, who would like to push it away (as in the case of obnoxious facilities which are either polluting or involve a risk to the environment, e.g. garbage incinerators, nuclear reactors, and storage tanks for highly flammable or poisonous substances). Alternatively, customers might be indifferent, in which case they can be left out of the model as they have no influence on the objective to be achieved. Finally, customers can judge the facility partly desirable and partly undesirable, as for the location of supermarkets or airports: their closeness is convenient but can cause some disadvantages due for examples to traffic, noise and so on. Customer preference about facility proximity is a crucial point in the location process, since their translation into objectives represents one of the major forces moving facilities to their optimal location.

3.4 Customer-Location Relationships

An essential part in the formulation of location problems is to identify an efficiency measure of the interaction occurring among the locations where the facilities are positioned and the customers using the service, so as to provide a tool for driving the location process towards a satisfactory result with respect to a number of different objectives.

The quality of the interactions is considered to be directly related to the relative spatial position of the interacting points (namely, customers and facility locations), and is usually expressed by some notion of distance. Many different distance measures may be of interest depending on the application, and the study and choice of adequate distance concepts have almost become a research field in its own right. The definition of distance measures represents the first step towards the specification of different efficiency criteria, which can be simply built by converting estimated distances into appropriate costs. For instance, distances can be adjusted to reflect travel times, by including factors such as physical or social barriers to travel, congestion and road conditions. In some cases distance-related elements do not appear directly as objectives but might be necessary to capture some particular aspects of the problem in the form of additional constraints.

In the following, we describe some distance measures which have been extensively analyzed and used in location theory to approximate distances between two spatial coordinates. For the sake of simplicity, we now consider the case of planar continuous spaces and denote by (x_1, x_2) and (y_1, y_2) the coordinates of two points x and y, identifying the position of the customer and the facility location for which we want to measure the distance.

The most familiar and widely used distance measure is the *straight-line* or *euclidean* measure, denoted by $l_2(x, y)$. It is derived from the euclidean norm and can be mathematically written as:

$$l_2(x,y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2} \tag{1}$$

Euclidean distance applies when movement is allowed homogeneously in all directions, as occurs for instance in some network location problems involving conveyors and air travel. Some electrical wiring problems and pipeline design problems are also examples of euclidean distance problems.

The second topper distance measure, $l_1(x, y)$, is the variously referred *rectangular, rectilinear, metropolitan*, or *Manhattan* distance. Rectilinear distances derive from the rectangular norm and can be mathematically expressed as:

$$l_1(x,y) = \mid x_1 - y_1 \mid + \mid x_2 - y_2 \mid \tag{2}$$

Interestingly, rectangular distances combine the feature of being very simple to treat analytically, thanks to their linearity properties, and the feature of being a very appropriate distance measure for a large number of location problems. In particular, they are adequate in some urban location analyses where travel occurs along an orthogonal set of streets.

Rectangular and straight-line distances are both special cases of l_p-distances (or *Minkowski*'s distances), whose general expression is given by:

$$l_p(x,y) = (\mid x_1 - y_1 \mid^p + \mid x_2 - y_2 \mid^p)^{1/p} \qquad 0 \leq p \leq \infty \tag{3}$$

Using the l_p distance function to model actual distances results in more accurate distance measures than restricting use to either of the special cases $p = 1$ (rectangular distance) or $p = 2$ (euclidean distance). Due to their fairly simple analytical expression, and stimulated by approximation studies on how to best fit real-world distances by theoretical ones, much of the current research directly assumes l_p-distances. Many papers can be found in the literature discussing techniques for empirically fitting the value of p to actual distance data. Some examples are given in the pioneering work by Love and Morris [69, 70], and in subsequent papers such as [5, 6, 14].

Different kinds of gauges have been proposed by other authors as distance predicting functions, but the weighted l_p norm seems to remain the most accurate. Some examples are the *block norms* described in [102], and the skewed norms [81], particularly useful for applications involving movement on an inclined plane [54] or flight under a steady wind.

When problems can be modeled as network problems, distances are defined by the length of the shortest path linking the corresponding points in the network. The shortest path can be computed with respect to different measures, such as actual distances defined on each arc, travel times, travel costs and so on. For a description of the basic properties of the distance function used in network models, we refer the reader to [62].

Finally, the overall relationship among all customers and locations can be stated through the use of norms. In this case, the norms commonly used are either the L_1 or the L_∞ norms, which represent respectively the sum of all the distances among the customers and the locations they have been assigned to, and the maximum distance among users and locations. Both norms have been extensively used as tools to measure the system performance for each facility location alternative. Further details on this point will be provided in later sections.

3.5 Customer-Customer Relationships

The relation among customers is especially relevant in those problems presenting market externality features, which are often associated with variable demand in *user-choice* environments. Location models incorporating externalities have almost universally assumed the externality to be a queuing delay (or congestion). A congestion externality exists when the utility of one customer is affected negatively or positively by the actions of other customers. Examples of negative congestion externalities include waiting time in service facilities, traffic delay in transportation systems, the bed occupancy levels in hospitals, noise pollution in residential areas, and so on. Congestion can also act as a positive externality, for example in locating nightclubs. Externalities imply that, when deciding which facility to patronize, customers consider not only the distance they have to travel, but also the cost associated with the congestion externalities. In this sense, the attractiveness of a facility, and hence the customer demand for the service, is affected by the other customers behavior. Problems which incorporate market externalities often include queuing theory elements to model the service of the new facilities. For a broader insight of models involving market externalities the reader is referred to [11, 12, 13].

Other examples of interaction among customers arise in some location-related problems, as for instance the location-routing problems [63, 66, 73, 79]. In a wide variety of services, such as delivery, customer pick-up, repair and maintenance services, a service unit regularly visits several customers during a given service tour. In location-routing problems that arise in such services, the objective is to find the home location for the service unit and design tours for units so as to minimize the total costs. The relative position of the service users affects both the selection of the service station and the choice of the journey scheduled for each server. Therefore, the interdependence among customers leads to an externality which can significantly increase the complexity of the problem.

4 Combination of basic elements into models

A critical issue in siting facilities is how to display and present alternatives for decisions making. This task amounts to finding an efficient way to combine all the aspects relating facilities, locations, customers and their interrelationships, as described before, and to state them mathematically as an optimization problem. The way elements are merged and integrated to form a location model with specific requirements will become clearer in the next section, which is devoted to the description of some major models and their variations.

In particular, the efficiency criterion driving the site selection process is usually defined through the overall customer-location relation, which hence becomes the objective function of the model. However, many other measures of goodness are possible [17]. Many practical location problems, for instance, are multicriteria in nature, and require solutions which represent efficient and acceptable compromises among several, often conflicting criteria. The tradeoffs usually involve cost (such as fixed facility setup costs, variable operating costs, etc.) and quality of service (for example fraction of demand served, response time or average distance traveled). A very customary approach to deal with more than one criterion simultaneously is to identify an objective of primary importance, which becomes the single objective of the model, and to use side constraints to account for the secondary objectives. An alternative approach is the use of methods borrowed from multicriteria optimization theory [88, 108]. An analysis in this direction has been conducted by Ross and Soland [87]. Multicriteria aspects in location analysis will be further explored in Section 5.5.

5 Representative problems from the literature

In the previous sections, we described the three basic components shared by the wide range of location problems, their properties and interdependencies, and for each of these features, we designed a menu of choices to specifically account for the most common problem variations. The aim of this section is to illustrate how these features can be combined to form the structure of some major location models.

To state the models mathematically, we will use the following notation:

$I = \{1, \ldots, n\}$: set of candidate facility locations.
$J = \{1, \ldots, m\}$: set of customers.

s_i : capacity of a facility located at site i.

w_j : demand for the service of customer j.

f_i : fixed cost for establishing a facility at location i.

d_{ij} : minimum distance between customer j and candidate location i.

c_{ij} : cost of supplying all the demand of customer j from a facility located at location i.

y_i : decision variable which takes value either 1 or 0 according to whether facility location i is established or not.

x_{ij} : fraction of customer j's demand supplied from facility i.

As already mentioned in section 3.4, the cost c_{ij}, which expresses the efficiency measure of the interaction among customers and locations, might simply coincide with the shortest distance d_{ij} between facility sites and customers, or be an adjustment of the distance which reflects travel costs or transportation costs, and includes some exogenous factors. Hence, in some models they are used interchangeably, in others they assume a slightly different meaning.

5.1 p-Median Problems

One of the most studied problems in locational analysis is the so called *p-median problem*.

The p-median problem, in its most general and simplest form, is characterized by the following facility, location and customer features and relations. Facilities to be located do not have capacity restrictions, their number is fixed to p and they all provide the same kind of service. Locations have a network-based spatial representation with customers positioned at the nodes and facilities anywhere on the links and at the vertices. Customers require a fixed amount of service for a single commodity and they always choose to be served by the closest facility among the ones being established. Their relation to locations is expressed through a distance function which represents the shortest path on the network to reach the location. The cost for establishing a facility is not dependent on the particular site where it is located and all the facilities are assumed to have the same setup cost which can then be neglected in the problem formulation.

The overall customer-location relationship is expressed through the L_1 norm, also known as the *minisum* objective, and it represents the efficiency criteria driving the selection of facility sites.

One of the most remarkable results in the study of p-median problems is due to Hakimi [43], who proved that the search for the set of p optimal locations for the facilities can be limited to the node set of the graph instead of the infinite number of points that lie on the links. This important result made it possible to study the problem in a discrete space rather than in the more complex continuous setting, leading to the following mathematical formulation. It is worth noticing that in this model the set I of candidate sites for facility location coincides with the set J of demand nodes, and that the variables x_{ij} take value either 0 or 1, since the demand of each customer is always entirely allocated to the closest facility. Further, x_{ij} operationalizes the allocation rule by defining the set of demand points served by each center.

$$\min \quad \sum_{j \in J} w_j \sum_{i \in I} d_{ij} x_{ij} \tag{4}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij} = 1 \qquad \forall\ j \in J \tag{5}$$

$$y_i - x_{ij} \geq 0 \qquad \forall\ i \in I,\ j \in J \tag{6}$$

$$\sum_{i \in I} y_i = p \tag{7}$$

$$x_{ij} \in \{0,1\} \qquad \forall\ i \in I,\ j \in J \tag{8}$$

$$y_i \in \{0,1\} \qquad \forall\ i \in I \tag{9}$$

Expression (4) states that the objective to be minimized is the weighted sum of distances from each demand node to its closest facility, the weight being the total demand arising from the community located at that node. Constraints (5) ensure that all customers are allocated to exactly one center; constraints (6) guarantee that no user is allocated to a site that has no facility; constraint (7) forces the number of open facilities to be exactly p.

The p-median model was formulated in this classical version in the early 1960s as an extension of the single-facility Weber problem [104] to the multiple supply points case.

Many variations of the classical p-median model exist which derive from different choices for some element features and/or for the kind of relations among some basic components. As an example, variations of the facility features can produce *median shortest path problems* (MSPP) in the case of facilities presenting specialized shapes such as trees [28], or *mobile facility location problems* [94] dealing with the location of mobile facilities that travel in the space and stop at several service points where users can receive services. Also, a network version of the p-median problem exists which includes interaction among new facilities, i.e. the *p-median problem with mutual communication* (PMMC) [19, 95].

Very extensive is also the study of variations of the classical model arising from the uncertainty underlying some customer and location features. p-median problems dealing with uncertainty are usually referred to as *stochastic network p-medians*. Uncertain parameters can involve customer features such as demands [75]; customer-location relations such as travel time [74, 75]; customer-facility relations such as server availability [7, 10, 62].

Also the spatial distribution of candidate locations can differentiate among p-median problems. A special case is the one in which the underlying metric space is a tree [96, 109]. Further, p-median problems have been defined in a continuous space, in which case they are referred to as *multi-Weber* problems [22, 106].

Solutions to p-median problems maximize consumer accessibility to server facilities, since access is usually strictly related to distances. As a consequence, this model is applicable in those location contexts where maximizing consumer access to supply centers is a major objective and it is reasonable to assume that consumers visit the nearest facility. This is likely to be the case for convenience stores, fast food outlets, and services such as banks and post offices. More generally, minisum objectives are especially appropriate in the context of facility construction for delivery of nonemergency services. However, this criterion tends to favor customers which are clustered together to the detriments of customers who are spatially dispersed. This flaw induced some authors [31] to question the adequacy of minisum objective to those public sector applications where fast accessibility represents a critical point and must be guaranteed to all clients. As a result, alternative objective functions have been proposed and adopted in subsequent studies, as we will describe in the next sections.

5.2 p-Center Problems

Another class of very well studied problems in location analysis, which are quite simple to characterize, is the class of *p-center problems*. Under many aspects, p-center models are almost identical to p-median models. The major feature differentiating the two classes concerns the overall customer-location relationship.

As previously remarked, the minisum efficiency criterion used in p-median problems is not appropriate if system performance is directly related to extreme distances, as for locating emergency urban facilities, designing detection or signaling systems, or locating rapid delivery industrial services and so on. A more applicable objective for this class of problems is the *minimax* criterion, expressed through the L_∞ norm, which attempts to locate a facility so as to make the longest customer-facility distance as short as possible.

As for p-median problems, the classical version of p-centers is characterized by unlimited facility capacities, fixed number of facility to be opened, network solution space, and unit customer demands. The problem consists in finding the locations of p facilities that can cover all demand nodes in the minimum possible distance, which is endogenously determined and is the objective function of the problem. Unlike p-median problems, however, even the unweighted version of the p-center model does not have the Hakimi property, i.e. an optimal solution does not necessarily exist in the set of vertices of the graph. This gives rise to a further partition of the class of p-center problems into *vertex p-center problems* as opposed to *absolute p-center problems*. The first subgroup requires the facilities to be located on the nodes of the network; the second allows the facilities to be located anywhere on the graph. We now provide a mathematical formulation for the unweighted vertex p-center problem. The notation is the same as the one used in the p-median case. The variable W indicates the maximum distance to be minimized.

$$\min \quad W \tag{10}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij} = 1 \qquad \forall\ j \in J \tag{11}$$

$$y_i - x_{ij} \geq 0 \qquad \forall\ i \in I,\ j \in J \tag{12}$$

$$\sum_{i \in I} y_i = p \tag{13}$$

$$\sum_{i \in I} d_{ij} x_{ij} \leq W \qquad \forall\ j \in J \tag{14}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\ i \in I,\ j \in J \tag{15}$$

$$y_i \in \{0, 1\} \qquad \forall\ i \in I \tag{16}$$

The new set of constraints (14) simply forces the variable W to be at least the longest customer-location distance.

Vertex p-center and absolute p-center problems have been extensively studied in both the cases of general networks and tree networks, with unit or fixed customer demand weights. A variety of complexity results and a number of algorithms are surveyed by [62, 97].

By analogy with the p-median problem with mutual communications, there is a version of the p-center problem which involves direct facility-facility relationships (*p-center problem with mutual communication*) [98].

A portion of the network location literature involves also probabilistic cases of p-center problems [10, 62]. Interesting variations of vertex p-center problems derive from the introduction of special singularities about locations and their eligibility for facility placements. These singularities involve additional restrictions on acceptable points as in the case of the *alternative* p-center problems [56], or the *set p-center* problems [53]. Finally, a version of the p-center problem exists which assumes that candidate locations for facility site are distributed in the continuum of a plane (*continuous p-center problem*) [38, 105].

5.3 Warehouse Location Problems

Another major class of location problems is represented by *warehouse location problems* (also referred to as *distribution location problems*), whose name derives from the fact that they aim to determine best sites for intermediate stocking points or warehouses while planning physical distribution systems.

This model class differs from the p-median class under two main aspects related to the facility-location relation and to the facility features. The first refers to the presence of fixed costs for operating and locating (e.g. leasing) facilities which relates supply centers to the specific location where they are sited; the second concerns the number of facilities to be opened which is not known a priori. Additionally, the efficiency criteria used to drive the site selection is the minimization of the total cost, i.e. the sum of set up costs and transportation costs. Such an objective is usually referred to as *fixed-charge* objective.

The simplest in the class of distribution location models is the single-commodity case involving unlimited capacity, single-echelon and linear costs. This problem, usually denoted by UFLP for *Uncapacitated Facility Location Problem*, can be mathematically formulated as follows.

$$\min \quad \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i \tag{17}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} = 1 \qquad \forall\ j \in J \tag{18}$$

$$y_i - x_{ij} \geq 0 \qquad \forall\ i \in I, j \in J \tag{19}$$

$$x_{ij} \geq 0 \qquad \forall\ i \in I, j \in J \tag{20}$$

$$y_i \in \{0,1\} \qquad \forall\ i \in I \tag{21}$$

Formula (17) establishes that the objective of the problem is the minimization of the sum of fixed costs for facility opening and shipment costs for providing demands to customers. Constraints (18) and (19) have the same meaning as in the p-median and p-center problems. The formulation is single-echelon since the facilities to be located are the supply points (that is, there are no transshipment points). Observe that the interaction between facilities and customers is expressed through some prespecified costs.

Often, warehouse location problems also assume the presence of facility capacities which depend on the particular location where they are established. In this case the problem is referred to as the *capacitated facility location problems* (CFLP), and the following additional constraints must be added to the mathematical formulation:

$$\sum_{j \in J} w_j x_{ij} \leq s_i y_i \qquad \forall\ i \in I, \tag{22}$$

where w_j is the demand of customer j and s_i is the maximum amount of demand that can be allocated to a plant located at i.

Warehouse location problems began to appear in the Operations Research literature with such papers as [4]. Since then on, they have been extensively studied by many other authors who proposed different variations of the original model and several solution methodologies [20, 23, 36, 57, 92, 93].

A little variation to the classical model concerns the type of facilities to be located, especially with reference to the number of warehouse echelons [58]. For a mathematical formulation of the *simple uncapacitated multi-echelon facility location model* and a survey of solution approaches, the reader is referred to [2]. In particular, two-level models have been analyzed in more recent works by [1, 3, 40].

Another problem variation results from the diversification of the type of service required by customers, i.e. commodities. Several papers appeared handling this special case, usually denoted as *multicommodity facility location problem* (MFLP) [24, 25, 42, 59, 103].

Finally, warehouse facility location has been considered which takes uncertainty into account. Customer demands, as well as selling prices, production and transportation costs may appear in location models as random elements [64, 67].

5.4 Covering Problems

Another class of location problems which captures a big portion of location literature includes the so called *covering problems.* The idea of covering models is to identify locations that provide potential users access to service facilities within a specified distance or travel time, with the objective to "cover" customers in order to "capture" their demand.

Covering models were originally developed for public sector location problems, but they are also suitable in designing multifacility networks for service oriented retail firms (such as movie theaters, banks, or ice cream parlors), where access is a major determinant of patronage and where a good location strategy can have a significant impact on market share and profitability.

The concept of coverage can be integrated into the model in two different ways: it can be required or optimized. This distinction dichotomizes set covering problems into two main categories: *Location Set Covering Problems* (LSCP) and *Maximal Covering Problems.*

LSCP [99, 100] seeks to position the minimum number of facilities to meet some prespecified standards of performance. The typical model assumes that facilities have no capacity restrictions, they are homogeneous in the kind of service they provide and their cost does not depend on a particular location. Their number is unknown and becomes the objective of the model to be minimized. Customers have deterministic static demands. Candidate facility sites are located at the nodes of a network. The relation among customers and facility locations, which provides the performance measure, represents the basis for the definition of "coverage" and is usually expressed through the notion of distance.

Let us assume that an unambiguous threshold of performance has been specified (e.g. a maximum distance D) so that any facility location i can be seen as either satisfying or falling short of achieving that level of performance with respect to any customer j. Then, we say that a candidate facility site i *covers* a demand point j if the point i satisfies the threshold of performance with respect to j. We can hence define a coverage matrix $A = [a_{ij}]$, where each element a_{ij} takes value 1 if i covers j, 0 otherwise (for instance, if the threshold is a standard distance D, $a_{ij} = 1$ if $d_{ij} \leq D$). LSCP can now be stated simply as follows.

$$\min \quad \sum_{i \in I} y_i \tag{23}$$

$$\text{s.t.} \quad \sum_{i \in I} a_{ij} y_i \geq 1 \qquad \forall\ j \in J \tag{24}$$

$$y_i \in \{0,1\} \qquad \forall\ i \in I \tag{25}$$

The objective (23) minimizes the number of facilities required. Constraints (24) state that the demand of each customer j must be covered by at least one facility located within the time or distance standard.

One of the variations that has been proposed for this model addresses the issue of locating facilities which are not completely desirable with respect to customer preferences [78].

The LSCP objective of providing universal service may not be feasible in many situations due to the cost of operating an excessive number of facilities. A second group of covering models has then been studied where coverage is optimized with a limited budget. The budget restriction is reflected as a constraint on the number of facilities to be sited, which hence is known a priori. Customer demands is deterministic and fixed and becomes the objective of the problem to be maximized. The resulting model is the *Maximal Covering Location Problem* (MCLP), framed by Church and ReVelle [21, 107], whose formulation is given below.

$$\max \quad \sum_{j\in J} w_j x_j \tag{26}$$

$$\text{s.t.} \quad \sum_{i\in I} a_{ij} y_i \geq x_j \qquad \forall\ j \in J \tag{27}$$

$$\sum_{i\in I} y_i = p \tag{28}$$

$$x_j \in \{0,1\} \qquad \forall\ j \in J \tag{29}$$

$$y_i \in \{0,1\} \qquad \forall\ i \in I \tag{30}$$

The objective of maximizing the demand that is covered is operationalized through the definition of the variables x_j and the constraints (27). These constraints dictate that x_j is equal to 0 if demand zone j is uncovered. Hence, an uncovered zone does not contribute to the objective function. Constraint (28) has the usual meaning of limiting the number of facilities to a specified number p. The objective function maximizes the amount of demand that is covered by p facilities within the specified accessibility criterion.

The maximal covering idea was also generalized to account for some changes in facility features, such as in the *maximum covering/shortest path problem* (MCSP) [27], aiming to locate path-shaped facilities, or in the *FLEET model* [89], where the location of facilities providing different kind of service is considered.

Covering models have also been extensively studied to deal with the customer-facility relationship concerning facility availability when customers require the service. Many different models have been proposed which treat congestion and service availability aspects. For example, the *Probabilistic Location Set Covering Problem* (PLSCP) [86], which includes some probabilistic constraints to force the level of server availability to be greater than or equal to a preset value, while minimizing the total number of servers; the *Maximum Expected Covering Location Problem* (MEXCLP) [30], which aims to maximize the expected value of population coverage within the time standard, given a fixed number p of facilities to be located; the *Maximum Availability Location Problem* (MALP) [85], which seeks to maximize the population which has service available within a stated travel time with a specified reliability.

Both SCP and MCLP models have also been studied in the presence of facility capacity restrictions by Current and Storbeck [29].

For a complete survey of covering problems and new lines of evolution including covering, availability and queueing issues, the reader is referred to [72, 90].

5.5 Other Location Problems

The four classes of location models we have surveyed so far (p-median, p-center, warehouse, and covering) are the most traditional and extensively studied. Nevertheless, they do not exhaustively cover the broad variety of issues arising in location analysis. In this section, we will briefly scan some less classical models focused on particular problem domains.

A class of location problems which have been extensively studied in the last few years is the class of *hub location problems* [15, 37, 48, 77], arising when the facilities to be located are *hubs*, i.e. interconnected facilities which serve as consolidation centers, transshipments points or switching points of traffic between specified origins and destinations. The advantage of using the hubs is that, by consolidating the flow of airline passengers, data packets, mail and so forth, economies of scale can be achieved whereby transferring flow between hubs is cheaper than the cost of moving flow to and from non-hub nodes. As a consequence of this property, hub location models find very large applicability in the design of telecommunication networks, airline passenger networks, and postal-delivery networks. A wide-ranging review and classification of hub location problems can be found in [16].

Particular issues concerning facility-facility relationships produce other well known location model classes which have emerged in the field of logistic, industrial organization and spatial economics. An example is provided by the *quadratic assignment location problem* (QAP) [52, 71], in which a fixed number of new facilities interchanging goods or services must be assigned to an equal number of candidate locations so as to minimize the cost of direct activity between pairs of facilities. Common types of facilities involved in quadratic assignment location problems are machines to be located in a factory floor, offices to be arranged in a building, or departments to be housed in a productive plant. These problems are also known as *plant layout problems*.

The facility-facility relationship also characterizes the class of *competitive location problems*. These models, which include interaction among facilities due to market competition, have attracted the interest of many economists, geographers and regional scientists. The study of *competitive location models* finds its foundation in the seminal paper of Hotelling [55]. Since Hotelling's paper, a myriad of different competitive models have appeared involving more and more complex features. The interdependence among facilities in competitive settings is usually modeled and studied within the framework of *noncooperative game theory*. A particularly interesting competitive model is the *Maximum Capture Problem* (MAXCAP) [84], which finds natural applications in the location of ATM machines, bank branches, fast food restaurants and so on. A detailed treatment of competitive models can be found in [39, 91]. Eiselt et al. [32] provide an extensive bibliographic survey with over 100 citations on competitive location.

Finally, the last set of location models we quote in this context, the *multiobjective location problems*, is chosen to evidence how multiple objectives can be considered simultaneously in a model to face the complexity of some siting decisions. A model which reflects the inherent tradeoff between two different objectives is the well-known *cent-dian problem* [45, 46, 47, 50], where the cost defining the customer-location relation is expressed through a convex combination of total distance and maximum distance among customers and facility sites. This kind of efficiency measure combines the economic aspects of the *minisum* objective used in p-median problems and the more "socially equitable" aspects of the *minimax* objective characterizing p-center problems. The applicability of cent-dian formulations is very broad, especially in the public sector. The location of many municipal service facilities falls into this category, since their placement should not be too far from any segment of the population while maximizing accessibility to the "average" citizen.

Multicriteria models are particularly suitable to represent location decisions where customer preferences are such that facility proximity is considered undesirable. In such a case, issues like risk, efficiency and equity must be taken into account simultaneously. Several multicriteria models have been studied which deal with the location of obnoxious facilities [34, 35, 83]. Thorough and extensive reviews on a variety of multicriteria location problems can be found in survey works like [26, 87].

6 Conclusion

As evidenced by the rich body of literature dedicated to the theoretical study of location problems and by the growing number of real world applications using location analysis tools, the field studying the optimal location of facilities in different spatial settings is a very active research area, copious in ideas and challenges.

In this paper we have reviewed many classical location models and their variations, trying to fit them in a general framework arising from the description of the three building blocks of location problems: facilities, locations and customers.

Many other possibilities might exist to combine the main features characterizing location models. However, many models have not been analyzed yet and await further developments. We hope that the framework presented in this paper may provide an effective tool for stating and studying new challenging and complex models.

References

[1] K. Aardal, M. Labbè, J. Leung, and M. Queyranne. On the two-level uncapacitated facility location problem. *INFORMS J. Comput.*, 8:289–301, 1996.

[2] C.H. Aikens. Facility location models for distribution planning. *European Journal of Operational Research*, 22:263–279, 1985.

[3] A.I. Barros and M. Labbè. A general model for the uncapacitated facility and depot location problem. *Location Science*, 3:173–191, 1994.

[4] W.J. Baumol and P. Wolfe. A warehouse-location problem. *Operations Research*, 6:252–263, 1958.

[5] W. Berens. The suitability of the weighted l_p-norm in estimating actual road distances. *European Journal of Operational Research*, 34:39–43, 1988.

[6] W. Berens and F.J. Koerling. Estimating road distances by mathematical functions. *European Journal of Operational Research*, 21:54–56, 1985.

[7] O. Berman, S.S. Chiu, R.C. Larson, A.R. Odoni, and R. Batta. Location of mobile units in a stochastic environment. In *Discrete Location Theory*, pages 503–549. R.L. Francis and P.B. Mirchandani, Editors, Wiley-Interscience, New York, NY, 1990.

[8] O. Berman and C. Parkan. A facility location problem with distance dependent demand. *Decision Sciences*, 12:623–632, 1981.

[9] M.L. Brandeau and S.S. Chiu. An overview of representative problems in location research. *Management Science*, 35:645–674, 1989.

[10] M.L. Brandeau and S.S. Chiu. A unified family of queueing location models. *Operations Research*, 38:1034–1044, 1990.

[11] M.L. Brandeau and S.S. Chiu. Facility location in a user-optimizing environment with market externalities: analysis of customer equilibria and optimal public facility locations. *Location Science*, 2:129–147, 1994.

[12] M.L. Brandeau and S.S. Chiu. Location of competing private facilities in a user-optimizing environment with market externalities. *Transportation Science*, 28:125–140, 1994.

[13] M.L. Brandeau, S.S. Chiu, S. Kumar, and T.A. Grossman. Location with market externalities. In Z. Drezner, editor, *Facility Location. A*

Survey of Applications and Methods, chapter 7. Springer-Verlag, New York, Inc., 1995.

[14] J. Brimberg and R.F. Love. Estimating travel distances by the weighted l_p norm. *Naval Research Logistics*, 38:241–259, 1991.

[15] J.F. Campbell. Integer programming formulations of discrete hub location problems. *European Journal of Operational Research*, 72:387–405, 1994.

[16] J.F. Campbell. A survey of hub location. *Studies in Locational Analysis*, 6:31–49, 1994.

[17] E. Carrizosa, F.R. Fernandez, and J. Puerto. An axiomatic approach to location criteria. In F. Orban and J.P.Rasson, editors, *Proceeding of the Fifth Meeting of the EURO Working Group in Locational Analysis*, pages 40–53, FUNDP, Namur, Belgium, 1990.

[18] E.J. Carrizosa, E. Conde, M. Munoz, and J. Puerto. The generalized weber problem with expected distances. In *RAIRO*, volume 29, pages 35–57, 1995.

[19] D. Chhajed and T.J. Lowe. m-median and m-center problems with mutual communication: solvable special cases. *Operations Research*, Vol. 40 Supp. No. 1:S57–S66, 1992.

[20] N. Christofides and J.E. Beasley. Extension to a lagrangian relaxation approch for the capacitated werahouse location problem. *European Journal of Operational Research*, 12:19–28, 1983.

[21] R.L. Church and C. ReVelle. The maximal covering location problem. *Papers of the Regional Science Association*, 32:101–118, 1974.

[22] L. Cooper. Location-allocation problems. *Operations Research*, 11:331–343, 1963.

[23] G. Cornuejols, Fisher M.L., and G.L. Nemhauser. Location of bank accounts to optimiza float: an analytic study of exact and approximate algorithms. *Management Science*, 8:789–810, 1977.

[24] T.G Crainic, P.J. Dejax, and L. Delorme. Models for multimode multicommodity location with interdepot balancing requirements. *Ann. Oper. Res.*, 18:279–302, 1989.

[25] T.G Crainic, P.J. Dejax, and L. Delorme. A branch and bound method for multicommodity location with balancing requirements. *European Journal of Operational Research*, 65:368–382, 1993.

[26] J. Current, H. Min, and D. Schilling. Multiobjective analysis of facility location decisions. *European Journal of Operational Research*, 49:295–307, 1990.

[27] J.R. Current, C. ReVelle, and Cohon J.L. The maximum covering/shortest path problem: a multiobjective network design and routing formulation. *European Journal of Operational Research*, 21:189–199, 1985.

[28] J.R. Current, C.S. Revelle, and Cohon J.L. The median shortest path problem: a multiobjective approach to analyze cost vs. accessibility in the design of transportation networks. *Transportation Science*, 21:188–197, 1987.

[29] J.R. Current and J.E. Storbeck. Capacitated covering models. *Environment and Planning B.*, 15:153–163, 1988.

[30] M.S. Daskin. A maximum expected covering location model: Formulation, properties and heuristic solution. *Transportation Science*, 17:48–70, 1983.

[31] M. Dear. Planning for mental health care: a reconsideration of public facility location theory. *Int. Reg. Sci. Rev.*, 3:93–111, 1978.

[32] H.A Eiselt, Laporte G., and J.F. Thisse. Competitive location models: a framework and bibliography. *Transportation Science*, 27:44–54, 1993.

[33] H.A. Eiselt and G. Laporte. Objectives in location problems. In Z. Drezner, editor, *Facility Location. A Survey of Applications and Methods*, chapter 8. Springer-Verlag, New York, Inc., 1995.

[34] E. Erkut and S. Neuman. Analytical models for locating undesirable facilities. *European Journal of Operational Research*, 40:275–291, 1989.

[35] E. Erkut and S. Neuman. A multiobjective model for locating undesirable facilities. *Annals of Operational Research*, 40:209–227, 1992.

[36] D. Erlenkotter. A dual-based procedure for uncapacitated facility location. *Operations Research*, 26:992–1009, 1978.

[37] A.T. Ernst and M. Krishnamoorthy. An exact solution approach based on shortest-paths for p-hub median problems. *INFORMS Journal on Computing*, 10:149–162, 1998.

[38] R.L. Francis. Some aspects of a minimax location problem. *Operations Research*, 15:1163–1169, 1967.

[39] T. Friez, T. Miller, and R. Tobin. Competitive networks facility location models: a survey. *Pap. Reg. Sci. Assoc.*, 65:47–57, 1988.

[40] L.L. Gao and E.P. Robinson Jr. A dual-based optimization procedure for the two-echelon uncapacitated facility location problem. *Nav. Res. Log. Quart.*, 39:191–212, 1992.

[41] B. Gavish, D. Horsky, and K. Srikanth. An approach to the optimal positioning of a new product. *Management Science*, 29:1277–1297, 1983.

[42] B. Gendron and T.G. Crainic. A branch and bound algorithm for depot location and containerfleet management. *Location Science*, 3:39–53, 1995.

[43] S.L. Hakimi. Optimum location of switching centers and the absolute centers and medians of a graph. *Operations Research*, 12:450–459, 1964.

[44] T. Hale. Trevor Hale's location science references. Available at the address: http://www.ent.ohiou.edu/∼thale/thlocation.html.

[45] J. Halpern. The location of a cent-dian convex combination on an undirected tree. *Journal of Regional Science*, 16:237–245, 1976.

[46] J. Halpern. Finding minimal center-median convex combiantion (cent-dian) of a graph. *Management Science*, 24:534–544, 1978.

[47] J. Halpern. Duality in the cent-dian of a graph. *Operations Research*, 28:722–735, 1980.

[48] H.W. Hamacher, M. Labbé, S. Nickel, and T. Sonneborn. Polyhedral properties of the uncapacitated multiple allocation hub location problem. University of Kaiserslautern, Department of Mathematics, 2000. Report in Wirtschaftsmathematik Nr.67.

[49] H.W. Hamacher, S. Nickel, and A. Shneider. Classification of location models. *Location Science*, 6:229–242, 1998.

[50] G.Y. Handler. Medi-centers of a tree. *Transportation Science*, 19:246–270, 1985.

[51] G.Y. Handler and P.B. Mirchandani. *Location on Networks Theory and Algorithms.* MIT Press, Cambridge, 1979.

[52] F.S. Hillier and M.M. Connors. Quadratric assignment problem algorithms and the location of indivisible facilities. *Management Science*, 13:42–57, 1966.

[53] D.S. Hochbaum and A. Pathria. Generalized p-center problems: Complexity results and approximation algorithms. *European Journal of Operational Research*, 100:594–607, 1997.

[54] M.J. Hodgson, R.T. Wong, and J. Honsaker. The p-centroid problem on an inclined plane. *Operations Research*, 35:221–233, 1987.

[55] H. Hotelling. Stability in competition. *Econ. Journal*, 39:41–57, 1929.

[56] O. Hudec. On alternative p-center problems. *ZOR - Methods and Models of Operations Research*, 36:439–445, 1991.

[57] S. K. Jacobsen. Heuristic for the capacitated plant location model. *European Journal of Operational Research*, 12:253–261, 1983.

[58] L. Kaufman, M.V. Eede, and P. Hansen. A plant and warehouse location problem. *Operational Research Quarterly*, 28:547–554, 1977.

[59] J.G. Klincewicz and H. Luss. A dual-based algorithm for multiproduct uncapacitated facility location. *Transportation Science*, 21:198–206, 1987.

[60] M. Labbè and S.L. Hakimi. Market and locational equilibrium for two competitors. *Operations Research*, 39:749–756, 1991.

[61] M. Labbè and F.V. Louveaux. Location problems. In F. Maffioli M. Dell'Amico and S. Martello, editors, *Annotated Bibliography in Combinatorial Optimization*, chapter 16. John Wiley & Sons, Ltd., 1997.

[62] M. Labbè, D. Peeters, and J.F. Thisse. Location on networks. In *Handbooks in OR & MS*, volume 8, pages 551–624. Elsevier Science B. V., 1995.

[63] G. Laporte. Location-routing problems. In B.L. Golden and A.A. Assad, editors, *Vehicle Routing: Methods and Studies*, pages 163–197. Elsevier Science Publishers, North Holland, Amsterdam, 1988.

[64] G. Laporte, F.V. Louveaux, and L. Van Hamm. Exact solution to a location problem with stochastic demands. *Transportation Science*, 28:95–103, 1994.

[65] G. Laporte, F.V. Louveaux, and L. Van Hamme. Exact solution of a stochastic location problem by an integer L-shaped algorithm. *Transportation Science*, 28:95–103, 1994.

[66] G. Laporte, Y. Nobert, and S. Taillefer. Solving a family of multi-depot vehicle routing and location-routing problems. *Transportation Science*, 22:161–172, 1988.

[67] F.V. Lauveaux and D. Peeters. A dual-based procedure for stochastic facility location. *Operations Research*, 40:564–573, 1992.

[68] P.J. Lederer and J.-F. Thisse. Competitive location on networks under delivered pricing. *Oper. Res. Lett.*, 9:147–153, 1990.

[69] R.F. Love and J.G. Morris. Modelling inter-city road distances by mathematical functions. *Operational Research Quarterly*, 23:61–71, 1972.

[70] R.F. Love and J.G. Morris. Mathematical models of road travel distances. *Management Science*, 25:130–139, 1979.

[71] R.F. Love, J.G. Morris, and G.O. Wesolowsky. *Facilities Location: Models and Methods*. North Holland, NY, 1988.

[72] V. Marianov and C. ReVelle. Siting emergency services. In Z. Drezner, editor, *Facility Location. A Survey of Applications and Methods*, chapter 10. Springer-Verlag, New York, Inc., 1995.

[73] H. Min, V. Jayaraman, and R. Srivastava. Combined location-routing problems: A synthesis and future research directions. *European Journal of Operational Research*, 108:1–15, 1998.

[74] P.B. Mirchandani and A.R. Odoni. Location of mediands on stochastic networks. *Transportation Science*, 13:85–97, 1979.

[75] P.B. Mirchandani, A. Oudjit, and R.T Wong. Locational decisions on stochastic multidimensional networks. Technical report, Rensselaer Polytechnic Institute, Troy, New York, 1981.

[76] P. Nelson. New product pricing and positioning in an oligopolistic market, 1986. Graduate School of Management, Univeristy of Rochester.

[77] M. O'Kelly. The location of interacting hub facilities. *Transportation Science*, 20:92–106, 1986.

[78] C.S. Orloff. A theoretical model of net accessibility in public facility location. *Geographical Analysis*, 9:244–256, 1977.

[79] J. Perl and M.S. Daskin. A warehouse location-routing problem. *Transportation Research*, 19B:381–396, 1985.

[80] J. Perl and P.K. Ho. Public facilities location under elastic demand. *Transportation Science*, 24:117–136, 1990.

[81] F. Plastria. Continuous location anno 92: a progress report. In *Isolde VI Survey Papers, Studies in Locational Analysis*, volume 5, pages 85–127, 1993.

[82] F.P. Preparata and M.I. Shamos. *Computational Geometry. An Introduction.* Springer Verlag, 1985.

[83] S.J. Ratick and A.L. White. A risk-sharing model for locating noxious facilities. *Environment and Planning B*, 15:165–179, 1988.

[84] C. ReVelle. The maximum capture or sphere of influence problem: Hotelling reviseted on a network. *Journal of Regional Science*, 26:343–357, 1986.

[85] C. ReVelle and K. Hogan. The maximum availability location problem. *Transportation Science*, pages 192–200, 1989.

[86] C. ReVelle and K. Hogan. The maximum reliability location problem and α-reliable p-center problem: Derivatives of the probabilistic location set covering problems. *Ann. Oper. Res.*, pages 155–174, 1989.

[87] G.T. Ross and R.M. Soland. A multicriteria approach to the location of public facilities. *European Journal of Operational Research*, 4:307–321, 1980.

[88] Y. Sawaragi, H. Nakayama, and T. Tanino. *Theory of Multiobjective Optimization.* Academic, Orlando, 1985.

[89] D. Schilling, D. Elzinga, J. Cohon, R. Church, and C. ReVelle. The TEEM/FLEET models for simultaneous facility and equipment siting, 1979.

[90] D. Schilling, V. Jayaraman, and R. Barkhi. A review of covering problems in facility location. *Location Science*, 1:25–55, 1993.

[91] D. Serra and C. ReVelle. Competitive location in discrete space. In Z. Drezner, editor, *Facility Location. A Survey of Applications and Methods*, chapter 16. Springer-Verlag, New York, Inc., 1995.

[92] R. Sridharan. A lagrangian heuristic for the capacitated plant location problem with single source constraints. *European Journal of Operational Research*, 66:305–312, 1991.

[93] R. Sridharan. The capacitated plant location problem. *European Journal of Operational Research*, 87:203–213, 1995.

[94] A. Suzuki and A. Okabe. Using voronoi diagrams. In Z. Drezner, editor, *Facility Location. A Survey of Applications and Methods*, chapter 6. Springer-Verlag, New York, Inc., 1995.

[95] A. Tamir. Complexity results for the p-median problem with mutual communication. *Oper. Res. Lett.*, 14:79–84, 1993.

[96] A. Tamir. An $O(pn^2)$ algorithm for the p-median and related problems on tree graphs. *Oper. Res. Lett.*, 19:59–64, 1996.

[97] B.C. Tansel, R.L. Francis, and T.J. Lowe. Location on networks: A survey - part i: The p-center and p-median problems. *Management Science*, 29:482–497, 1983.

[98] B.C. Tansel, R.L. Francis, and T.J. Lowe. Location on networks: A survey - part ii: Exploiting the tree network structure. *Management Science*, 29:498–511, 1983.

[99] C. Toregas and C. ReVelle. Binary logic solutions to a class of location problems. *Geographical Analysis*, pages 145–155, 1973.

[100] C. Toregas, R. Swain, C. ReVelle, and L. Bergman. The location of emergency service facilities. *Operations Research*, 19:1363–1373, 1971.

[101] J.L. Wagner and L.M. Falkson. The optimal nodal location of public facilities with price-sensitive demand. *Geographical Analysis*, 7:69–83, 1975.

[102] J.E. Ward and R.E Wendel. Using block norms for location modelling. *Operations Research*, 33:1074–1090, 1985.

[103] A. Warszawsky. Multi-dimensional location problems. *Operational Research Quarterly*, 24:165–179, 1973.

[104] A. Weber. Uber den Standort der Industrien, 1909. traslated as *Alfred Weber's Theory of the Location of Industries*, University of Chicago, 1929.

[105] G.O. Wesolowsky. Rectangular distance location under the minimax optimality criterion. *Transportation Science*, 6:103–113, 1972.

[106] G.O. Wesolowsky. The Weber problem: History and perspectives. *Location Science*, 1:5–23, 1993.

[107] J. White and K. Case. On covering problems and the central facility location problem. *Geographical Analysis*, 281, 1974.

[108] P.L. Yu. Multiple criteria decision making: five basic concepts. In *Optimization, Handbooks of Operations Research and Management Science*, volume 1. G.L.Nemhauser, A.H.G.Rinnooy and M.J.Tood, 1989.

[109] B. Zelinka. Medians and the peripherians of trees. *Archivium Math*, 4:87–95, 1968.