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Abstract: A territory design problem motivated by a bottled beverage distribution company is addressed. The problem consists of finding a partition of the entire set of city blocks or basic units into a given number of territories subject to several planning criteria. Each unit has three measurable activities associated to it, namely, number of customers, product demand, and workload. The plan must satisfy planning criteria such as territory compactness, territory balancing with respect to each of the block activity measures, and territory connectivity, meaning there must exist a path between any pair of units in a territory totally contained in it. In addition, there are some disjoint assignment requirements establishing that some specified units must be assigned to different territories, and a similarity with existing plan requirement. An optimal design is one that minimizes a measure of territory dispersion and similarity with existing design. A mixed integer linear programming model is presented. This model is unique in the commercial territory design literature as it incorporates the disjoint assignment requirements and similarity with existing plan. Previous methods developed for related commercial districting problems are not applicable. A solution procedure based on an iterative cut generation strategy within a branch-and-bound framework is proposed. The procedure aims at solving large-scale instances by incorporating several algorithmic strategies. These strategies are evaluated and tested on some real-world instances of 5000, and 10000 basic units. The empircal results show the effectiveness of the proposed method in finding good quality solutions to these very large instances.

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RE: "Commercial Territory Design Planning with Realignment and Disjoint Assignment Requirements" by Roger Z. Ríos-Mercado and J. Fabián López-Pérez

Dear Professor Lev,

I would like to submit the above referenced paper to *Omega*. Please find an electronic version enclosed. The paper represents original work and is not under review elsewhere. Please address all future correspondence to me at the address below.

I look forward to hearing from you.

Sincerely,

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Highlights

> We study a commercial districting problem with realignment and disjoint assignment.

> We develop a hybrid branch-and-bound algorithm with cut generation heuristic.

> Exponential number of connectivity constraints is handled by a cut generation scheme.

> Strategies such as variable fixing and forced connectivity enhanced performance.

> Feasible designs to very large-scale real-world instances were successfully obtained.

Commercial Territory Design Planning with Realignment and Disjoint Assignment Requirements

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Abstract

A territory design problem motivated by a bottled beverage distribution company is addressed. The problem consists of finding a partition of the entire set of city blocks or basic units into a given number of territories subject to several planning criteria. Each unit has three measurable activities associated to it, namely, number of customers, product demand, and workload. The plan must satisfy planning criteria such as territory compactness, territory balancing with respect to each of the block activity measures, and territory connectivity, meaning there must exist a path between any pair of units in a territory totally contained in it. In addition, there are some disjoint assignment requirements establishing that some specified units must be assigned to different territories, and a similarity with existing plan requirement. An optimal design is one that minimizes a measure of territory dispersion and similarity with existing design. A mixed-integer linear programming model is presented. This model is unique in the commercial territory design literature

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as it incorporates the disjoint assignment requirements and similarity with existing plan. Previous methods developed for related commercial districting problems are not applicable. A solution procedure based on an iterative cut generation strategy within a branch-and-bound framework is proposed. The procedure aims at solving large-scale instances by incorporating several algorithmic strategies. These strategies are evaluated and tested on some real-world instances of 5000, and 10000 basic units. The empircal results show the effectiveness of the proposed method in finding good quality solutions to these very large instances.

Keywords: Bottled beverage distribution, Commercial districting, Mixed-integer programming, Branch-and-bound method, Heuristics

1. Introduction

Commercial TDP may be viewed as the problem of grouping basic units (i.e. city blocks, zip codes, or individual customers) into subsets according to specific planning criteria. These subsets are known as territories. There are some other spatial constraints as part of the geographic definition of the problem. Depending on the context of the problem, the concept "territory design" may be used as equivalence to "districting". Districting is a truly multidisciplinary research which includes several fields like geography, political science, public administration and operations research. However, all these problems have in common the task of subdividing the region under planning into a number of territories, subject to some capacity constraints. Indeed, territory design problems emerge from different type of real world applications. We can mention pick up and delivery applications, waste collection, school districting, sales workforce territory design and even some others related to geo-political concerns. Most public services including hospitals, schools, postal delivery, etc., are administered along territorial boundaries. We can mention either economic or demographic issues that may be taken in consideration for setup a balanced territory.

The problem addressed in this work is motivated by a real-world application in the bottled beverage distribution industry. As each territory is to be served by a single resource, it makes sense to use some planning criteria to balance the quantity of customers, product demand, and workload required by the dispatchers or truck drivers to cover each territory. Moreover, it is often required to balance the demand among the territories in order to delegate responsibility fairly. To this end, the firm wishes to partition the city area into disjoint territories that are suitable for their commercial purposes. In particular, given a set of city blocks or basic units (BUs), the firm wants to create a specific number of territories according to some planning criteria such as (i) compactness: customers as close to each other as possible, (ii) balancing with respect to each of the three activity measures (number of customers, product demand, and workload), (iii) territory connectivity: such that a truck assigned to a territory can deliver the goods without leaving the territory, (iv) disjoint BU assignment: that avoids assigning a specific subset of customers to the same territory, and (v) similarity with existing plan for a subset of BUs. In other words, the main objective of TDP is to group the customers into manageable sized territories in order to guarantee that BUs assigned to a territory are relatively close to each other and meeting the aforementioned planning criteria.

From the technical perspective, this combinatorial optimization problem is NP-hard [1]. To the best of our knowledge, the TDP version studied in this problem has not been tackled bafore. Related versions have been studied, though. State-of-the-art exact methods can solve instances of some simplified models of around 100-150 BUs. Typical real-world instances are very large and intractable by exact methods. There has been some heuristic appoaches for commercial TDPs. For instance, Ríos-Mercado and Fernández [2] developed a Reactive GRASP for a problem similar to ours; however, they measure territory dispersion based on the objective function of a p-Center Problem, and they do not consider the disjoint assignment constraints nor similarity with exiting plan. In our case, we are measuring dispersion by means of a function from a p-Median Problem. This of course leads to a different structure and make previuos approaches inapplicable. In addition, one of the main goals of our work is to develop a tool that can be relatively easy to implement in commercial off-the-shelve modeling languages and optimizers. This is of great value to the company.

Now, when this TDP is modeled as a mixed-integer programming problem, one of the main difficulties is that of the exponential number of connectivity constraints. These simply cannot be written out explicitly. On the other hand, this decision problem can be viewed as a two-level decison problem where at the top level one has to decide where to place territory centers (called location level) and at a second level one has to assign BUs to centers (called allocation level). Location-allocation approaches to TDP have been applied before. In our case, from a practical perspective there is a relatively fair knowledge of reasonable sites to act as territory centers.

Therefore, by assuming we have a good representation of these centers and fix them in advance, we focus on the allocation problem.

In this paper, we present a heuristic solution approach based on the iterative resolution of an associated mixed-integer programming model for the TDP aimed at obtaining high quality solutions to large-scale instances. The algorithm consists of iteratively solving a relaxed MILP model (relaxing the connectivity constraints), identifying violated constraints by solving an easy separation problem, and adding these violated cuts to the model. The procedure continues until no more connectivity constraints are needed. This is similar to the exact approach developed by Salazar-Aguilar et al. [1], except that they apply it to the complete model solving instances of up to 100-150 BUs. In our case, we apply this technique to the relaxed model which is solved considerable faster allowing the solution of larger instances. In addition, we have implemented some strategies that allow to fix some binary variables in advance. The solution method and algorithmic strategies were evaluated on a case study from industry. We found that this procedure is successful in finding good quality solutions for large-scale instances (i.e., 5000 BUs) in reasonable times. The results show the effectiveness of the proposed approach as it was able to obtain good quality solutions in terms of compactness and balancing.

The paper is structured as follows. In Section 2 we describe the problem. In Section 3 we present an overview of the most relevant work on models and algorithms for territory design. This is followed by Section 4, where the mathematical framework is presented in detail. The proposed solution approach is fully described in Section 5. In Section 6 we present the empirical evaluation of the proposed approach. We wrap up the paper in Section 7, with some conclusions and final remarks.

2. Problem Description

The territory design problem can be defined as the process of grouping small geographic areas, i.e. basic areas, into clusters or *territories.* called territories. As it is required, we define that every basic area should be contained in exactly one territory. Moreover, we require compactness and connectivity for the territories constructed. Indeed, connectivity means that the basic areas that conform a territory have to be geographically connected. It is easy to understand that in order to obtain contiguous territories, explicit neighborhood information for the basic areas is required. Our problem definition includes three measurable attributes or activities for each basic unit: (i) number of customers, (ii) product demand, and (iii) workload. The activity measure of a territory is the total sum of the activity measure of its indvividual basic units. As defined before, all territories should be balanced with respect to the three activity measures. Indeed, this balancing procedure takes into account each activity measure individually and simultaneously. It is interesting to point out that only a few authors consider more than one criterion simultaneously for designing balanced territories (e.g, Deckro [3], Zoltners [4], Zoltners and Sinha [5]).

The number of territories p to be constructed is fixed in advance. Our problem definition includes some prescribed and/or forbidden territories. That means that from the beginning we already have some basic areas which are required to be assigned to a specific territory. Furthermore, there are other basic areas which are not allowed to be assigned to the same territory. As can be verified, all these features could be easily extended to consider some territories that may already exist at the beginning of the planning process. That means that our method should be prepared to take the already existing territories into account and then add additional basic areas to them. This modeling feature could be applied to take into account geographical obstacles, e.g., rivers and mountains. We can generalize that the territory design problem is common to all applications that operate with a group of resources that need to be assigned in order to subdivide the work area into balanced regions of responsibility. The problem can be summarized as follows: partition the set V of basic areas into p territories which satisfy the specified planning criteria such as balance, compactness, connectivity, disjoint assignment, and similarity with existing BU assignment.

The problem specifications can be summarized as follows:

- Given a set of BUs (city blocks) for delivering bottled beverages, we need to partition this set into a given number p of disjoint territories.
- Each BU must be fully assigned to a single territory. It is not allowed to split BUs. That is, for each BU, the route that delivers product type 1, for instance, should be the same as the one that is responsible for delivering product type 2.
- For each BU, the following information is known with certainty: location coordinates (from the firm GIS), number of customers, product demand or sales volume measured by the number of 12-bottle boxes, and workload measured in time (min).

- The firm wants to design territories that are balanced (similar in size) with respect to each of the the three different activity measures in every BU. That is, the total number of customers, product demand, and workload assigned to each territory should be fairly distributed among the territories.

- Territories must be connected, that is, for any two BUs belonging to the same territory there must be a path connecting them totally contained in the territory.

- There is some pre-defined pairs of BUs that are required to be assigned to the same territory as much as possible. This is called similarity with existing plan. In a similar fashion, there are some predefined pairs of BUs that must be assigned to different territories. We called these disjoint assigment constraints.

- The goal of the design is to obtain territories that are as compact as possible, that is, the BUs in a given territory must be as close to each other as possible, and whose assigment includes as much as possible the similarity with existing subset of BUs.

3. Overview of Models and Solution Approaches

Depending on the context of the problem, Territory Design may be used as equivalent to Districting which is a truly multidisciplinary research field which includes several areas such as geography, political science, public administration, and operations research, as well. We can generalize that TDP is common to all applications that operate within a group of resources that need to be assigned in an optimal way in order to subdivide the work area into balanced regions of responsibility. We can mention pick up and delivery applications, waste collection, school districting, sales workforce territory design, and even some others related to geopolitical concerns. Most public services including hospitals, schools, and so on, are managed along territorial boundaries. We can mention either economic or demographic issues that may be considered for setting-up a well balanced territory.

In Operations Research the first work about districting can be traced back to Forrest [6]. The recent paper by Kalcsics, Nickel, and Schröder [7] is an extensive survey on approaches to TDP that gives an up to date state of the art and unifying approach to the topic. For a more extensive review related to sales districting see Zoltners and Sinha [8]. Another recent survey on districting models is the one by Duque, Ramos, and Suriñach [9].

One of the first mathematical programming approaches was proposed by Hess et al. [10]. The approach they applied was to decompose the location and allocation procedures into two independent phases. In the location phase the centers of the territories are chosen. For that purpose they use a capacitated p-median facility location method. Afterwards, on the second allocation phase the basic areas are assigned to these centers. Taking in mind the capacity of the locations selected, the objective is to assign each basic area to a unique location among the candidates, such that the demands of the basic areas are satisfied efficiently. The balancing requirement was modeled as a side constraint. Compactness and contiguity were tried to be obtained by minimizing the sum of distances between basic areas and territory centers. Due to its combinatorial complexity, the computational implementation of this model was limited.

In general for solving large scale problems, the allocation phase can be tackled by relaxing the integrality constraints on the assignment variables. However, this procedure usually assigns portions of basic areas to more than one territory center which is not desired. Hess and Samuels. [11] proposed a simple rule, which exclusively assigns the so called split areas to the territory center which “owns” the largest share of the split area. Fleischmann and Paraschis [12] report poor results with this heuristic. For about 50% of the resulting territories the activity measure of the territories was violated. Moreover, Zoltners and Sinha [5] model the allocation problem assigning basic areas to the closest territory center. This procedure yields compact and often connected territories, however, usually not well balanced. There are other types of methods named as “Divisional”. The idea of these types of methods is to iteratively partition the region under consideration into smaller and smaller subproblems. The iteration stops if a level has been reached where each subproblem can be solved.

Marlin [13] observes that using squared Euclidean instead of straight line distances produces compact but disconnected territories. Hojati [14] shows that a good selection for territory centers may impact on the resulting territories. We can mention another procedure named as “Multi-kernel growth”. This method starts by selecting a certain number of basic areas as “seeds” (centers) for the territories. Furthermore, one territory after the other is extended at its boundary through successively adding yet unassigned, adjacent basic areas to the territory. Here the procedure could check for minimal distance and/or better balance. This procedure stops until the territory constructed satisfies the activity measures constraints. See, e.g. Mehrotra, Johnson, and Nemhauser [15]. develop a column-generation method for districting where a decision binary variable is associated to a complete design. They are able to solve instances of relatively small- to medium-size.

Algorithms based on simulated annealing are proposed by Browdy [16], and D'Amico et al. [17]. Tabu search has been successfully applied in the recent papers of Bozkaya et al. [18] and Blais et al. [19]. Genetic algorithms for solving territory design problems have been introduced recently by Forman and Yue [20] and Bergey, Ragsadale, and Hoskote [21]. They encode the solution as it used to solve Traveling Salesman Problems. The encoding is a path representation through each basic area. As the basic areas are traversed, territories are formed by this sequence. Haugland, Ho, and Laporte [22] work with stochastic data which they argue is frequently present in territory design decisions, e.g., uncertain demand for basic areas.

There have been some studies on territory realignment that consists of developing territory designs subject to some constraints that attempt to keep an existing plan to the best possible extent. This issue have been studied in the context of political districting [16], school districting [23], and sales territory design [24]. In our problem, there is an interest on having a similarity, at least partially, not with an entire existing design, but with a given set of BUs. To the best of our knowledge, our model is the first to consider this issue within commercial districting.

As far as commercial territory design is concerned, Vargas-Suarez, Ríos-Mercado, and López [25] address a related commercial TDP with a variable number of territories, using as an objective a weighted function of the activity deviations from a given goal. No compactness criterion was considered. A basic GRASP was developed and tested in a few instances obtaining relatively good results. Ríos-Mercado and Fernández [2] studied the problem by considering compactness and contiguity but without joint assignment constrains. They used the objective function of the p-Center Problem for modeling territory dispersion. In that work, the authors proposed and developed a reactive GRASP algorithm for handling large instances. They evaluated their algorithm on 500- and 1000-node instances with very good results. More recently, Salazar-Aguilar et al. [1] develop an exact optimization scheme for solving the TDP with double balancing and connectivity constraints. They used their framework for solving models with both types of dispersion functions: the one based on the p-Center Problem (pCP) and the one based on the p-Median Problem (pMP). They observed that models with a pMP objective function were solved faster than the ones using a pCP objective. Furthermore, they also observed that solutions obtained from the relaxation of the pMP based models had a very high degree of connectivity. Still, the largest instance they could solve for the pMP based models was about 150 BUs. Our idea is to use a similar framework than the one they used in their work, except that we will be focusing in the allocation phase aiming at large instances. More recently, several approaches have been developed for multiobjective versions of the commercial TDP, including both exact optimization approaches [26] and metaheuristic methods [27].

4. Modeling Framework

The problem is modeled by a graph $G = (V, E)$, where a city block or basic unit (BU) i is associated with a node, and an edge connecting nodes i and j exists in E if blocks i and j are adjacent to each other. Now each node $i \in V$ has several associated parameters such as geographical coordinates (c_i^x, c_i^y), and three measurable activities. Let w_i^a be the value of activity $a \in A = \{1, 2, 3\}$ at node i, where a = 1, 2, and 3, referes to number of customers, product demand, and workload, respectively. A territory is a subset of nodes $V_k \subset V$. The number of territories is given by the parameter p. It is required that each node is assigned to only one territory. Thus, the territories define a partition of V. One of the properties sought in a solution is that the territories are balanced with respect to each of the activity measures. Thus, let us define the size of territory V_k with respect to activity a as: $w^a(V_k) = \sum_{i \in V_k} w_i^a$, $a \in A$. Due to the discrete structure of the problem and to the unique assignment constraint, it is practically impossible to have perfectly balanced territories with respect to each activity measure. To account for this, we measure the balance degree by computing the relative deviation of each territory from its average size μ^a, given by $\mu^a = w^a(V)/p$, $a \in A$. Another important feature is that all of the nodes assigned to each territory are connected by a path contained totally within the territory. In other words, each of the territories V_k must induce a connected subgraph of G. As mentioned before, due to strategic or political reasons, there are some BUs that are required to be assigned to different territories. Let H_d be set that contains all pairs of units that must be assigned to different territories, that is, $H_d = \{(j_1, j_2) \mid j_1 \text{ and } j_2 \text{ must be assigned to different territories}\}$.

This set will be used to represent these disjoint assignment constraints.

The company is also interested in keeping certain similarity with a subset of BUs from an existing plan. The concept of territory realignment [16, 23, 24] considers somehow either as a constraint or a term in the objective function a measure of dissimilarity with respect to previous plan. In this particular case, the company wishes to keep a similarity not with an entire existing design but with a subset of BUs. Let F^i denote the pre-specified subset of BUs associated to center i from an existing plan. Then the firm wishes that the new plan assigns to the new territory with center in i a significant proportion of the BUs from set F^i taking into account of course the corresponding distance measure. For instance, if two given units, say i and j belong to F^k, preference for assigning either of this to the new territory with center in k should be given to the unit nearest to k. This may be achieved by introducing a penalty term in the objective function q_{ij}. In addition, it is required that at least certain number of these BUs meet this assignment. This can be achieved by introducing a corresponding constraint. These can be seen in the model below.

Finally, industry demands that in each of the territories, blocks must be relatively close to each other. One way to achieve this is for each territory to select an appropriate node to be its center, and then to define a distance measure such as $D = \sum_{k=1}^{p} \sum_{j \in V_k} d_{c(k),j}$ where $c(k)$ denotes the index of the center of territory k so $d_{c(k),j}$ represents the Euclidean distance from node j to center of territory k. So maximizing compactness is equivalent to minimizing this dispersion function D. All parameters are assumed to be known with certainty. The problem can be thus described as finding a p-partition of V

satisfying the specified planning criteria of balancing, connectivity, and disjoint assigment, that minimizes the above distance-based dispersion measure and partial similarity with existing set of BUs.

4.1. MILP Formulation

Indices and sets

n	number of blocks (BUs)
p	number of territories
i, j	block indices; $i, j \in V = \{1, 2, \ldots, n\}$
a	activity index; $a \in A = \{1, 2, 3\}$
k	territory index; $k \in K = \{1, 2, \ldots, p\}$
E	edge set of adjacent blocks
H_d	set of pairs of BUs that must be assigned to different territories
F^i	set of BUs that are assigned to territory with center in i under a current design
N^i	$(= \{j \in V : (i, j) \in E \vee (j, i) \in E\})$ set of nodes which are adjacent to node i; $i \in V$

Parameters

w_i^a	value of activity a in node i; $i \in V$, $a \in A$
d_{ij}	Euclidean distance between i and j; $i, j \in V$

q_{ij} Weight of assigning unit j to center i equal to $0.5d_{ij}$ if $j \in F^i$; 0, otherwise; $i, j \in V$

τ^a relative tolerance with respect to activity a; $a \in A$, $\tau^a \in [0, 1]$

Computed parameters

$w^a(X_k)$ $(= \sum_{j \in X_k} w_j^a)$ size of set X_k with respect to a; $a \in A$, $X_k \subset V$

μ^a $(= w^a(V)/p)$ average (target) value of activity a; $a \in A$

Decision variables

In the original problem we are not concerned with territory centers; however, we introduce binary variables based on centers for modeling the dispersion measure.

$$x_{ij} = \begin{cases} 1 & \text{if unit } j \text{ is assigned to territory with center in } i;\ i, j \in V \\ 0 & \text{otherwise} \end{cases}$$

Note that $x_{ii} = 1$ implies that unit i is a territory center.

Model (TDP)

$$\begin{aligned} \min \quad & f(x) = \sum_{i,j \in V} d_{ij} x_{ij} + \sum_{\substack{i \in V \\ j \in F^i}} q_{ij}(1 - x_{ij}) && (1) \\ \text{s. t.} \quad & \sum_{i \in V} x_{ij} = 1 \qquad j \in V && (2) \\ & \sum_{i \in V} x_{ii} = p && (3) \end{aligned}$$

$$\sum_{j \in V} w_j^a x_{ij} \leq (1+\tau^a)\mu^a x_{ii} \qquad i \in V,\ a \in A \tag{4}$$

$$\sum_{j \in V} w_j^a x_{ij} \geq (1-\tau^a)\mu^a x_{ii} \qquad i \in V,\ a \in A \tag{5}$$

$$\sum_{j \in \cup_{v \in S} N^v \setminus S} x_{ij} - \sum_{j \in S} x_{ij} \geq 1 - |S| \qquad i \in V, \quad S \subset V \setminus (N^i \cup \{i\}) \tag{6}$$

$$x_{ij} + x_{ih} \leq 1 \qquad i \in V,\ (j,h) \in H_d \tag{7}$$

$$\sum_{i \in V} \sum_{j \in F^i} x_{ij} \geq \alpha |\cup_i F^i| \tag{8}$$

$$x_{ij} \in \{0,1\} \qquad i,j \in V \tag{9}$$

Objective (1) incorporates a term that measures territory dispersion and a term that favors the assignment of a subset of units from existing plan. Constraints (2) guarantee that each node j is assigned to a territory. Constraint (3) sets the number of territories. Constraints (4)-(5) represent the territory balance with respect to each activity measure as it establishes that the size of each territory must lie within a range (measured by tolerance parameter τ^a) around its average size. In particular, the upper bound balancing constraints (4) also assure that if no center is placed at i, no customer can be assigned to it. Constraints (6) guarantee the connectivity of the territories. These constraints, proposed by Drexl and Haase [28], are similar to the constraints used in routing problems to guarantee the connectivity of the routes. Note that, as usual, there is an exponential number of such constraints. The disjoint assignment is represented by constraints (7). Constraints (8) assure that at least a minimum number of BUs from existing plan is assigned, where α is a user-specified parameter usually set to 0.10 to 0.20 in practice.

Computational complexity: This TDP is $\mathcal{NP}$-hard. It can be reduced from the commercial TDP as follows. It is clear that a given solution can be checked for feasibility in polinomial time. Now, if we consider a special case where $F = \emptyset$, for all $i \in V$, and $H_d = \emptyset$, we are left with the commercial TDP which is known to be $\mathcal{NP}$-hard [1]. It follows our TDP is $\mathcal{NP}$-hard too.

Allocation Model

Now, we have attempted to solve Model A with a branch-and-bound method with very limited success. While instances of up to 150-nodes are somewhat tractable, it is no longer possible to solve larger instances within a few hours of CPU. The model has n^2 binary variables and a very weak LP relaxation.

The problem can be decomposed into a two-stage hierarchy problem. One can see a location phase, which has to do with placing the territory centers, and then an allocation phase, which has to do with assigning nodes to centers. Since our aim is to provide solutions to very large instances (in the order of 5,000-10,000 nodes), we make the assumption that the set of centers is given and focus our effort in the allocation phase. Let V_c be the set of centers. This set can be approximately obtained by means of previous knowldege, a heuristic, or a truncated branch and bound. The allocation phase model becomes.

$$\text{(AM)} \quad \min \qquad f(x) = \sum_{\substack{i \in V_c \\ j \in V}} d_{ij} x_{ij}$$

$$
\begin{aligned}
& & + \sum_{\substack{i\in V_c\\ j\in F^i}} q_{ij}(1-x_{ij}) & & & (10)\\
\text{s. t.} & \sum_{i\in V_c} x_{ij} & = & 1 & j\in V & (11)\\
& \sum_{j\in V} w_j^a x_{ij} & \leq & (1+\tau^a)\mu^a & i\in V_c,\ a\in A & (12)\\
& \sum_{j\in V} w_j^a x_{ij} & \geq & (1-\tau^a)\mu^a & i\in V_c,\ a\in A & (13)\\
& \sum_{j\in \cup_{v\in S} N^v\setminus S} x_{ij} & & & & \\
& -\sum_{j\in S} x_{ij} & \geq & 1-|S| & i\in V_c & \\
& & & & S\subset V\setminus (N^i\cup\{i\}) & (14)\\
& x_{ij}+x_{ih} & \leq & 1 & i\in V_c,\ (j,h)\in H_d & (15)\\
& \sum_{i\in V}\sum_{j\in F^i} x_{ij} & \geq & \alpha|\cup_i F^i| & & (16)\\
& x_{ij} & \in & \{0,1\} & i\in V_c, j\in V & (17)
\end{aligned}
$$

Model AM has pn binary variables. In typical location-allocation methods (Hess et al. [10], Kalcsics, Nickel, and Schröder [7]), the allocation model to be addressed has single balancing constraints, no contiguity constraints and no disjoint assignment constraints. The way this allocation problem is solved is by replacing the single balancing constraints by a single equation (i.e., making the tolerance parameter equal to zero) and relaxing the integrality restriction of the binary variables. The result is a transportation problem that is solved relatively efficiently. In this solution, which of course has perfect balance, there might fractional variables, i.e, a variable may be partially assigned to two or more centers. This new problem is named the split reslution problem and need to be solved according to certain criteria depending on the specific context. After the split resolution has been solved, the solution obtained may no longer necesarily satisfy the balancing constraints.

In our allocation model, we must deal simultaneously with multiple balancing constraints, connectivity constraints, and disjoint assignment constarints, which makes typical location-allocation procedures not applicable. So instead, our goal is to deal directly with the allocation model by developing several strategies within a branch-and-bound framework that would allow us to solve relatively large instances. By relaxing the connectivity constraints (14) from Model AM, we are left with the following relaxed model.

$$\text{(AMR)} \quad \min \quad f(x) = \sum_{\substack{i \in V_c \\ j \in V}} d_{ij} x_{ij} + \sum_{\substack{i \in V_c \\ j \in F^i}} q_{ij}(1 - x_{ij}) \tag{18}$$

$$\text{s. t.} \quad \sum_{i \in V_c} x_{ij} = 1 \qquad j \in V \tag{19}$$

$$\sum_{j \in V} w_j^a x_{ij} \leq (1 + \tau^a)\mu^a \qquad i \in V_c,\ a \in A \tag{20}$$

$$\sum_{j \in V} w_j^a x_{ij} \geq (1 - \tau^a)\mu^a \qquad i \in V_c,\ a \in A \tag{21}$$

$$x_{ij} + x_{ih} \leq 1 \qquad i \in V_c,\ (j,h) \in H_d \tag{22}$$

$$\sum_{i \in V} \sum_{j \in F^i} x_{ij} \geq \alpha |\cup_i F^i| \tag{23}$$

$$x_{ij} \in \{0,1\} \qquad i \in V_c, j \in V \tag{24}$$

In the following section we describe in detail the solution procedure.

5. Solution Approach

In this section we present a solution strategy for solving the Allocation Model (AM) given by (10)-(17). One main difficulty in the exponential number of connectivity constraints (14), which implies it is practically impossible to write them out explictly. Therefore, we consider instead the relaxation AMR of AM that consists of relaxing these connectivity constraints. The basic idea of our method is to solve model AMR and then check if the solutions obtained satisfy the connectivity constraints. To determine the violated connectivity constraints, a relatively easy separation problem is solved, and these cuts are added to model AMR. This procedure iterates until no additional connectivity constraints are found and therefore an optimal solution to model AM is obtained. This is guaranteed because the separation problem for identifying violated cuts is solved exactly. A general overview of the method is depicted in Figure 1.

[Figure 1 goes about here]

In Step 1, a branch-and-bound method is used (since we are not relaxing the integrality requirements of the binary variables). This approach is motivated by the fact that model AMR can be solved optimally by current branch-and-bound methods relatively fast for relatively large instances. For instance, 2000-node instances can be solved in a few seconds of CPU time in a PC. In addition, identifying and generating the violated cuts in Step 2 can also be done in polynomial time, so the overall procedure may be suitable as long as the number of iterations needed to reach optimality is not too large. The algorithm delivers an optimal solution to model AM. Several issues are of particular interest. We would like to investigate the empirical behavior of the method in terms of the number of iterations/cuts required to reach optimality. In addition, the fact we are assuming a fixed set of centers can be further exploited to develop several algorithmic strategies like variable fixing in Step 1. These are further ellaborated below.

Algorithmic strategies for speeding up convergence

- *Variable fixing: Eliminating assignments of relatively far units.* We proceed now to reduce the complexity of our problem by eliminating some unncessary binary variables x_{ij}. This idea is based upon the fact that in an optimal solution, from a practical standpoint it makes no sense to assign a BU that is very far away from a given territory center. Making this assignment will have a very negative impact in the objective function. It is clear that theoretically one can build a pathological instance where this might be the case; however, given the particular data distribution for this problem this never happens in practice. Thus, for each BU i we determine a reduced feasible subset R_i, such that we fix $x_{ij} = 0$ for all $j \in \bar{R}_i$. For each i we have reduced the number of binary variables from n to $|R_i|$. This is done as follows. First, for each center $i \in V_c$ we sort all the remaining nodes by nondecreasing order of d_{ij}. Let (j) denote the j-th nearest BU to i breaking ties arbitrarilyy, that is $d_{i(j)}$ denote the distance from BU i to the j-th nearest BU. Then, given a user specified parameter $\beta \in (0, \infty)$ the set R_i is given by $R_i = \{(j) \in V \ : \ \sum_{k=1}^{j} w_{i(j)}^a \leq \beta\mu^a$ for at least one $a \in A\}$. That is R_i is formed by the nearest BUs to i such that their accumulated sum of weights with respect to all activities do not exceed this threshold for at least one activity. A very large value of β implies $R_i = V$ so no reduction takes place. As β gets smaller, the number of variables fixed at 0 grows. A relatively small value of β means only a few binary variables will be considered (as many will be fixed at 0) overcompromising the optimality of the solution. An issue to investigate is precisely the sensibility and trade-off between solution quality and computing time as a function of β.

- *Variable fixing: Preassigning relatively close units.* Applying a similar rationale as in the previous point, it is possible to find a set of relative close units to a given center i such that in any optimal solution, all the units belonging to this set are always assigned to i. Again, while one can build an example where this might not happen, in practice we always see a considerable portion of relative close BUs being assigned to a center i. To this end, we determine a set K_i such that $x_{ij} = 1$ for all $j \in K_i$. Given a user-specified parameter γ the set K_i is given by $K_i = \{(j) \in V \;:\; \sum_{k=1}^{j} w^a_{i(j)} \leq \gamma\mu^a \text{ for all } a \in A\}$. That is K_i is formed by the nearest BUs to i such that their accumulated sum of weights with respect to all activities do not exceed this threshold for all activities. Here a value of $\gamma = 0$ implies $K_i = \emptyset$ and no reduction is applied. The larger γ the larger the numer of binary variables will be fixed at 1. So again, it is important to investigate the trade-off between solution quality and time as a function of γ.

- *Strengthening of connectivity constraints.* One way to strength the formulation of the relaxed model is by introducing the following constraints

$$x_{ij} \leq \sum_{q \in N^j} x_{iq}, \qquad i \in V_c,\ j \in V \tag{25}$$

These valid inequalities can be interpreted as follows. If BU j is assigned to territory with center in i at least one of its neighbors ($q \in N^j$) needs to be assigned to the same territory as that of BU j. These constraints avoid territories with just a single BU unconnected. The motivation for this stems from the fact that previous research [1] has shown that optimal solutions of the relaxed model contain most of the unconnected subsets S with cardinality equal to 1, that is $|S| = 1$. As can be seen, there is a polynomial number of these new constraints (25), so these can be easily added to the model without imposing a considerable computational burden. Of course, Step 2 of the algorithm still checks for violated constraints with subsets of cardinality $|S| > 1$.

- *Finding violated inequalities.* Step 2 of the method can be efficiently done in polynomial time. Let $X = (X_1, \ldots, X_p)$ be a design found in Step 1. For each territory k, there is associated subgraph induced by X_k given by $G_k(X_k, E(X_k))$. It is well known that finding all connected components of a graph can be donde by breadth first search (BFS) in $O(|E(X_k)|)$. So we apply BFS to graph G_k and find its r connected components $(G_k^1, \ldots, G_k^r)$, with corresponding node sets $(X_k^1, \ldots, X_k^r)$. It is clear that $r = 1$ implies G_k is connected; otherwise let us assume without loss of generality that the BU center of X_k, named $c(k)$ belongs to G_k^1. Clearly, each of remaining subsets $X_k^2, \ldots, X_k^r$ is disconnected from the center. Each of these corresponds to a violated constraint (14) where set X_k^q plays the role of set S in (14). By repeating this procedure for every set X_k one can efficiently solve this separation problem optimally and add all found violated cuts to set C in Step 3.

- *Forced connectivity strategy for faster convergence.* Step 2 is key for the effciency of the proposed methods. It has been observed that the number of iterations needed to find a connected solution in instances of up to 5000 BUs is very reasonable. However, for larger instances up to 10,000 BUs the computational effort grows considerable. The main cause for this is that it takes a significant large amount of iterations to converge. This stems from the fact the combinatorial nature of all possible unconnected subsets make the algorithm find and add a large number of different cuts. Therefore, we have implemented a heuristic strategy that can be employed as part of the method when faced with large-scale instances.

  To motivate this strategy, it is important to note that if we keep track of a single BU throughout the execution of the algorithm, it can happen that this node may or may not belong to an unconnected subset in the following iteration. In many cases, an oscillating behavior between being part of an unconnected subset and being part of the connected territory is followed by many of these nodes. Therefore to avoid this nasty behavior, instead of solving the AMR model we add a penalty term to the objective function that would favor the assignment of BUs that are already found to belong to a connected territory. The basic idea of this strategy is to take advantage of the connectivity information from a given iteration to attempt to avoid the oscillating behavior expecting to reduce at every iteration the number of BUs that belong to unconnnected subsets.

Let us define Z_t as the number of BUs that are disconnected at iteration t. This parameter is dynamic because the number of disconnected BUs changes at each iteration. In fact, it is expected that this parameter Z_t will tend to zero as the number of iterations grow. Let U_i^t the set of BUs that are connected to territory with center in i at iteration t. Then we add a term to the objective function (18) as follows.

$$\text{Minimize } g(x) \quad = \quad f(x) + \delta \sum_{i \in V_c} \sum_{j \in U_i^t} r_{ij}(1 - x_{ij})/(Z_t + 1) \qquad (26)$$

In this added term, a penalty $r_{ij} = \bar{d} - d_{ij}$, where $d^{\max} = \max_{ij}\{d_{ij}\}$ implies closer assignments are preferred, dividing by $Z_t + 1$ avoids division by zero and makes the preference fo the assignemt of already connected units stronger as the number of iterations grow, and parameter δ allows the user to control the weight of this added term with respect to the original objective function. Naturally, setting $\delta = 0$ implies deactivation of this strategy.

6. Empirical Evaluation

We implement our model on X-PRESS MIP Solver from FICO$^{\text{TM}}$ (Fair Isaac, Dash Optimization before). The method was executed on a PC with 2 Intel Cores at 1.4GHz and Win X64 operating system. For evaluation the proposed method, we use some real-world instances of 5000 and 1000 BUs and 50 territories. In all experiments we set $\tau^a = 0.10$ for all $a \in A$ and a relative optimality gap of 0.1 % as stopping criterion. These instances are available at: *http://yalma.fime.uanl.mx/˜roger/ftp/tdp/*.

[Table 1 goes about here]

Table 1 shows the effect on problem reduction by using different values of parameters β and γ, discussed in Section 5. The first two column reflect the size of the original instance in terms of its number of BUs, number of territories and number of binary variables. The third and fourth column display values of parameters β and γ, respectively. The fifth column (RNBV) displays the number of binary variables after the reduction, and the last column the relative reduction with respect to the original size given by 100 (NBV - RNBV)/NBV. It can be seen how the number of binary variables in the reduced problem grows as β increases and γ decreases. Note that the case $\beta = 50.0$ and $\gamma = 0.0$ implies no reduction is applied. In summary, the strategy we adopt is to decrease the feasible solution space to deal with a reduced problem that can be solved more efficiently without a significant loss on optimality. This trade-off on optimality is evaluated next.

We now apply the proposed method to instances of 5000 BUs with 50 territories. In this experiment we set $\delta = 0.0$, that is no forced connectivity strategy applied. The goal of this experiment is to assess the trade-off between solution quality and execution time for different values of β and γ.

Table 2 and displays the results for the 5000-BU instance. The first two columns display the values of β and γ used. The third and fourth column show the number of iterations (NI) and CPU time (sec.). The fifth column shows the optimal solution (OptSol) and the last column displays the relative optimality gap between this solution and the best known solution (corresponding to the row $\beta = 8.0$ and $\gamma = 0.0$).

[Table 2 goes about here]

As can be seen the quality of the results is very good reporting gaps of less than 0.02 % in less than 6 minutes in all cases.

In the following experiment we assess the effect of implementing the forced connectivity strategy for the solution of large instances. So we apply the method for different values of the parameters, to an instance with 10000 BUs and 50 territories, fixiing $\beta = 3.0$. Table 3 and displays the results for the 10000-BU instance. As can be seen the introduction of this strategy speeds up the algorithm considerably. The quality of the solution is not over-compromised. In fact, sometimes a better solution was found in less computational effort. For instance, for the ($\gamma = 0$) case it was observed how the soution improved from 124.150 to 124.142 when switching from $\delta = 0$ (no forced-connectivit strategy in action) to $\delta = 1.0$. This better solution was obtained in almost 50 % of the time employed by the $\delta = 0$ case. As we penalize more, moving from $\delta = 1.0$ to $\delta = 5.0$ we can see that the solution deteriorates sligthly (less than 0.01 %) but it is 90% faster. A similar behavior is observed when we look at the $\gamma = 0.1$ and $\gamma = 0.25$ cases separately. Here, the best solution 124.122 is obtained when activating the forced-connectivity strategy with $\delta = 1.0$. Finally, in the $\gamma = 0.50$ case, it was not even possible to find a feasible solution when $\delta = 0$, so activating the strategy with $\delta \geq 1.0$ helped obtain feasible designs. Overall the best solution was obtained when using $\delta = 1.0$ and $\gamma = 0.1$ or 0.25.

[Table 3 goes about here]

In order to show the behavior of the solution method in terms of solution quality versus computational time we plot the following measures: (i) number of unconnected BUs, (ii) number of unconnected territories, (iii) number of cuts added, and (iv) objective function value as a function of the iterations for several configurations of the parameters. Figures 2 to 5 display these results for (β, γ, δ) values of (3.0, 0.25, 50.0), (3.0, 0.25, 35.0), (3.0, 0.25, 25.0), and (3.0, 0.10, 7.0), respectively.

[Figure 2 goes about here]

[Figure 3 goes about here]

[Figure 4 goes about here]

[Figure 5 goes about here]

As it can be verified in Figures 2 and 3, the first two runs with a very high value on parameter δ have a similar behavior. The number of unconnected BUs, unconnected territories, and cuts added to the model decrease with the number of iterations. Something similar happens with the objective function value but in the opposite direction. On the other two cases (Figures 4 and 5) with a lower value of parameter δ, we have a very different behavior. Particularly, the objective function value moves slowly as the time grows. Indeed, this is the reason why lower objective function values are obtained. Either way, it is important to point out that our methodology presents a MIP model that ensures integral assignments at each iteration. Thus, it is interesting to verify how rapidly our heuristic implemented on the allocation MIP model can evolve and converge on solutions with very high degree of connectivity.

We now evaluate the method efficiency when applied to the solution of a 10000-BU instance with a smaller territory balance tolerance. For this case, we set $\tau^a = 0.05$. This new value for parameter significantly lower than the previous one of 0.10. Thus, we have a very large scale instance with a very narrow tolerance for territory balancing. This makes the problem extraordinarily difficult to solve. Our results are presented on Table 4

[Table 4 goes about here]

As can be seen, even in this more difficult case it was possible to obtain feasible design in very competitive times. The best solution was found under the $\gamma = 5.0$ and $\delta = 5.0$ settings showing the success of the introduced strategies for speeding up convergence, and improving solution quality.

[Figure 6 goes about here]

Finally, Figure 6 displays the graphical solution of a 5000-BU, 50-territory instance under tolerances of 0.05. This is a feasible solution satsifying all of the planning constraints. The legend besides the graph indicates teh number of BUs contained in each territory.

7. Conclusions

In this paper we have addressed a commercial territory design problem motivated by a real-world aplication in the bottled beverage distribution industry. Planning criteria includes territory compactness, territory balancing with respect to three activity measures, and territory connectivity. In addition, our model incorporates new issues such as disjoint assignment requirements and partial similarity with existing plan. We present a new MILP model in literature that consideres all these issues. A solution framework based on a cut generation strategy within a branch-and-bound algorithm for solving the allocation stage for a fixed set of territory centers is developed. This method is intended for solving large-scale instances. The method is enhanced through several algrithmic strategies that help reduce the size of the problem and search space. One added value is a very effective tool can be relatively easily implemented with off-the-self optimization modeling software such as X-PRESS, GAMS, AMPL. The method and its algorithmic strategies were assessed on large-scale real-world instances. Previous work on heuristics for some related commercual territory design models had reported empirical evidence on instances of up to 2000 BUs in some simplified models. We found the proposed method very successfull on handling instance of 5000-, and 10000-BUs, obtaining solutions of very good quality.

There are naturally opportunities for future research. For instance, in this work we focused on solving the allocation problem; however, the results obtained in our research can be used to extend this work to a location-allocation approach where the centers are dynamically updated in an iterative way. This has been done in other similar simpler models. Deriving models and methods for problems with both territory design and routing decisions simultaneously is another very challenging research area. In fact, when one looks at the districting literature in general, one can barely find a very few applications addressing this issue. Finally, in this work we are assuming a deterministic model; therefore the introduction of stochastic models to deal with some parameters such as the product demand becomes a natural extension worthile exploring.

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function `method( )`

Input: An instance of the TDP problem.

Output: A feasible solution X.

1 Solve model AMR and obtain solution X;

2 Identify a set C of violated constraints of model AM for solution X;

3 If $|C| > 0$, add these constraints to model AMR and go to Step 1;

4 Return X;

end `method`

Figure 1: A pseudocode of solution procedure.

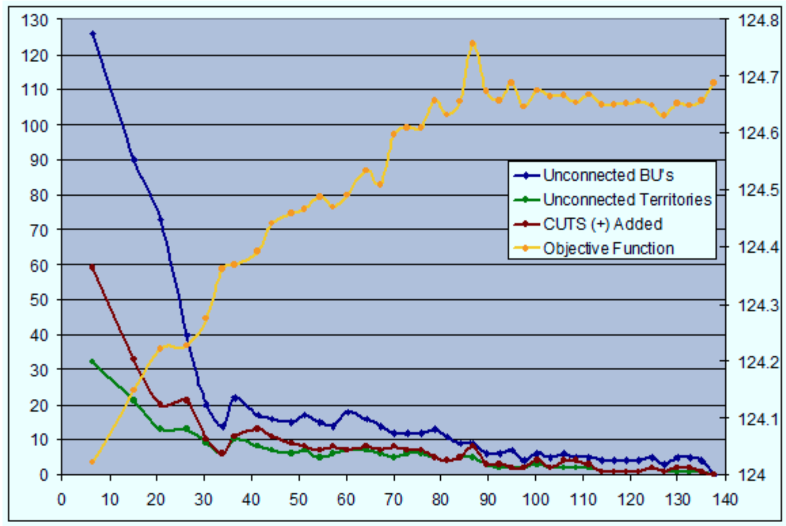

Figure 2: Algorithm performance for a (10000,50) instance with $\beta = 3.0$, $\gamma = 0.25$, $\delta = 50.0$.

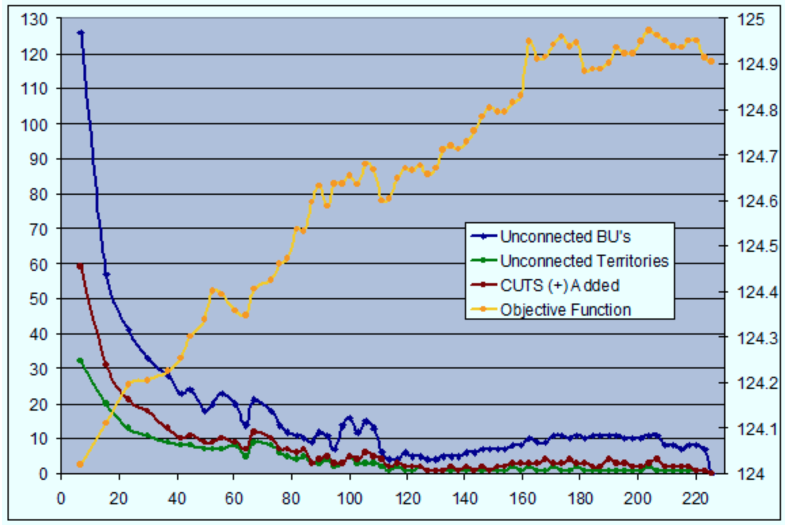

Figure 3: Algorithm performance for a (10000,50) instance with $\beta = 3.0$, $\gamma = 0.25$, $\delta = 35.0$.

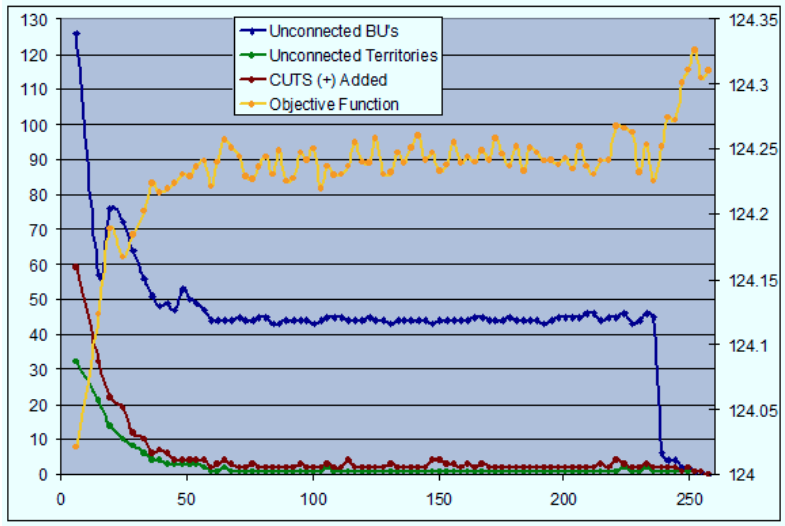

Figure 4: Algorithm performance for a (10000,50) instance with $\beta = 3.0$, $\gamma = 0.25$, $\delta = 25.0$.

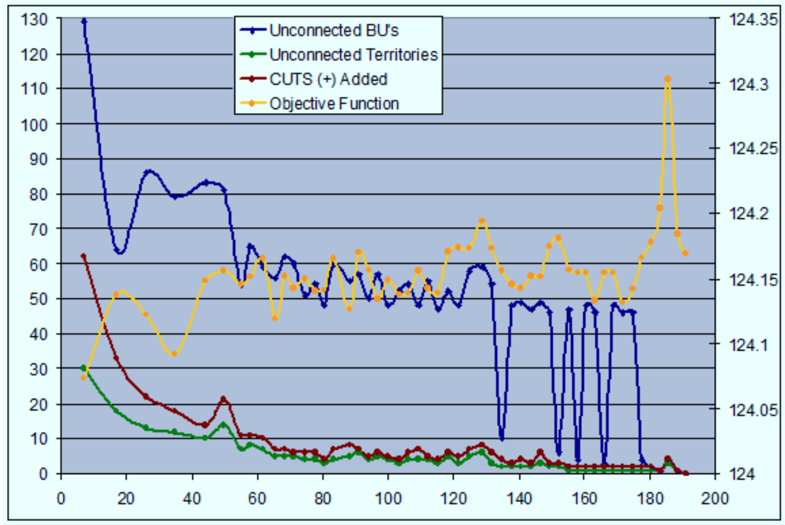

Figure 5: Algorithm performance for a (10000,50) instance with $\beta = 3.0$, $\gamma = 0.10$, $\delta = 7.0$.

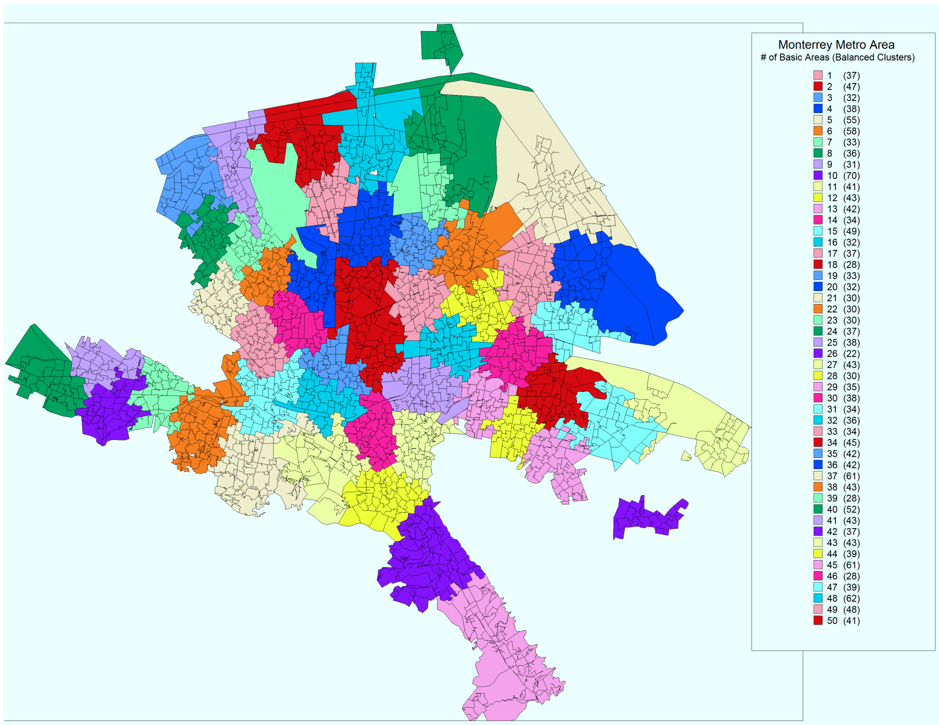

Figure 6: Visual results of an optimal territory design in Monterrey with 5000 BUs.

Table 1: Problem reduction effect

Size (n, p)	NBV (np)	β	γ	RNBV	Reduction (%)
(5000, 50)	250,000	3.0	0.50	7,191	97.1
		3.0	0.25	10,501	95.8
		3.0	0.10	12,428	95.0
		3.0	0.00	13,542	94.6
		4.0	0.50	9,702	96.1
		4.0	0.25	14,612	94.1
		4.0	0.10	17,545	93.0
		4.0	0.00	19,400	92.2
		8.0	0.50	20,484	91.8
		8.0	0.25	30,365	87.8
		8.0	0.10	36,253	85.5
		8.0	0.00	39,755	84.1
		50.0	0.00	250,000	0.0
(10000, 50)	500,000	3.0	0.50	14,615	97.1
		3.0	0.25	21,227	95.8
		3.0	0.10	25,027	95.0
		3.0	0.00	30,202	94.0
		4.0	0.50	19,968	96.0
		4.0	0.25	29,609	94.1
		4.0	0.10	35,352	92.9
		4.0	0.00	39,244	92.1
		8.0	0.50	41,531	91.7
		8.0	0.25	60,810	87.8
		8.0	0.10	72,214	85.5
		8.0	0.00	79,693	84.1
		50.0	0.00	500,000	0.0

Table 2: Results for instance (5000, 50).

β	γ	NI	Time	OptSol	Gap (%)
3.0	0.50	25	58	62.5027	0.011
	0.25	38	118	62.5056	0.016
	0.10	46	158	62.4972	0.003
	0.00	50	186	62.4978	0.004
4.0	0.50	44	146	62.5011	0.009
	0.25	60	262	62.4986	0.005
	0.10	48	223	62.4972	0.003
	0.00	54	264	62.4957	0.000
8.0	0.50	48	330	62.5101	0.023
	0.25	63	457	62.5002	0.007
	0.10	37	305	62.4976	0.003
	0.00	61	576	62.4956	0.000

Table 3: Results for instance (10000, 50).

β	γ	δ	NI	Time	OptSol	Gap (%)
3.0	0.50	50.0	305	947	124.732	0.49
	0.50	10.0	76	243	124.443	0.26
	0.50	5.0	97	404	124.373	0.20
	0.50	1.0	120	1062	124.296	0.14
	0.50	0.0	(*)	(*)	(*)	xx
	0.25	50.0	139	139	124.688	0.46
	0.25	10.0	114	114	124.248	0.10
	0.25	5.0	233	233	124.185	0.05
	0.25	1.0	1965	1965	124.122	0.00
	0.25	0.0	7442	7442	124.185	0.05
	0.10	50.0	46	161	124.670	0.44
	0.10	10.0	33	110	124.225	0.08
	0.10	5.0	47	203	124.171	0.04
	0.10	1.0	106	1026	124.122	0.00
	0.10	0.0	140	4132	124.168	0.04
	0.00	50.0	87	257	124.467	0.28
	0.00	10.0	41	145	124.244	0.10
	0.00	5.0	52	193	124.165	0.03
	0.00	1.0	136	1516	124.142	0.02
	0.00	0.0	94	3040	124.150	0.02

Table 4: Results for instance (10000, 50) with $\tau^a = 0.05$.

β	γ	δ	NI	Time	OptSol	Gap (%)
3.0	0.25	15.0	55	424	127.633	0.079
	0.25	10.0	54	689	127.770	0.187
	0.25	7.0	45	800	127.595	0.049
	0.25	5.0	35	615	127.587	0.043
	0.25	3.0	54	874	127.626	0.073
	0.10	15.0	478	1697	127.929	0.311
	0.10	10.0	154	812	127.698	0.130
	0.10	7.0	100	545	127.626	0.074
	0.10	5.0	61	694	127.532	0.000
	0.10	3.0	75	2261	127.543	0.009