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Classification of Spatial Properties for Spatial Allocation Modeling

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Abstract

Given a set of spatial units, such as land parcels and grid cells, how to allocate subsets of it to activities of interest while satisfying certain criteria? Such a decision process is here called spatial allocation. Though many problems of spatial allocation share this generic construct, each may have a quite unique set of criteria and interpret even the same criteria in its own way. Such diversity makes it difficult to model spatial allocation problems in unambiguous terms that are amenable to algorithmic solution. This paper proposes a classification scheme for spatial properties that helps to address a variety of spatial properties in establishing spatial allocation criteria. The implication of the paper is that a number of spatial properties and spatial allocation criteria can be decomposed into a few kinds of primitive spatial properties and their relations.

Key Words: spatial properties, spatial allocation, mathematical programming

1. Introduction

Efficient use of limited resources of space is a common interest for almost all who live on the earth. Farmers look for suitable sites for growing particular types of crops. Planners spend countless hours and meetings to design zoning plans for their jurisdictions. A politician may be interested in "gerrymandering" [50], i.e., redrawing boundaries of voting districts in favor of herself or against her opponents. With the continuing growth of computer technology and information science, these problems are now more efficiently tackled than before. In order for digital devices such as geographic information systems (GIS) to handle the infinite continuous nature of spatial variation, however, space must be discretized into a manageable number of elements [24]. This representation of space enables us to generalize the problems described above to what we call "spatial allocation:" the grouping of discrete spatial units into larger clusters according to specified criteria. Spatial units involved in spatial allocation may be socio-economic ones such as counties, Zip-code areas, census tracts, and land parcels [17], or results of systematic sampling such as pixels and grid cells [45]. There are different terms for clusters to which spatial units are allocated, such as "zones" [45], "regions" [6], "districts" [28], "territories" [27], and "turfs" [38] depending on context. They are herein referred to as "objects" for generality. Since such objects are not natural ones but products of human mental acts, they should be regarded as fiat in terms of Smith [42].

Spatial allocation encompasses a wide variety of applications ranging from school districting [4], [19], [53] and political districting [21], [28], [29], [36], to market analysis [18], [27], [34], [38], [39], [47], [54], to land use planning [13], [14], [46] and site selection [5], [6], [12], [22], [52]. While required criteria vary from one application to another, they tend to relate to spatial properties such as population "size," compact "shape," and distant "spatial relation." Nevertheless, such spatial properties, too, seem too diverse to handle in a systematic manner.

This paper then attempts to classify spatial properties in a way that facilitates mathematical modeling of spatial allocation criteria. The same topic was partially addressed by Shirabe and Tomlin's *Decomposing Integer Programming Models for Spatial Allocation* [41]. It was mainly concerned with implementation issues including a prototype language for spatial allocation modeling within a GIS environment. The present paper modifies it to place emphasis more on theoretical aspects, that is, how spatial properties are classified, formulated, and combined based on an assumption that some spatial properties are more primitive than others. In addition, this paper discusses a possible extension of the proposed scheme to handle spatial allocation with temporal considerations. The rest of the paper is organized as follows. Section 2 describes the diversity of spatial properties, which discourages a unified modeling approach. Section 3 discusses the complexity of existing models of spatial properties, which makes the models unsuitable—not easy to be inverted—for spatial allocation purposes. To (at least partially) resolve these issues, Section 4 decomposes spatial allocation models into two basic components: data and variables. Section 5 introduces and formulates four classes of spatial properties we consider primitive. As illustrated in Section 6, these primitives can be combined for complex spatial properties in establishing spatial allocation criteria. Section 7 summarizes the paper and suggests an agenda for future research.

2. Spatial properties

Spatial properties, as opposed to non-spatial counterparts (e.g., name, color, and texture), depend on where and how objects are situated in space. They are often informally and intuitively characterized in terms of size, shape, and spatial relation. As illustrated below, however, it is virtually impossible to enumerate, much less formulate, all conceivable such properties.

2.1. *Size*

Spatial properties relating to size are relatively easy to articulate. In many cases the size of an object refers to geometric quantity such as length, area, and volume. When an object is situated in a heterogeneous background like geographic space, however, a variety of "non-geometric" size properties arise. Take a city as an example. Its size may refer to its population, tax base, municipal waste production, vacant lot acreage, and so on. Furthermore, even the same attribute can be summarized differently to meet different purposes and contexts. For instance, the value of the most affordable real estate may characterize the city's livability better than the total value of all real estates or the average of them. As such, an object in geographic space has a virtually unlimited number of size-related properties depending on how and what attribute to summarize.

2.2. Shape

Shape is more difficult to express mathematically [48]. Though there were some early efforts for general description of shape [8], [10], most attempts have been directed toward the measurement of only selected aspects of shape. This is because no one parameter [32] or set of parameters [20] serves to describe all possible shapes. White and Renner [49] identified five types of shape properties, namely, compactness, attenuation (or elongation), proruption (or indentation), fragmentation, and perforation (or punctuatedness). Frolov [20] regarded compactness, indentation, and dissection as three key independent aspects of shape. Wentz [48] defined shape by three independent elements of edge, perforation, and elongation.

Obviously different applications are concerned with different properties of shape, but compactness "has been given the greatest attention due to its potential applicability to a broad range of geographic problem" [33] and may well be seen as "the most important single property of shape" [7] in the geographic context. In fact, a number of seemingly different shapes share the most compact shape (e.g., a circle or a square) as their extreme case (see Figure 1).

Compactness is a relatively simple concept. It can be intuitively understood as "consolidated rather than spread out" [28], "grouped or packed around its central point" [7], or "round in shape rather than long or thin" [26]. It is not difficult for the human eye to tell which of two objects is more compact than the other. Nonetheless, there is no universally-accepted description or measure of compactness.

Perhaps the simplest measure of compactness is based on the comparison between the area and the perimeter of an object. It may take on several variations such as area/perimeter, $\sqrt{\text{area}}/\text{perimeter}$, and $\text{area}/(\text{perimeter})^2$. These values are easy to compute and interpret, but their insufficient accuracy in representing compactness has been pointed out by many researchers [20], [33], [48].

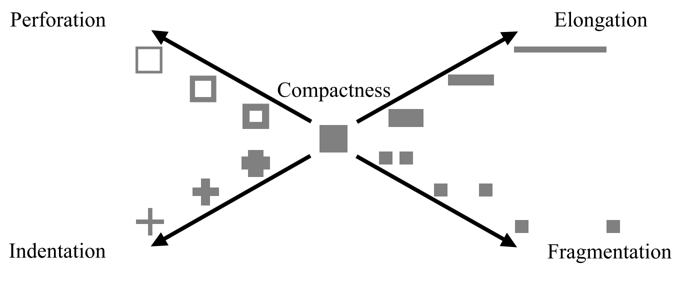

Figure 1. Four types of shape divergence from compactness. Source: P. J. Taylor's *Distances within Shapes: An Introduction to a Family of Finite Frequency Distributions* [43], [44]. With permissions from the University of Iowa's Department of Geography and Blackwell Publishing.

Moment of inertia [7], [26]–[28], [35] is often considered as one of the most accurate compactness measures [20], [33]. The moment of inertia of a body, in the physical sense, represents the degree of dispersion of the body's mass. As a measure of compactness, smaller values indicate more compact shapes. When an object is seen as a finite set of discrete spatial units, its moment of inertia, I, is formulated as:

$$I = \sum_{i \in I} a_i d_i^2$$

where

I is the set of spatial units constituting the object of interest,
a_i is the area of spatial unit i, and
d_i is the distance from the object's gravity center to spatial unit i.

There may be cases where this basic model needs modification. To neutralize its inherent scale dependence (i.e., the larger an object is, the greater its moment of inertia is), moment of inertia may be divided by area. Also, when emphasizing a functional meaning of compactness rather than a geometric one, one may want to replace the area of each spatial unit with another attribute (e.g., population [26]) of that unit.

Though moment of inertia is generally a useful concept for modeling compactness, it is possible to think of a more comprehensive alternative. According to Taylor's observation, "any areal shape will have associated with it a frequency distribution of distances between all points within its boundaries" [43], [44], compactness can be evaluated in terms of such parameters as the total, average, and maximum distances between all pairs of spatial units. An advantage of this concept is that no spatial unit needs to be specially designated as the object center, as all spatial units are equally taken into account.

Contiguity—the quality of being connected—is another key aspect of shape, since many other shape properties assume objects to be contiguous. Again, the human eye is generally good to discern this binary property. For the sake of formal modeling, however, it would be useful to adopt a graph-theoretic definition of contiguity or "connectedness" [2]. That is, by equating each spatial unit with a vertex and each adjacency relationship between a pair of units with an edge, a set of spatial units is said to be contiguous if there is a path between any two vertices. Contiguity, too, has variations depending on the nature of the adjacency relationship. While some are due to the difference in mathematical definition—e.g., two spatial units are adjacent if they share a certain geometric element such as an edge or a vertex, others are due to the difference in practical interpretation—e.g., two spatial units are adjacent if they share a certain geographic feature such as a water body or a forest patch.

2.3. *Spatial relation*

Properties concerning spatial relation are not attributed to individual objects, but to pairs (or groups) of objects. They are generally classified into two kinds: topological and metric. Topological relations are preserved under topological transformations such as translation, rotation, and scaling [16], while metric relations do not survive them. Egenhofer and Franzosa [15] showed that any two simply-connected 2-dimensional objects embedded in 2-dimensional space has one (and only one) of the following topological relations: “disjoint,” “meet,” “contains,” “covers,” “equal,” “overlap,” “inside,” and “coveredBy” (Figure 2).

Metric relations tend to involve measures of distance or direction, and thus tend to be readily quantifiable. Apart from difference in unit of measurement (e.g., meter vs. foot and degree vs. radian), distances and directions between points in Euclidean space can in fact be expressed in terms that are universally accepted. Even such straightforward concepts, however, can be ambiguous in two ways. First, distance, as a degree of geographic separation, can be evaluated in terms of various attributes such as time, cost, and utility. Second, distance measurement can involve different functions. For example, the distance between two objects, which consist of multiple spatial units, may refer to the distance between their nearest spatial units, that between their central spatial units, or that between their farthest spatial units. The same idea applies to direction, as well as other metric relations such as inter-visibility (i.e., the extent to which two objects can see each other). The differentiation is again due to how and what attribute to process.

3. Prescriptive modeling of spatial properties

The models described above are primarily for describing spatial properties of existing objects, and it is generally not easy to construct new objects of specified spatial properties. Nonetheless, it is the latter “prescriptive” [45] task that spatial allocation needs to address.

In modeling such a problem, what properties to be realized are often specified by narratives such as “the proposed site is compact” at the outset, but ultimately must be

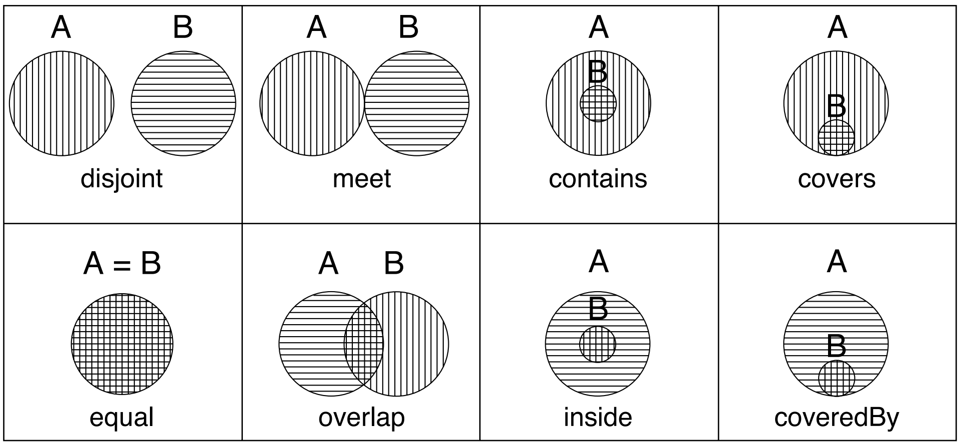

Figure 2. Topological relations between a pair of simply-connected 2-D objects [16].

expressed as a set of mathematical equations for systematic solution. These equations are then collectively and simultaneously inverted to solve for the unknowns. There exist general techniques for such inversion, of which mixed integer programming (MIP) is particularly useful for the problem under consideration. A MIP model consists only of linear functions with integer and continuous variables, and thus can effectively represent the process that involves indivisible, discrete spatial units. MIP models can be solved exactly by existing tools such as branch-and-bound algorithms, if they are not exceedingly large-scale. Alternatively spatial allocation problems may be approached by heuristic methods, which are designed to find approximate solutions [1], [9], [14]. These methods are often used when MIP modeling would require nonlinear functions and/or a large number of variables. They do not, however, guarantee to find best possible solutions and may not even tell how good/poor obtained solutions are. Thus, if heuristic solutions need to be correctly evaluated, an exact problem formulation like MIP model is still important.

Among the aforementioned three kinds of spatial properties, size seems relatively amenable to MIP formulation. For example, the geometric area of an object is the summation of the areas of each spatial unit within the object. On the contrary, while extensive research on shape description has been done, relatively little progress has been made in prescriptive modeling of shape. For instance, until recently a seemingly simple shape, contiguity, has been unable to be expressed in MIP terms [51], unless it is linked to other conditions such as compactness or a fixed object center or root [12], [36], [54]. As to spatial relations, they tend not to be given much attention in spatial allocation. Obviously, no spatial relation is relevant to a problem concerning a single object such as facility siting. Even in the case of multiple-object allocation like school/political districting, the non-overlap (or disjoint) relation seems to be the only frequent spatial relation explicitly taken into account. Yet, as problems involve more complex processes, however, other spatial relations can play equally important roles.

It seems that the complexity and the diversity of spatial properties are rather unmanageable in spatial allocation modeling. One possible remedy is to abstract a relatively few basic spatial properties from complex spatial properties and then to formulate each of such primitive properties in a way that they can be recombined into complex ones associated with specific spatial allocation criteria. To do so, we first identify building blocks of spatial allocation models.

4. Elements of spatial allocation models

As with other prescriptive models, spatial allocation models contain givens and unknowns. We call the former “data” and the latter “variables.”

4.1. Data

Data in a spatial allocation model represent the condition of a discretized space. The smallest entities of space are referred to as spatial units. Unlike Frank et al. [17], this paper assumes that spatial units do not change their identity, location, or attribute during the allocation process. Each spatial unit is herein denoted by $i \in I$ where I is a set of all spatial units. When two different units need to be referred to, one is denoted by i and the other by j. Each spatial unit is characterized by one or more attributes. Attributes may be size, distance to a particular target, acquisition cost, suitability for a land use, and so on.

While spatial units are the most fundamental entities of a spatial allocation model, it is convenient to regard pairs of spatial units (or spatial unit pairs) as another type of basic entity as Goodchild [23] originally proposed. Spatial unit pairs, too, can have attributes. Examples include distance, direction, and adjacency. It is certainly possible to think of attributes pertaining to three or more spatial units. This is conceptually trivial, but computationally problematic since higher levels of combination would result in exceedingly large volume of data.

To distinguish attributes for individual spatial units from those for spatial unit pairs, we call the former "1-unit attributes" and the latter "2-unit attributes." Each 1-unit attribute is herein denoted by $k \in K$ where K is a set of all l-unit attributes, and the value of l-unit attribute k for unit i is denoted by a_{ik}. Meanwhile, each 2-unit attribute is herein denoted by $l \in L$ where L is a set of all 2-unit attributes, and the value of 2-unit attribute l for the pair of spatial units i and j is denoted by l_{ijl}. For simplicity, the present paper assumes that 2-unit attributes are symmetrical for each of a spatial unit pair, i.e., $r_{ijl} = r_{jil}$; so that asymmetrical attributes such as direction is not considered in the rest of the paper.

4.2. *Variables*

Variables in a spatial allocation model concern objects (i.e., sets of spatial units). More precisely, a variable corresponds to a decision whether a certain spatial unit is allocated to a certain object. Each object is herein denoted by $m \in M$. When two different objects need to be referred to, one is denoted by m and the other by n. Note that since the present scheme does not presuppose objects are mutually exclusive, a spatial unit may be allocated to two objects.

Similarly to spatial units, objects can be paired for an object pair. This pairing is useful when spatial properties are not attributed to individual objects but to relations between two objects. Again, an object grouping is limited to no more than two to keep models small enough to be practically solvable.

Objects and object pairs are determined by the following variables:

x_{im} : equal to 1 if unit i is allocated to object m, 0 otherwise
x_{imn} : equal to 1 if unit i is allocated to objects m and n, 0 otherwise
y_{ijm} : equal to 1 if units i and j are allocated to object m, 0 otherwise
y_{ijmn}: equal to 1 if units i and j are allocated to objects m and n, respectively, 0 otherwise

These variables are subject to the following constraints.

$$x_{im} \in \{0, 1\} \qquad \forall i, \in I, m \in M \tag{1}$$

$$x_{imn} \in \{0, 1\} \qquad \forall i \in I; m, n \in M | m \neq n \tag{2}$$

$$y_{ijm} \in \{0, 1\} \qquad \forall i, j \in I | i \neq j; m \in M \tag{3}$$

$$y_{ijmn} \in \{0, 1\} \qquad \forall i, j \in I | i \neq j; m, n \in M | m \neq n \tag{4}$$

$$x_{imn} \geq x_{im} + x_{in} - 1 \qquad \forall i \in I; m, n \in M | m \neq n \tag{5}$$

$$x_{imn} \leq x_{im} \qquad \forall i \in I; m, n \in M | m \neq n \tag{6}$$

$$x_{imn} \leq x_{in} \qquad \forall i \in I; m, n \in M | m \neq n \tag{7}$$

$$y_{ijm} \geq x_{im} + x_{jm} - 1 \qquad \forall i, j \in I | i \neq j; m \in M \tag{8}$$

$$y_{ijm} \leq x_{im} \qquad \forall i, j \in I | i \neq j; m \in M \tag{9}$$

$$y_{ijm} \leq x_{jn} \qquad \forall i, j \in I | i \neq j; m \in M \tag{10}$$

$$y_{ijmn} \geq x_{im} + x_{jn} - 1 \qquad \forall i, j \in I | i \neq j; m, n \in M | m \neq n \tag{11}$$

$$y_{ijmn} \leq x_{im} \qquad \forall i, j \in I | i \neq j; m, n \in M | m \neq n \tag{12}$$

$$y_{ijmn} \leq x_{jn} \qquad \forall i, j \in I | i \neq j; m, n \in M | m \neq n \tag{13}$$

Constraints (1)–(4) reflect the fact that all the variables are associated with binary decisions. The value 0 indicates "no" or "false," while the value 1 indicates "yes" or "true." Constraints (5)–(7) together set x_{imn} = 1 when unit i is allocated to objects m and n, and 0 otherwise. Constraints (8)–(10) together set y_{ijm} = 1 when units i and j are allocated to object m, and 0 otherwise. Constraints (11)–(13) together set y_{ijmn} = 1 when units i and j are allocated to objects m and n, respectively, and 0 otherwise. Because constraints (1) and (5)–(13) always guarantee x_{imn}, y_{ijm}, and y_{ijmn} to be integer, constraints (2)–(4) can be replaced with those that simply bound the variables between 0 and 1 inclusive. Note that the relations between the variables could be formulated in quadratic terms, e.g., $y_{ijmn} = x_{im}x_{jn}$. This, however, would make a nonlinear MIP model, which is generally considered as a difficult class of mathematical programming model to solve.

5. Primitive spatial properties

In this section, we first define what we consider primitive spatial properties and propose a systematic scheme for classifying them, and then formulate them in linear algebraic terms. A primitive spatial property in the scheme is merely a *function*, which takes attribute values of spatial units as input. Each such primitive by itself does not necessarily correspond to a spatial property one might be interested in, but can be combined with other primitives to represent a complex spatial property.

5.1. Classification

A primitive spatial property is defined as a single value as a function of the values of one l-unit (or 2-unit) attribute from all spatial units associated with one object (or object pair). Four classes of such properties are then possible, which are named: 1-unit 1-object, 2-unit 1-object, l-unit 2-object, and 2-unit 2-object. They are abstract quantities and independent of what they might indicate in specific applications.

A 1-unit 1-object property is returned by a function of the values of a 1-unit attribute from all spatial units belonging to an object. A 2-unit 1-object property is returned by a function of the values of a 2-unit attribute from all spatial unit pairs belonging to an object. A 1-unit 2-object property is returned by a function of the values of a 1-unit attribute from all spatial units belonging to both of an object pair. A 2-unit 2-object property is returned by a function of the values of a 2-unit attribute from all spatial unit pairs, one of which belongs to one of an object pair, the other to the other object. See Figure 3 for illustration.

We currently consider three functions shared by all classes of primitive properties as follows:

– *sum* function that returns the summation of all input values
– *max* function that returns the maximum of all input values
– *min* function that returns the minimum of all input values

Additionally, the following function is used for a 2-unit 1-object property, contiguity, which cannot be realized by any combination of the preceding functions:

– *con* function that returns a Boolean value of TRUE (or 1) if an object is contiguous (or connected) in terms of the values of a 2-unit attribute that represent the adjacency

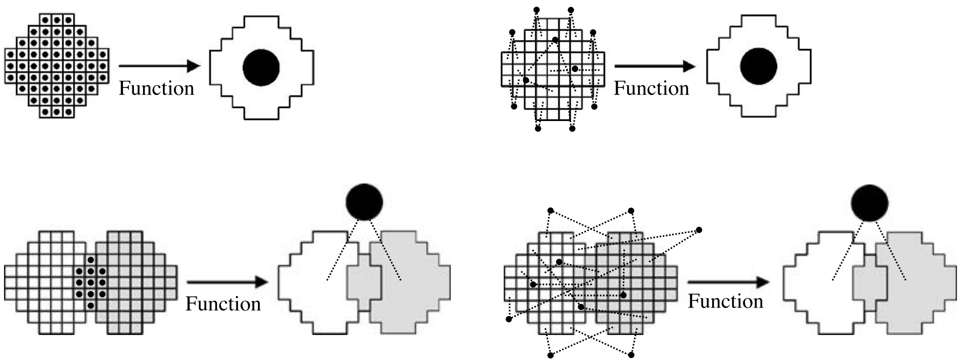

Figure 3. Four classes of primitive spatial properties. Each small square represents a spatial unit, each dot enclosed by a square represents an attribute of the corresponding spatial unit, and each dot bridging two squares (not all are shown) represents an attribute of the corresponding spatial unit pair. Each large polygon represents an object, each dot enclosed by a polygon represents a property of the corresponding object, and each dot bridging two polygons represents a property of the corresponding object pair.

relationships of all spatial unit pairs within the object, and a Boolean value of FALSE (or 0) otherwise.

This set of functions seems fairly general and versatile. It, however, should be considered an arbitrary choice, since the present paper does not intend to find a complete set of primitive functions but demonstrate the expressiveness of our modeling approach. In theory, a spatial property could be designated as a primitive property if it is found not decomposable into any combination of other functions.

5.2. *Formulation*

The four functions described above can be formulated in terms that lend themselves to MIP models. The sum, max, and min functions are here only associated with 1-unit 1-object properties, while the con function is by definition a 2-unit 1-object property. Formations for other types of properties are easily derived by changing variables and attributes. In the following, to facilitate our discussions, we employ a unified notation for all primitive property functions such that:

$$\text{function}\,(\text{attribute}, \text{object}\,(\text{pair}))$$

Examples are sum(k, m), con(l, m), max(k, m, n), and min(l, m, n).

The sum function for a 1-unit 1-object property is formulated as:

$$\text{sum}(k, m) = \sum_{i \in I} a_{ik} x_{im} \tag{14}$$

The max function for a l-unit l-object property is formulated as:

$$\max(k, m) = \sum_{i \in I} a_{ik} x_{imk}^{\max} \tag{15}$$

subject to

$$\max(k,\, m) \geq a_{ik}\, x_{im} + M(1 - x_{im}) \quad (\forall i \,\in\, I) \tag{16}$$

$$\sum_{i \in I} x_{imk}^{\max} = 1 \tag{17}$$

$$x_{imk}^{\max} \leq x_{im} \qquad (\forall i \,\in\, I) \tag{18}$$

$$x_{imk}^{\max} \,\in\, \{0, 1\} \qquad (\forall i \,\in\, I) \tag{19}$$

where

$x_{imk}^{\max}$ is a variable. It is equal to 1 if unit i has the maximum value of attribute k, 0 otherwise.
M is a given number smaller than any a_{ik}.

As you can see, the formulation of this function relies on several constraints. Constraints (16) ensure that the maximum value of the attribute in the object is greater than or equal to any other value in the object. Constraint (17) designates one and only one spatial unit as the carrier of the maximum value. Thus if more than one spatial units share the maximum value in the object, only one will be arbitrarily chosen. Constraints (18) demand that the spatial unit with the maximum value belongs to the object.

The min function for a l-unit l-object property is formulated as:

$$\min(k, m) = \sum_{i \in I} a_{ik}\, x_{imk}^{\min} \tag{20}$$

subject to

$$\min(k, m) \le a_{ik}\, x_{im} + M(1 - x_{im}) \qquad (\forall i \in I) \tag{21}$$

$$\sum_{i \in I} x_{imk}^{\min} = 1 \tag{22}$$

$$x_{imk}^{\min} \le x_{im} \qquad (\forall i \in I) \tag{23}$$

$$x_{imk}^{\min} \in \{0, 1\} \qquad (\forall i \in I) \tag{24}$$

where

$x_{imk}^{\min}$ is a variable. If it is equal to 1, unit i has the minimum value of attribute k, 0 otherwise.

M is a given number greater than any a_{ik}.

Again, the formulation of this function relies on several constraints. Constraints (21) ensure that the minimum value of the attribute in the object is smaller than or equal to any other value in the object. Constraint (22) designates one and only one spatial unit as the carrier of the minimum value. Thus if more than one spatial units share the minimum value in the object, only one will be arbitrarily chosen. Constraints (23) demand that the spatial unit with the minimum value belongs to the object.

The con function is here designed to be callable when enforcing contiguity on an object. Thus it can be used only in the form of "con(l, m) = 1," which is formulated as the following set of equations:

$$\sum_{\{j \mid (i,j) \in A\}} f_{ijm} - \sum_{\{j \mid (j,i) \in A\}} f_{jim} \ge x_{im} - N w_{im} \quad (\forall i \in I) \tag{25}$$

$$\sum_{i \in I} w_{im} = 1 \tag{26}$$

$$f_{jim} \le (N - 1) x_{im} \quad (\forall i \in I, j \in A_i) \tag{27}$$

$$f_{ijm} \ge 0 \tag{28}$$

$$w_{im} \in \{0, 1\} \tag{29}$$

where

A_{il} is a set of spatial units adjacent to unit i indicated by 2-unit attribute l.
N is the cardinality of I.
f_{ijm} is a variable indicating the amount of flow (see below) from unit i to unit j in object m.
w_{im} is a variable, equal to 1 if unit i is object m's sink (see below), 0 otherwise.

This formulation is based on the observation that if a contiguous subset of spatial units is represented by a network in which one and only one unit serves as a sink and all other units are sources supplying one (or more) piece while no inflow is allowed to those units outside the network, then every piece of supply must ultimately reach the sink. The interested reader may see [40] for details of this formulation, and [51] for an alternative formulation.

It is important to note that some of the above formulations may be able to be simplified depending on how they are desired to be controlled. For example, if a max function is to be minimized, its simpler equivalent is:

$$\text{minimized}\,\text{sum}(k,m)$$

subject to

$$\text{sum}(k,m) \geq a_{ik}x_{im} + M(1 - x_{im}) \quad (\forall i \in I) \tag{30}$$

where

M is a given number smaller than any a_{ik}.

Note that if all a_{ik}'s are nonnegative, the second terms on the right-hand side may be dropped.

6. Primitive spatial properties to complex spatial properties

The utility of the present scheme lies in the ability to address a variety of spatial properties in establishing spatial allocation criteria. To illustrate this, we analyze selected properties from the literature, which are commonly referred to as size, shape, or spatial relation, in terms of our primitive properties. This section relies on the following notation.

m : object of interest (e.g., sales territory)
n : another object of interest
$\overline{m}$: set of all spatial units that do not belong to m (called m's complement)
$\overline{mn}$: set of all spatial units that do not belong to m or n
k_s : 1-unit attribute indicating the sales potential of each spatial unit

k_d : 1-unit attribute indicating the area-weighted squared distance from a given center of m
k_1 : 1-unit attribute equal to one for every spatial unit
l_d : 2-unit attribute indicating the distance between each spatial unit pair
l_a : 2-unit attribute indicating the adjacency of each spatial unit pair in the form of 0–1 value

6.1. *Size*

Size properties often involve 1-unit 1-object properties only. Still, many different size properties can be specified simply by applying different 1-unit attributes to different functions. Likewise, spatial allocation criteria concerning size properties, too, can be varied depending on how and what primitive properties are related. For example, the following statement demand that object *m* be at least *c* times larger in terms of sales potential than object *n*:

$$\mathrm{sum}(k_s, m) \geq \mathrm{c} \cdot \mathrm{sum}(k_s, n) \tag{31}$$

6.2. *Shape*

Shape properties tend to be modeled in a more complex manner than size properties. Of White and Renner's five shapes [50], compactness, fragmentation, perforation and elongation are formulated below.

Compactness in terms of moment of inertia is related to a 1-unit 1-object property. To achieve a certain degree of compactness, object *m*'s moment of inertia is minimized:

$$\text{minimize } \mathrm{sum}(k_d, m)$$

or limited to a specific value, *d*:

$$\mathrm{sum}(k_d, m) \leq d$$

If there is no fixed object center, a 2-unit 1-object property may be used. The following statement, for instance, will drive the object compact:

$$\text{minimize } \max(l_d, m)$$

Fragmentation is attained by excluding all adjacent spatial unit pairs from an object, *m*, which is formulated as follows:

$$\mathrm{sum}(l_a, m) = 0$$

Elongation can be seen as a combination of two primitive properties: one for stretching an object out and the other for keeping it contiguous. For example,

$$\begin{aligned}&\text{maximize } \mathrm{sum}(k, m)\\ &\mathrm{con}(l_a, m) = 1\end{aligned}$$

Perforation is not straightforward to model. The following statement, which enforces contiguity on both object m and its complement $\overline{m}$, prevents holes:

$$\mathrm{con}(l_a, m) = 1$$
$$\mathrm{con}(l_a, \overline{m}) = 1$$

Indentation, however, does not seem to be able to be modeled by the present scheme. It may be worthwhile to explore whether this property results from an overlooked combination of our primitive properties; and if not, whether (and how) it can be formulated independently. Certainly there are more shapes that cannot be covered by the present scheme. In general, the more specific a shape description is, the more difficult it is to model. For example, a star-shaped object is harder (if not impossible) to model than a round-shaped object.

6.3. Spatial relation

As for topological relations, Egenhofer and Franzosa's eight topological relations (see Figure 2) are formulated below. Each formulation examines the number of spatial units shared by two objects and/or the number of adjacent spatial unit pairs linking two objects.

Objects m and n are disjoint:
$\mathrm{sum}(k_1, m, n) = 0$
$\mathrm{sum}(l_a, m, n) = 0$
Objects m and n meet:
$\mathrm{sum}(k_1, m, n) = 0$
$\mathrm{sum}(l_a, m, n) \geq 1$
Object m contains object n:
$\mathrm{sum}(k_1, m, n) = \mathrm{sum}(k_1, n)$
$\mathrm{sum}(l_a, n, \overline{mn}) = 0$
Object m covers object n:
$\mathrm{sum}(k_1, m, n) = \mathrm{sum}(k_1, n)$
$\mathrm{sum}(l_a, n, \overline{mn}) \geq 1$
Objects m and n are equal
$\mathrm{sum}(k_1, m, n) = \mathrm{sum}(k_1, m)$
$\mathrm{sum}(k_1, m, n) = \mathrm{sum}(k_1, n)$
Objects m and n overlap
$\mathrm{sum}(k_1, m, n) \geq 1$
$\mathrm{sum}(k_1, m, n) \leq \mathrm{sum}(k_1, m)$
$\mathrm{sum}(k_1, m, n) \leq \mathrm{sum}(k_1, n)$
Object m is inside of object n:
$\mathrm{sum}(k_1, m, n) = \mathrm{sum}(k_1, m)$
$\mathrm{sum}(l_a, m, \overline{mn}) = 0$

Object m is covered by object n:
$\text{sum}(k_1, m, n) = \text{sum}(k_1, m)$
$\text{sum}(l_a, m, \overline{mn}) \geq 1$

Metric relations often involve 2-unit 2-object properties concerning distance attribute. Since distance is not necessarily measured in Euclidean terms, various metric relations can be addressed by coupling different 2-unit attributes with different functions. Here we only give one example, which states that the shortest distance between objects m and n must be greater than or equal to d:

$$\min(l_d, m, n) \geq d$$

7. Conclusion

Spatial allocation—allocation of discrete spatial units to objects of interest—is a difficult process to model. It is partly because there is no common approach to formulating spatial allocation criteria that involve a literally countless number of complex spatial properties. To evade this difficulty, we have assumed that some spatial properties are more primitive than others and shown that various spatial properties can be decomposed into four types of primitive properties: one concerning individual spatial units and a single object, one concerning spatial unit pairs and a single object, individual spatial units and an object pair, and spatial unit pairs and an object pair. This simplification enables one to express spatial allocation criteria in a clear and open-ended manner.

The present scheme is not ready to be incorporated in any computer system, though general-purpose spatial allocation systems may sound appealing. The major obstacle is that such systems would allow (or even encourage) users to build MIP models too large to be solved in reasonable time. This seems an inherent problem in the integration of mathematical programming models and GIS (see, e.g., [11]).

The present scheme can be extended at least in two ways: three dimensional (e.g., ore mining) and temporal (e.g., timber harvesting). The three-dimensional extension seems relatively straightforward since the primitive properties presented here are not associated with any particular dimension. The temporal extension, on the other hand, is more complicated, as it involves two types of unknowns: where to do something and when to do it. A typical example is a timber harvesting problem, where a series of harvesting decisions are to be made while forests grow over time. To model this kind of decision process, it might be useful to picture space and time as a three-dimensional space with two horizontal spatial dimensions and one vertical temporal dimension (see, e.g., [25]), and discretize it into a finite set of "spatio-temporal units" defined by a spatial extent and a temporal interval (see, e.g., [30]). In this setting, we can cast spatial allocation with time considerations as what might be called "spatio-temporal allocation"—aggregation of spatio-temporal units into clusters or "spatio-temporal objects" according to specific criteria. It, however, does not resemble spatial allocation any further for at least two reasons. First, distance in space and interval in time do not share the same unit of measurement. Second, the temporal dimension is asymmetric as time goes in one direction—from the past to the future. As a result, properties in a spatio-temporal context cannot be evaluated in the same way as in a spatial context. To see this, imagine two objects in space and time. Which is bigger? Is one object more compact than the other? Are they both contiguous? How far are they from each other? These questions are ambiguous (if not meaningless) because properties associated with spatio-temporal objects are not correctly specified. This kind of issue arises not only in evaluating spatio-temporal patterns, but in establishing spatio-temporal allocation criteria. As such, analyses of spatio-temporal properties should be approached from both descriptive and prescriptive perspectives. A preliminary research is underway to answer the following questions: what kinds of problems are concerned with spatio-temporal allocation, what kinds of criteria they require, and whether those criteria are expressed in terms of relationships between the present primitive spatial properties. What will follow is to identify typical properties in spatio-temporal contexts and to decompose them into primitive properties. An ultimate goal is to design a general approach to spatio-temporal allocation modeling with these primitives. A crucial factor to be addressed elsewhere is the development of quality spatio-temporal databases (see, e.g., [3], [31], [37] for overviews), since no model of spatio-temporal allocation would be useful unless data on space and time are correctly represented.

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References

1. J.C.J.H. Aerts and G.B.M. Heuvelink. "Using simulated annealing for resource allocation," *International Journal of Geographical Information Science*, Vol. 16:571–587, 2002.
2. R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. Network Flows: Theory, Algorithms, and Applications. Prentice Hall: Englewood Cliffs, NJ, 1993.
3. K.K. Al-Taha and R. Barrera. "Temporal data and GIS: An overview," in *Proc. of GIS/LIS '90*, pp. 244–254, 1990.
4. P.C. Belford and H.D. Ratliff. "A network-flow model for racially balancing schools," *Operations Research*, Vol. 20:619–628, 1972.
5. S. Benabdallah and J.R. Wright. "Shape considerations in spatial optimization," *Civil Engineering Systems*, Vol. 8:145–152, 1991.
6. S. Benabdallah and J.R. Wright. "Multiple subregion allocation models," *ASCE Journal of Urban Planning and Development*, Vol. 118:24–40, 1992.
7. D.J. Blair and T.H. Bliss. "The measurement of shape in geography: An appraisal of methods and techniques," *Bulletin of Quantitative Data for Geographers*, No. 11, 1967.

8. R.B. Boyce and W.A.V. Clark. “The concept of shape in geography,” *Geographical Review*, Vol. 54:561–572, 1964.
9. C.J. Brookes. “A parameterized region-growing programme for region allocation on raster suitability maps,” *International Journal of Geographical Information Science*, Vol. 11:375–396, 1997.
10. W. Bunge. Theoretical Geography, Lund Studies in Geography, Series C, General and Mathematical Geography, No. 1. Gleerup: Lund, 1966.
11. R.L. Church. “Location modelling and GIS,” in P.A. Longley, M.F. Goodchild, D.J. Maguire, and D.W. Rhind (Eds.), *Geographical Information Systems, Vol. I, Principles and Technical Issues*, 2nd edition. John Wiley & Sons: New York, 293–303, 1999.
12. T.J. Cova and R.L. Church. “Contiguity constraints for single-region site search problems,” *Geographical Analysis*, Vol. 32:306–329, 2000.
13. S.C. Crema. “A comparison between linear programming and a choice heuristic approach to multi-objective decision making using GIS,” in *Proc. of GIS/LIS ’96*, pp. 954–963, 1996.
14. J.R. Eastman, P.A.K. Kyem, and J. Toledano. “A procedure for multi-objective decision making in GIS under conditions of conflicting objectives,” in *Proc. of EGIS ‘93*, pp. 438–447, 1993.
15. M. Egenhofer and R. Franzosa. “Point-set topological spatial relations,” *International Journal of Geographical Information Systems*, Vol. 5:161–174, 1991.
16. M.J. Egenhofer and J.R. Herring. “Categorizing binary topological relations between regions, lines, and points in geographic databases,” Technical Report, Department of Surveying Engineering, University of Maine, Orono, ME, 1990.
17. A. Frank, J. Raper, and J.P. Cheylan. Life and Motion of Socio-Economic Units. Taylor & Francis: London, 2001.
18. B. Fleischmann and J.N. Paraschis. “Solving a large scale districting problem: A case report,” *Computers and Operations Research*, Vol. 15:521–533, 1988.
19. A.D. Franklin and E. Koenigsberg. “Computed school assignments in a large district,” *Operations Research*, Vol. 21:413–426, 1973.
20. Y. Frolov. “Measuring the shape of geographical phenomena: A history of the issue,” *Soviet Geography: Review and Translation*, Vol. 16:676–687, 1974.
21. R.S. Garfinkel and G.L. Nemhauser. “Optimal political districting by implicit enumeration techniques,” *Management Science*, Vol. 16:B495–B508, 1970.
22. K.C. Gilbert, D.D. Holmes, and R.E. Rosenthal. “A multiobjective discrete optimization model for land allocation,” *Management Science*, Vol. 31:1509–1522, 1985.
23. M.F. Goodchild, “Towards an enumeration and classification of GIS functions,” in *Proc. of International Geographic Information Systems (IGIS) Symposium: The Research Agenda*, NASA, Washington, Vol. 2:67–77, 1988.
24. M.F. Goodchild. “Geographical data modeling,” *Computers & Geosciences*, Vol. 18:401–408, 1992.
25. T. Hägerstrand. “What about people in regional science?” *Papers of the regional Science Association*, Vol. 24:7–21, 1970.
26. S.W. Hess. “Compactness-what shape and size? Conflicts among possible criteria for rational districting,” *National Municipal League*, 15–23, 1969.
27. S.W. Hess and S.A. Samuels. “Experiences with a sales districting model: Criteria and implementation,” *Management Science*, Vol. 18(No. 4, Part II):41–54, 1971.
28. S.W. Hess, J.B. Weaver, H.J. Siegfeldt, J.N. Whelan, and P.A. Zitlau, “Nonpartisan political redistricting by computer,” *Operations Research*, Vol. 13:998–1006, 1965.
29. M. Hojati. “Optimal political districting,” *Computers & Operations Research*, Vol. 23:1147–1161, 1996.
30. M. Kavouras. “Understanding and modeling spatial change,” in A. Frank, J. Raper, and J.P. Cheylan (Eds.), *Life and Motion of Socio-Economic Units*. Taylor & Francis: London, 2001.
31. G. Langran. Time in Geographic Information Systems. Taylor & Francis: London, 1992.
32. D.R. Lee and G.T. Sallee. “A method of measuring shape,” *Geographical Review*, Vol. 60:555–563, 1970.
33. A.M. MacEachren. “Compactness of geographic shape: Comparison and evaluation of measures,” *Geografiska Annaler*, Vol. 67B:53–67, 1985.
34. P.G. Marlin. “Application of the transportation model to a large-scale “districting” problem,” *Computers and Operations Research*, Vol. 8:83–96, 1981.

35. B.H. Massam and M.F. Goodchild. “Temporal trends in the spatial organization of a service agency,” *Canadian Geographer*, Vol. 15:193–206, 197l.
36. A. Mehrotra, E.L. Johnson, and G.L. Nemhauser. “An optimization based heuristic for political districting,” *Management Science*, Vol. 44:1100–1114, 1998.
37. D.J. Peuquet. “Time in GIS and geographical databases,” in P.A. Longley, M.F. Goodchild, D.J. Maguire, and D.W. Rhind (Eds.), *Geographical Information Systems, Vol. I, Principles and Technical Issues*, 2nd edition. John Wiley & Sons: New York, pp. 91–103, 1999.
38. M. Segal and D.B. Weinberger. “Turting,” *Operations Research*, Vol. 25:367–386, 1977.
39. R.J. Shanker, R.E. Turner, and A.A. Zoltners. “Sales territory design: An integrated approach,” *Management Science*, Vol. 22:309–320, 1975.
40. T. Shirabe. “A model of contiguity for spatial unit allocation,” in revision.
41. T. Shirabe and C.D. Tomlin. “Decomposing integer programming models for spatial allocation,” in M.J. Egenhofer and D.M. Mark (Eds.), *Geographic Information Science, Proc. of GIScience 2002, Lecture Notes in Computer Science*, Vol. 2478:300–312, 2002.
42. B. Smith. “On drawing lines on a map,” in A.U. Frank and W. Kuhn (Eds.), *Spatial Information Theory, Proc. of COSIT ’95, Lecture Notes in Computer Science*, Vol. 988:475–484, 1995.
43. P.J. Taylor. “Distances within shapes: An introduction to a family of finite frequency distributions,” Discussion Paper, No. 16, University of Iowa, 1–20, 1970.
44. P.J. Taylor. “Distances within shapes: An introduction to a family of finite frequency distributions,” *Geografiska Annaler*, Vol. 53B(1):1971.
45. C.D. Tomlin. Geographic Information Systems and Cartographic Modeling. Prentice Hall: Englewood Cliffs, NJ, 1990.
46. C.D. Tomlin and K.M. Johnston. “An experiment in land-use allocation with a geographic information system,” in D.J. Peuquet and D.F. Marble (Eds.), *Introductory Readings in Geographic Information Systems*. Taylor & Francis: London, 159–169, 1990.
47. M.L. Vidale. “A graphical solution of the transportation problem,” *Operations Research*, Vol. 4:193–203,1956.
48. E.A. Wentz. “A shape definition for geographic applications based on edge, elongation, and perforation,” *Geographical Analysis*, Vol. 32:95–112, 2000.
49. C.L. White and G.T. Renner. College Geography: Natural Environment and Human Society. New York, pp. 590–599, 1957.
50. J.C. Williams. “Political districting: A review. Papers in regional science,” *Papers in Regional Science: The Journal of the Regional Science Association International*, Vol. 74:13–40, 1995.
51. J.C. Williams, “A zero–one programming model for contiguous land acquisition,” *Geographical Analysis*, Vol. 34:330–349, 2002.
52. J. Wright, C. ReVelle, and J. Cohon. “A multiple objective integer programming model for the land acquisition problem,” *Regional Science and Urban Economics*, Vol. 13:31–53, 1983.
53. M. Yeates. “Hinterland Delimitation: A distance minimizing approach,” *The Professional Geographer*, Vol. 15(No. 6):7–10, 1963.
54. A.A. Zoltners and P. Sinha, “Sales territory alignment: A review and Model,” *Management Science*, Vol. 29:1237–1256, 1983.

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