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Spin systems and Political Districting Problem

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Abstract

The aim of the Political Districting Problem is to partition a territory into electoral districts subject to some constraints such as contiguity, population equality, etc. In this paper, we apply statistical physics methods to Political Districting Problem. We will show how to transform the political problem to a spin system, and how to write down a q-state Potts model-like energy function in which the political constraints can be written as interactions between sites or external fields acting on the system. Districting into q voter districts is equivalent to finding the ground state of this q-state Potts model. Searching for the ground state becomes an optimization problem, where optimization algorithms such as the simulated annealing method and Genetic Algorithm can be employed here.

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1. Introduction

The aim of the Political Districting Problem is to partition a territory into electoral districts subject to some constraints such as contiguity, population equality, etc. It is a real-world political problem in which the politicians always will do some things in their own favor. In 1812, Massachusetts governor, Elbridge Gerry, got help from his political party by crafting a district and won the election. At the time, someone produced an illustration of the districting and emphasized its similarities with a salamander. The term Gerrymander was then coined from putting together Gerry and mander. Nowadays, Gerrymandering refers to the practice of drawing district lines to maximize the advantage of a political party [1].

From a mathematical point of view, the Political Districting Problem belongs to what is known as the Districting (or zone design) Problem in which n units are grouped into k zones such that some cost function is optimized, subject to constraints on the topology of the zones, etc. and has been shown to be NP-Complete [2]. Thus, it is best to be treated by some optimization methods. The Districting Problem is a geographical problem which is present in a number of geographical tasks, such as school districting, design of sales territories, etc. Most people nowadays agree that the basic constraints of the Problem are population equality, contiguity and compactness.

2. The model

We will map the Political Districting Problem onto a q-state Potts model [3] which has been applied to graph coloring problems such as the "q-partitioning of a graph problem" and the "chromatic number problem" [4]. In this model, we use "precinct" as the smallest unit and identify it as a spin. The constraints can be written as interactions between sites or external fields acting on the system. The optimal solution to districting into q voter districts is equivalent to finding the ground state of this q-state Potts model. The total number of spins (precincts) is equal to N and each spin can have q-states. Each spin can have its state function as $S_i = 1, 2, \ldots, q$. The goal here is to find the ground state of this q-state Potts model with the interaction given by the constraints.

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We now write down the interaction terms of the Hamiltonian for various constraints. For the population equality, we associate a random field to the site (precinct) p_i. When $p_i = \text{const.}$, the population for each site (precinct) is equal. Therefore, the magnetization (the total voter population) of a voter district P_l can be written as

$$P_l = \sum_{i=1}^{N} p_i\, \delta_{s_i,l},$$

where

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{else} \end{cases},$$

the total population is $P_0 = \sum_{l=1}^{q} P_l$ and the average population for each voter district is $\langle P \rangle = P_0/q$. Hence we can write the average population for each voter district as

$$E_P = \sum_{l=1}^{q} \left| 1 - \frac{P_l}{\langle P \rangle} \right|.$$

The smaller this energy is, the closer it is to the average value.

We next consider the constraint of compactness. Define a connection table for each spin as

$$C_{i,j} = \begin{cases} 1, & \text{if connected} \\ 0, & \text{else} \end{cases},$$

where $C_{i,j}$ equals to 1 when spins i and j are connected to each other and zero otherwise(Fig. 1). We can then write down the energy function of the compactness constraint. We here define the boundary of the domain of spins as

$$E_D = \sum_{i,j} (1 - \delta_{S_i,S_j}) C_{i,j}.$$

When this function is at its minimum, the district will have the smallest number of precincts on its boundary. Notice that the way we introduce the compactness term in the Hamiltonian also guarantees contiguity in the optimal solution.

One can also consider other constraints such as the administrative zones that the area already has. We here use the external field to represent the effect of the administrative zones. We define the external field function of the spin as $H_i = 1, 2, \ldots, n_d$ to indicate the administrative zone that a precinct belongs. The energy function E_A can then be written as $E_A = \sum_{i,j;j<i} -\delta_{S_i,S_j}\,\delta_{H_i,H_j}\,C_{i,j}$, smaller E_A means fewer administrative zones. The total energy will be given by $E = \lambda_P E_P + \lambda_D E_D + \lambda_A E_A$. Varying λ will affect the contribution of each constraint to the total energy function.

In the above section, we have shown how to map the Districting Problem to a q-state Potts Model and rewrite the constraints into interaction between different sites and external fields acting on the system. We will here use the problem of determining the districting for the Taiwan Legislature seats as an example, though the method can be equally applicable to any districting problem.

3. Result

Once we have the energy function, we can try to solve this problem by using optimization algorithms such as the simulated annealing method and Genetic Algorithm. Below are some best results we have. In our example here, we set $\lambda_A = 0$ in the following discussion. In Table 1, E_{min} is the minimum energy, E_P the population deviation from average (the smaller the better), E_D the boundary energy of the district (the smaller the better), ΔP_{max} the population difference from the average and $\langle P \rangle$ the average population/district (Fig. 2).

Table 1
Voter districting in Taipei city with different λ_P and λ_D

λ_P	λ_D	E_{min}	E_P	E_D	ΔP_{max}	$\Delta P_{max}/\langle P \rangle$
1	1	86.715	4.715	82	390925	1.5693
5	1	97.073	1.615	89	143919	0.5778
10	1	104.075	0.507	99	47145	0.1893
50	1	110.537	0.031	109	1765	0.0071
100	1	111.998	0.020	110	1846	0.0074
500	1	145.778	0.014	139	801	0.0032

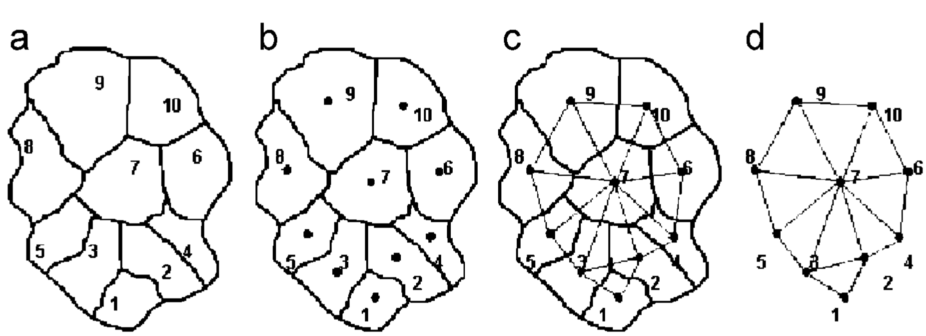

Fig. 1. (a) A district of 10 precincts in our model; (b) each red dot represents one precinct; (c) a network of precincts connected by green arcs; (d) the network extracted from (c).

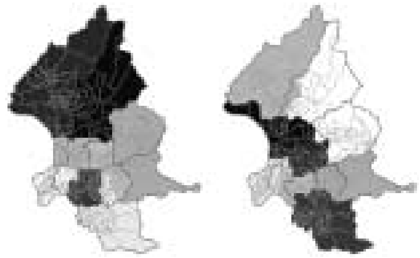

Fig. 2. Two optimal solutions of districting Taipei city.

4. Conclusion

We have here demonstrated how a social economics problem can be transformed into a problem in physics and carry out an optimization study to look for the optimal solution of the problem. We believe one can easily use this method to study more related problems, such as "school districting problem", "floor planning problem", etc.

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