A districting procedure for social organizations

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The problem dealt with in the paper refers to the decomposition of a geographical region into an unspecified number of non-overlapping regional partitions. Each partition is given by a connected aggregation of indivisible elementary areas and must contain at least one source of a certain social service characterized by assigned capacity and location. Each area is characterized by a service demand and by a measure quantifying its dimension.

Two heuristic methods are proposed to obtain a preliminary aggregation of the elementary areas, in order to reduce the computational effort involved in generating the feasible partitions. A fuzzy sets approach is presented for rating and ranking the regional partitions. This approach seems to be particularly suited to planning problems, when some kind of compromise between qualitative and quantitative issues must be sought. A health-care districting problem is then discussed as a case study.

1. Introduction

A problem that is frequently encountered in regional planning can be stated as follows: given a set of undivisible areas (typically communes) constituting a geographical region, cluster these areas into an unspecified number of connected non-overlapping districts in such a way that any district contains at least one existing source of a certain social service characterized by a given capacity. Such sources may correspond to hospitals with an assigned number of beds, to schools of fixed size, etc. A problem of this type has been considered, for instance, in [4].

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Since, in general, the way of decomposing a given region is not unique, the planner must face a second problem: rank the obtained feasible regional partitions according to some preference ordering suggested by a certain number of quantitative and qualitative issues. Two kinds of conflicting difficulties are then encountered:

(1) the presence of more than one issue should require that the problem is stated within the framework of multiobjective optimization,

(2) the fact that some issues cannot be expressed but in a qualitative form hinders one from defining a sufficiently sound vector-valued objective function.

In order to take these qualitative attributes into an analytic decision process, a fuzzy sets method is proposed in the paper. Our approach follows the one presented in [1], where any alternative in a decision set (regional partitions in our case) is characterized by a fuzzy rating. A preference ordering is then defined on the ground of these ratings.

A case study based on a health-care districting (HCD) problem is carried on throughout the paper, which is organized as follows. Some preliminary definitions and problem statements are presented in Section 2. Heuristic methods to restrict the admissible class of regional partitions are discussed in Section 3. These methods, which clearly give suboptimal solutions (if the problem is posed in optimization terms), are required whenever the complexity of the problem grows beyond certain limits. This practically occurs when the number of elementary areas to be clustered reaches a few hundred units. The fuzzy sets rating procedure is described in Section 4. Computational results and possible methods for ranking the regional partitions are finally discussed in Sections 5 and 6.

2. Statement of the problem

The definitions and the problem presented in this section are primarily intended to state the HCD case in a formal way but, for their generality, they can easily suit a wide range of districting problems. Observe that, from a graph theory point

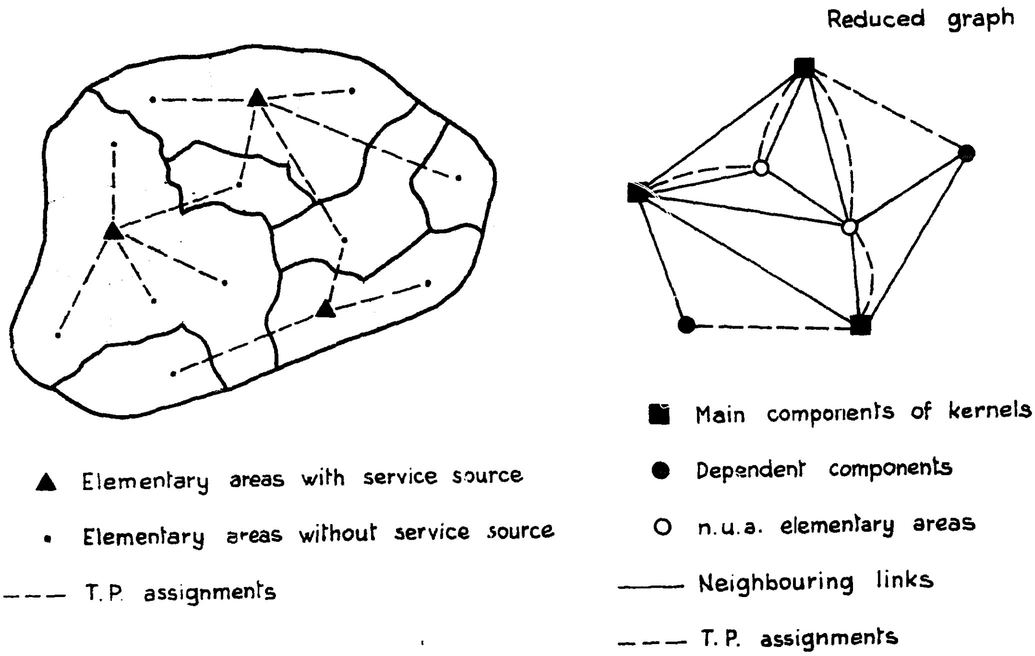

Fig. 1. Reduction of the original graph by means of the transportation problem.

of view, the region to be partitioned may be regarded as a network in which the indivisible areas constitute the set $\mathfrak{N}$ of nodes and the arcs correspond to neighbouring relationships (Fig. 1). Each area k ($k = 1,\dots,n$) is characterized by a service demand a_k and by a measure p_k quantifying its dimension (e.g., population, geographical extent, etc.). Let m be the number of service sources available in the region and b_j ($j = 1,\dots,m$) the capacity of source j.

Then, we introduce the following:

Definition 1. A connected subset of nodes $u_i \subset \mathfrak{N}$ is said to be a *feasible district* if it contains at least one source and if the following conditions are met:

$$p_{\mathrm{MIN}} \leqslant \sum_{k \in C(u_i)} p_k \leqslant p_{\mathrm{MAX}}, \tag{1}$$

$$\sum_{k \in C(u_i)} a_k \leqslant \sum_{j \in H(u_i)} b_j. \tag{2}$$

$C(u_i)$ and $H(u_i)$ are the subsets of elementary areas and social services belonging to district u_i, respectively. p_{MIN} and p_{MAX} are lower and upper bounds imposed by technical reasons.

Let $\mathfrak{U}$ be the set of all feasible districts. Then the following definition establishes the geometrical properties required by the partitioning procedure.

Definition 2. A set S_i of districts $u_j \in \mathfrak{U}$ is said to be a *feasible regional partition* (i.e., a feasible solution of the districting problem) if the following conditions are met:

$$\bigcup_{u_j \in S_i} u_j = \mathfrak{N}, \tag{3}$$

$$u_r \cap u_s = \emptyset, \quad \forall u_r, u_s \in S_i. \tag{4}$$

On the basis of the above definitions, we can state the following:

Problem 1. Find the set $\mathfrak{S}$ of all feasible regional partitions.

In formulating Problem 1, we want to stress explicitly the difficulty involved in stating our districting problem in optimization terms. Actually, while Definitions 1 and 2 are based on constraints which are formally sound and generally acceptable from a planning point of view, it is much more difficult to define a scalar or a vector-valued objective function, since this function should be the result of many qualitative planner's issues.

It follows that the knowledge of $\mathfrak{S}$ might be quite helpful by itself for intuitive judgements on the solution to be selected provided that the number of solutions is not too large. If it is not so, we must face the much more questionable problem of comparing solutions, and state the following.

Problem 2. Define a preference ordering $\succcurlyeq$ on the set $\mathfrak{S}$ and rank the solutions with respect to $\succcurlyeq$.

The symbol $\succcurlyeq$ denotes a given preference among solutions so that $S_a \preccurlyeq S_b$ is to be read 'solution S_a is not preferred to solution S_b'. Later on we shall discuss an attempt at defining a preference ordering on the solution set by using fuzzy sets theory.

Problem 1 can also be restated in an algebraic form which will be used in the following. Define a matrix A, whose element a_{ij} takes on the value 1 if the elementary area i belongs to district j and the value 0 otherwise. The number of rows of A is given by the number n of elementary areas. The number of columns of A is given by the number of all feasible districts. Let $\mathbf{1}$ be an n-dimensional vector of all 1's. Then, solving Problem 1 is equivalent to the following:

Problem 1′. Find all zero–one vectors x that solve the system

$$Ax = \mathbf{1}. \tag{5}$$

Observe, in passing, that if the preference ordering $\preccurlyeq$ is described by a scalar-valued linear cost function $f(\boldsymbol{x}) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$, to be minimized under constraints (5), a set partitioning problem can be stated [2].

3. Heuristic methods for reducing the set of feasible districts

Solving Problems 1 and 2 involves formidable computational difficulties whenever, as is typical in regional planning, the number of elementary areas reaches a few hundred units. Actually, these difficulties are mainly encountered in the preliminary phase generating the set $\mathfrak{A}$ of feasible districts or, equivalently, the columns of matrix A.

In this section, we then propose heuristic criteria to reduce the set $\mathfrak{A}$, that is, to 'disregard' some solutions of Problem 1. This reduction is performed by introducing some new conditions defined by a vector $\boldsymbol{y}$ which reflects some additional information with which the decision maker is provided. For a given choice of $\boldsymbol{y}$, a parametrized subset $\mathfrak{A}_y \subset \mathfrak{A}$ is obtained, from which a reduced set $\mathfrak{S}_y \subset \mathfrak{S}$ of feasible partitions is derived. Sensitivity analysis can be performed on $\mathfrak{S}_y$ by varying vector $\boldsymbol{y}$. If this vector contains the additional information postulated above, the decision maker can benefit by an interactive tool to choose reasonable solutions by 'moving' the subset $\mathfrak{S}_y$ in the complete set $\mathfrak{S}$.

We shall now discuss two possible criteria for the reduction of set $\mathfrak{A}$ which have proved quite efficient in simplifying the HCD problem. The main lines of these methods, however, can be easily extended to other kinds of districting problems.

3.1. Preliminary clustering via the transportation problem

A first criterion to obtain a preliminary clustering of the elementary areas can be suggested by the reasonable requirement of minimizing the total cost needed to connect the service sources to the demand points (we suppose that the demand of an elementary area can be concentrated at a single point).

Let z_{kj} be the flow of service units from source j to the elementary area k and c_{kj} the cost per service unit to connect these two points. Then, the following linear program (transportation problem) can be stated

$$\min \sum_{k=1}^{n} \sum_{j=1}^{m} c_{kj} z_{kj}, \quad z_{kj} \geqslant 0, \tag{6}$$

$$\sum_{j=1}^{m} z_{kj} = a_k, \quad k = 1, \dots, n, \tag{7}$$

$$\sum_{k=1}^{n} z_{kj} \leqslant b_j, \quad j = 1, \dots, m. \tag{8}$$

As a result of this linear program, some elementary areas will be assigned to a unique source, while the remaining ones will be shared among two or more sources. Denote them as *not univocally assigned (n.u.a.)* elementary areas. All elementary areas univocally assigned to the same source are grouped together to constitute a new unit, which will be termed a *kernel*. A kernel may be geometrically connected or not. In the second case, the kernel is made up of the following components, each of which is connected: a *main* component, including the elementary area that contains the source and a certain number of *dependent* components containing only demand areas. Then the region is partitioned into kernels and n.u.a. elementary areas.

A reduced graph can now be considered whose nodes are given by the main components of the kernel, the dependent components and the n.u.a. elementary areas. The links of the reduced graph correspond to the neighbouring relationships between its nodes (see Fig. 1). On the basis of the reduced graph and of the assignments defined by the transportation problem, we may restate Definition 1 as follows:

Definition 1′. A connected subset u_i of nodes of the reduced graph is said to be a *feasible district* if it contains at least one kernel, if conditions (1) and (2) are satisfied, and if the following conditions are met:

$$\Gamma(n_j) \subseteq u_i \Rightarrow n_j \in u_i \tag{9}$$

$$n_j \in u_i \Rightarrow \Gamma(n_j) \cap u_i \neq \emptyset, \tag{10}$$

where n_j is an n.u.a. elementary area and $\Gamma(n_j)$ is the set of kernels to which n_j is assigned.

Condition (9) means that a district must include an n.u.a. elementary area n_j if it includes all the kernels that serve n_j. Condition (10) means that if a district includes n_j, then it must include at least one kernel serving n_j.

Though the transportation problem outlined by (6), (7), (8) reduces the set of feasible districts, it does not lead to a 'parametrized' reduction of $\mathcal{U}$ as postulated by the first requirement introduced at the beginning of this section. This can be obtained by exploiting some additional information suggested by the peculiarity of the problem dealt with. The HCD problem carried on throughout the paper will clarify this point. Consider then a districting case in which the elementary areas are communes and the service sources are hospitals [5]. Demand a_k of commune k can be expressed in terms of its population (this quantity will also be used as a measurement of its extent, i.e., $a_k = p_k$); and the capacity of service source j can be specified by the number of beds l_j. Since variables a_k and l_j are non-homogeneous quantities, some conversion coefficient is needed to meet the demand and offer of the health-care system.

Define a coefficient α such that $b_j = \alpha l_j$, where the value of α specifies the number of inhabitants which are supposed to be assisted by the health services corresponding to a bed-unit. Then, substitute b_j in (8). A reduced graph parametrized by α can thus be obtained by solving the transportation problem. A parametrized subset $\mathcal{U}_\alpha \subset \mathcal{U}$ is finally derived from this graph. Districts belonging to $\mathcal{U}_\alpha$ satisfy conditions (1), (9), (10) and

$$\sum_{k \in C(u_i)} p_k \leq \alpha_{\mathrm{MAX}} \sum_{j \in H(u_i)} l_j \tag{2′}$$

which replaces condition (2) (α_{MAX} is a coefficient with the same dimension as α).

Coefficient α can then be used as the parametrizing variable y to generate subsets $\mathcal{U}_y$ and $\mathcal{S}_y$.

A combinatorial procedure to derive all feasible districts from the reduced graph (or from a general one) is detailed in [7]. This procedure actually requires a major effort in this reduction method. Observe that α_{MAX} is a constraint coefficient which has been set by the decision maker to specify the maximum number of inhabitants that can be assisted by the health services corresponding to one bed-unit. On the contrary $y = \alpha$ is to be considered as a free parameter ranging in the interval $[\bar{\alpha}, \alpha_{\mathrm{MAX}}]$, where

$$\bar{\alpha} = \sum_{k=1}^{n} a_k \Big/ \sum_{j=1}^{m} l_j \tag{11}$$

It is worth noting that increasing α results in a trend toward least-distance assignments in the transportation problem, if in (6) c_{kj} corresponds to the distance between commune k and hospital j.

3.2. Progressive aggregation of neighbouring elementary areas

A second criterion for obtaining preliminary clusters and subset $\mathcal{U}_y$ may consist in aggregating progressively the nearest neighbouring elementary areas around one or more service sources until the clusters violate the second inequality of (1) or inequality (2).

Define for a district u_i:

$$l(u_i) \triangleq \sum_{j \in H(u_i)} l_j,$$

$$p(u_i) \triangleq \sum_{k \in C(u_i)} p_k. \tag{12}$$

Consider then a subset $H(u_i)$ of hospitals such that

$$l(u_i) \geqslant p_{\mathrm{MIN}}/\alpha_{\mathrm{MAX}} \quad (13)$$

and add neighbouring communes following an order of increasing distances. Feasible districts u_i are obtained as soon as

$$p(u_i) \geqslant p_{\mathrm{MIN}} \quad (14)$$

and as long as

$$p(u_i) \leqslant p_{\mathrm{MAX}} \quad \text{and} \quad p(u_i) \leqslant \alpha_{\mathrm{MAX}} l(u_i). \quad (15)$$

This procedure leads to a subset of feasible districts the number of which can be parametrized as follows. For a given subset $H(u_i)$ satisfying condition (13), consider all the districts that can be found by adding a commune at a time, starting the aggregation when condition (14) is met and stopping it when one of constraints (15) is violated. A reasonable clustering rule may consist in defining a new district whenever the number of the cluster inhabitants increases by a certain step Δp. This quantity can then be used as the parametrizing variable y.

Observe that the subset $\mathfrak{U}_y$ obtained by means of this procedure can be described in an algebraic form by introducing a matrix A_y with the usual meaning. This matrix can be partitioned as follows:

$$\tilde{A}_y = \begin{bmatrix} \tilde{A}'_y \\ --- \\ \tilde{A}''_y \end{bmatrix}$$

The rows of $\tilde{A}'_y$ correspond to all the communes containing a hospital, while the rows of $\tilde{A}''_y$ correspond to the remaining ones. This type of progressive aggregation leads the columns of $\tilde{A}_y$, which correspond to districts originating from the same subset $H(u_i)$ of hospitals, to have the same elements in the higher part of the matrix (that is, in the columns of matrix $\tilde{A}'_y$), and to differ in the lower part (columns of $\tilde{A}''_y$). In the next section, a possible use of this structure of $\tilde{A}_y$ will be sketched to derive $\mathfrak{S}_y$.

4. A fuzzy sets approach for evaluating regional partitions

Suppose that a subset $\mathfrak{U}_y$ has been derived for a given value of parameter y. Then solving Problem 1 requires the determination of the set $\mathfrak{S}_y$ of all feasible regional partitions. A different procedure must be used depending on which of the two criteria proposed in Section 3 has been chosen.

If the transportation problem reducing method has been selected, finding $\mathfrak{S}_y$ means to determine all zero–one solutions of the system

$$A_y \boldsymbol{x}_y = \mathbf{1}_y \quad (16)$$

which has the same meaning as system (5) in Problem 1′.The rows of the 'reduced' matrix A_y correspond to the main components of the kernels, to the dependent components and to the n.u.a. communes. $\boldsymbol{x}_y$, $\mathbf{1}_y$ are vectors whose reduced dimensions are specified by the dimensions of A_y. A quite efficient method for deriving $\mathfrak{S}_y$ consists in performing a search on the columns (districts) of matrix A_y by following the same technique that is used in the implicit enumeration algorithm proposed by Garfinkel–Nemhauser, Pierce, etc. to solve set partitioning problems (see, for instance, [2]).

If the second criterion to derive $\mathfrak{U}_y$ has been chosen, matrix $\tilde{A}_y$ can be used in a system like (16). Observe, however, that the dimensions of $\tilde{A}_y$ are now different as compared with the dimensions of the matrix $\tilde{A}_y$ derived by means of the first criterion. Actually, for proper choices of y, the number of columns of $\tilde{A}_y$ turns out to be reduced if compared with the one of the 'complete' matrix A, but the number of rows is still given by all the elementary areas. It follows that system (16) might not admit any solution. In this case, however, it is not difficult to exploit the aggregation procedure used to derive $\mathfrak{U}_y$ so as to suitably modify the columns of the lower matrix $\tilde{A}''_y$, defined in the preceding section, thus obtaining a nonempty subset $\mathfrak{S}_y$ of feasible partitions.

Once $\mathfrak{S}_y$ has been found by means of one of the two criteria, the decision maker may deem that the obtained results contain sufficient information to conclude his analysis provided that

(1) $\mathfrak{S}_y$ is composed of a sufficiently small number of regional partitions, thus enabling direct comparisons among partitions on the basis of intuitive judgements,

(2) parameter y may be given a significant value from a planning point of view so that no other values need to be considered.

If the first condition is not met, a ranking procedure is required (see Problem 2). If the second condition is not met, different values of y must be taken into account or, equivalently, subset $\mathbb{S}_y$ must be 'moved' in the complete set $\mathbb{S}$. At this point, it is worth making some comments.

Ranking the regional partitions of subset $\mathbb{S}_y$ requires two phases:

(1) to define a preference ordering $\succcurlyeq$ on set $\mathbb{S}$ (and then on $\mathbb{S}_y$),

(2) to rank the solutions with respect to $\succcurlyeq$.

If a scalar-valued cost function could be accepted to define the preference ordering, an integer programming problem should be faced. This is not our case, however, since the preference ordering must in general be derived on the basis of many heuristic planner's issues. Two kinds of conflicting difficulties are then encountered:

(1) 'many' issues require that the problem be stated within the framework of multiobjective programming,

(2) 'heuristic' issues hinder one from defining a sufficiently sound vector-valued objective function and from deriving, for example, the set of all nondominated solutions (on the other hand, this set might not contain sufficient information for the planner's ultimate decisions, unless further reductions of the set are obtained [8].

These difficulties may be circumvented, at least in part, by considering a subset $\mathbb{S}_y$ of feasible partitions (obtained in solving Problem 1 for a given y) and by evaluating these solutions on the basis of other qualitative attributes, which do not appear explicitly in the constraints leading to $\mathbb{S}_y$. In order to take these attributes into an analytic decision process, fuzzy sets theory seems to constitute an adequate tool. To be more specific, our approach follows the method proposed in [1], where any alternative in a decision set (regional partitions, in our case) is characterized by a fuzzy rating. A preference ordering will then be defined on the ground of these ratings.

Let $S_1,\ldots,S_N$ be all regional partitions constituting set $\mathbb{S}$ (i.e., all feasible solutions of Problem 1′). Later on, comparison among partitions will be restricted to $S_{y1},\ldots,S_{yN_y}$, which constitute subset $\mathbb{S}_y$. Now consider a certain number of aspects $1, 2,\ldots, G$, which enter into the evaluation of each solution. For a given S_i, the relative merit of aspect j is assessed by a fuzzy rating $\mu_{\mathrm{R}ij}(r_{ij})$, where the scalar r_{ij} takes values in the interval $[0,1]$. $\mu_{\mathrm{R}ij}$ are the membership functions (m.f.) by which we define the fuzzy sets

$$\{r_{ij}, \mu_{\mathrm{R}ij}\}, \quad r_{ij} \in [0,1]; \; i = 1,\ldots,N; \; j = 1,\ldots,G. \tag{17}$$

We assume that $\mu_{\mathrm{R}ij}$ takes the value 1 for at least one value of its argument. For the basic notions of fuzzy sets theory, we refer to Zadeh's papers (see, for instance, [3]). As will be seen later on, the definition of membership function $\mu_{\mathrm{R}ij}$ may be the result of 'judgements' referring to aspect j, which have been given by the planner to the districts constituting the regional partition S_i. Also the aggregation of these judgments will be obtained through fuzzy sets theory.

Generally speaking, our fuzzy sets approach fits in well with the requirement of taking into consideration qualitative (though conveying information) statements like 'a district should comprehend about 55 000 inhabitants' or 'a suitable balance is required in any district, between offer and demand of health care-service'.

Since aspect j may enter into the evaluation of S_i with different importance, we introduce the fuzzy weights

$$\{w_j, \mu_{\mathrm{W}j}\}, \quad w_j \in [0,1]; \quad j = 1,\ldots,G. \tag{18}$$

Then, following [1], partition S_i is characterized, in a fuzzy sense, by the final rating r_i with the m.f.

$$\mu_{\mathrm{R}i}(r_i) = \sup_{z_i : g(z_i) = r_i} \mu_{\mathrm{Z}i}(z_i), \quad r_i \in R, \quad i = 1,\ldots,N, \tag{19}$$

where vector $z_i = (w_1,\ldots,w_G, r_{i1},\ldots,r_{iG})$ has the m.f.

$$\mu_{\mathrm{Z}i}(z_i) = \left[\bigwedge_{j=1}^{G} \mu_{\mathrm{W}j}(w_j)\right] \wedge \left[\bigwedge_{k=1}^{G} \mu_{\mathrm{R}ik}(r_{ik})\right]. \tag{20}$$

The symbols $\bigwedge$, $\wedge$ denote the operation of taking the minimum. $g(z_i)$ is the average rating

$$g(z_i) = \sum_{j=1}^{G} w_j r_{ij} \Big/ \sum_{j=1}^{G} w_j. \tag{21}$$

The computational aspects involved in determining m.f. $\mu_{\mathrm{R}i}(r_i)$ and the use of these functions in establishing the order of preference of partitions S_i will be considered in the HCD problem discussed in the next section. Besides the fuzzy sets approach described above, other methods can be used in multiple-aspect decision processes based on intuitive feelings and subjective judgements. A probabilistic method which is strictly related to the approach discussed in this section is proposed in [6]. See [1] for a comparison between the two methods.

5. The health-care districting problem

We shall now consider the HCD problem outlined in Section 3. This will enable us to gain an insight into the transportation problem simplifying procedure presented in Section 3 (this criterion has been chosen to obtain the reduced subset $\mathcal{U}_y$) and into the fuzzy sets rating method described in Section 4. As a case study, some results will be given for the province of Savona, belonging to the Italian region Liguria.

An outcome of the transportation problem is presented in Fig. 2, where the transportation problem assignments, the main and the dependent components of the kernels and the n.u.a. communes are shown for the parametrizing variable $y=\alpha=1.1\bar{\alpha}$, $p_{\text{MIN}}=30\,000$ inhabitants, $p_{\text{MAX}}=135\,000$ inhabitants (see (1)), $\alpha_{\text{MAX}}=290$ inhabitants per bed unit (see (2')). The cost coefficients c_{kj} appearing in (6) are simply given by the distances between commune k and hospital j. Observe that the mountainous structure of the region is reflected by the rather 'scattered' assignments of the transportation problem outcome.

Table 1
Transportation problem results for different values of $y=\alpha$

$\alpha/\bar{\alpha}$	N	ND	number of feasible districts	number of regional partitions
1	9	4	150	14
1.05	8	5	54	8
1.1	8	8	70	4
1.2	5	1	67	8
1.4	4	0	50	11
1.5	2	0	30	11

Some results of the transportation problem are summarized in Table 1 for different values of α. Two values of p_{MAX} are introduced, namely $p^{\text{U}}_{\max}=135\,000$ for urban and $p^{\text{R}}_{\text{MAX}}=80\,000$ for rural districts. N is the number of the transportation problem links less the number of communes with no hospital. ND is the number of dependent components of the kernels. The numbers of feasible districts and of regional partitions (i.e., the numbers of elements of $\mathcal{U}_y$ and of $\mathcal{S}_y$, respectively) are also given.

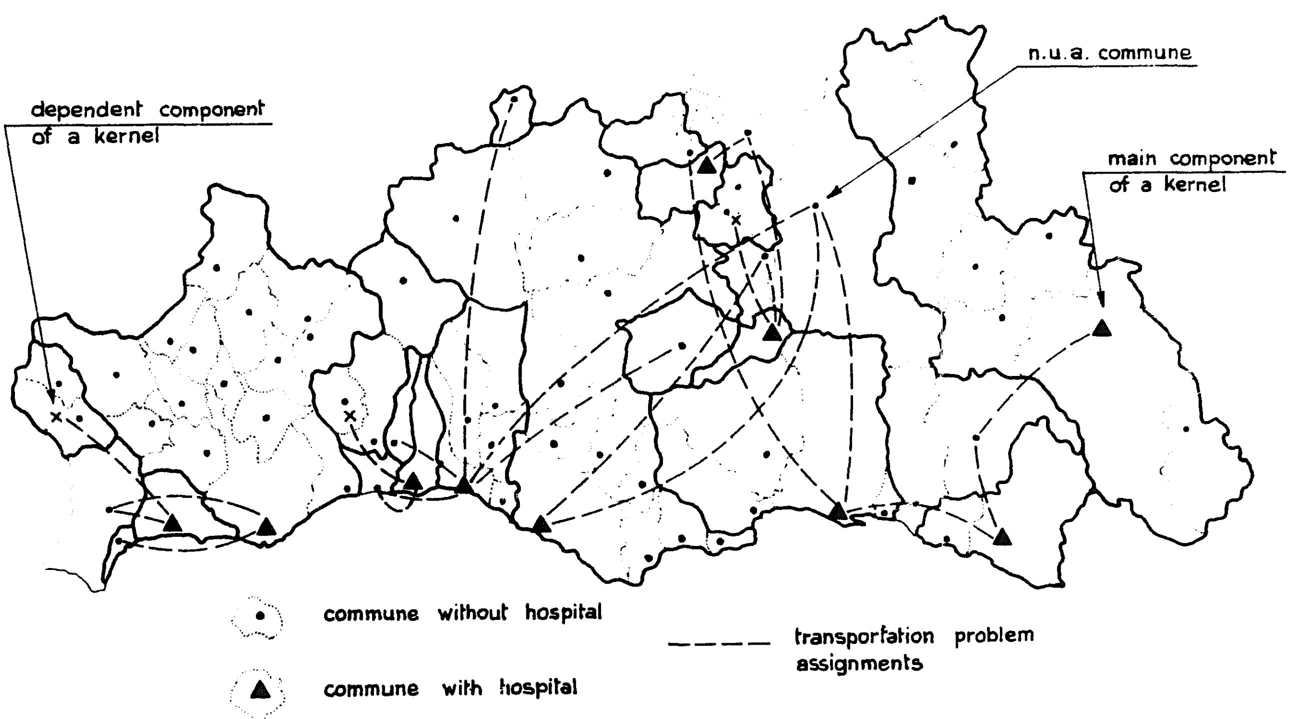

Fig. 2. Output of the transportion problem for $\alpha=1.1\bar{\alpha}$.

We shall now describe a possible assessment of m.f. $\mu_{Rij}(r_{ij})$. All m.f. will be derived from the aggregation of qualitative judgements assigned to each district constituting partition S_i.

Let us first define the following fuzzy ratings to evaluate each district u_k:

$$\{d_{kj}, \mu_{Dkj}\}, \quad d_{kj} \in [0,1];$$
$$k = 1,\ldots,U_y; \quad j = 1,\ldots,G \tag{22}$$

where U_y is the number of districts contained in $\mathfrak{A}_y$. Then, denote by $\boldsymbol{d}_{ij}$ $(j = 1,\ldots,G)$ a vector whose components are the ratings characterizing the district constituting S_i, that is, the elements of the set $\{d_{tj}: t \in I_i\}$, for which we have defined the set of integers $I_i = \{t: u_t \in S_i\}$.

On the space of vector $\boldsymbol{d}_{ij}$, define the m.f.

$$\mu_{Dij}(\boldsymbol{d}_{ij}) = \bigwedge_{t \in I_i} \mu_{Dtj}(d_{tj}). \tag{23}$$

Then, by following the same procedure outlined at the end of Section 4, we obtain the global rating μ_{Rij} for regional partition S_i as to aspect j

$$\mu_{Rij}(r_{ij}) = \sup_{\boldsymbol{d}_{ij}:\, g_{ij}(\boldsymbol{d}_{ij}) = r_{ij}} \mu_{Dij}(\boldsymbol{d}_{ij}), \quad r_{ij} \in \mathrm{R} \tag{24}$$

where

$$g_{ij}(\boldsymbol{d}_{ij}) = \sum_{t \in I_i} p(u_t)\, d_{tj} \Big/ \sum_{t \in I_i} p(u_t). \tag{25}$$

In (25), the numbers of district inhabitants have been used as nonfuzzy weights to define the rating $\{r_{ij}, \mu_{Rij}\}$ of S_i.

The relationships given in this section and in Section 4 define completely the mathematical procedure for obtaining the fuzzy ratings of each partition S_i, once the m.f. μ_{Wj} and μ_{Dkj} have been established (the computational aspects involved in deriving the final ratings are detailed in [1]). However, the choice of μ_{Wj} and μ_{Dkj} is actually a crucial point in the overall fuzzy sets procedure. A possible way of generating these m.f. (as well as a clear interpretation of their meaning) is given in [1], where the existence of a 'committee' is postulated dealing with the decision problem. Consider, for instance, the district u_k and the aspect j. Then the value of μ_{Dkj} for a particular rating d_{kj} is determined as the fraction of the committee that is willing to accept each d_{kj} as a possibly correct value. The same can be repeated for the other m.f.

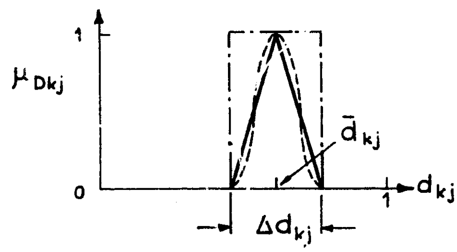

Fig. 3. Examples of membership functions for the ratings of district u_k (criteria $j = 1, 2, 3, 4$).

However, since we are more interested in the methodological aspects of the fuzzy sets approach than in its socio-administrative motivations, a much simpler model is dealt with. A unique decision maker is supposed to be given the task of establishing the m.f. As to μ_{Dkj}, this is done by first deciding a nonfuzzy rating $\bar{d}_{kj}$ according to the rules specified below. These ratings are then 'fuzzified' by means of m.f. of given shape centered around $\bar{d}_{kj}$, which express the qualitative nature of the decision maker's assessment. Examples of these functions are given in Fig. 3. The width of the support Δd_{kj} reflects the planner's uncertainty.

It is worth noting that such a 'mechanistic' procedure is a rather poor interpretation of the fuzziness concept. Nevertheless, it is quite helpful in gaining an insight into the computational aspects of the method, and in discussing the final results. As to the HCD problem, the following four criteria are taken into account.

(1) *Populations of urban and rural districts.* While the number of inhabitants of each feasible district must range between the values p_{MIN} and $p^{\mathrm{U}}_{\mathrm{MAX}}$ or $p^{\mathrm{R}}_{\mathrm{MAX}}$ defined above, socioadministrative reasons lead to prefer intermediate levels of population. Therefore, a nonfuzzy rating $\bar{d}_{k1}$ is defined for any admissible level of population $p(u_k)$ of district u_k according to the diagrams shown in Figs. 4(a) and 4(b) and a fuzzy rating $\{d_{k1}, \mu_{Dk1}\}$ is obtained by using one of the membership functions shown in Fig. 3.

(2) *Balance between demand and offer of healthcare service.* Define the quantities

$$\alpha_k = p(u_k)/l(u_k), \quad k = 1,\ldots,U_y \tag{26}$$

From (2′) it is $\alpha_k \leqslant \alpha_{\mathrm{MAX}}$. Consider a standard coefficient α_{st}, by which the Regional Hospital Plan prescribes the number of inhabitants who should be assisted by the health services corresponding to a bed-unit. Clearly, it should be $\bar{\alpha} \leqslant$

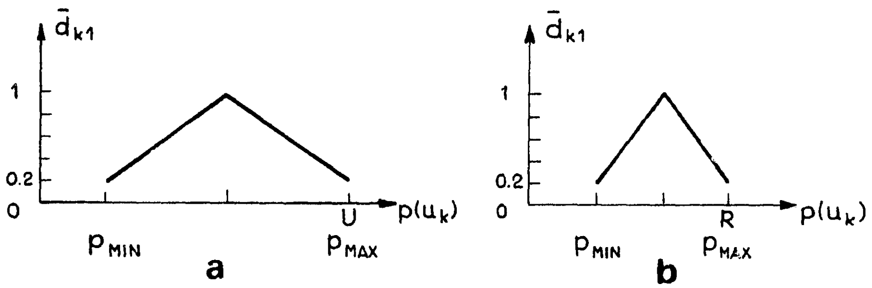

Fig. 4. (a) Nonfuzzy rating for urban district u_k and criterion 1; (b) Nonfuzzy rating for rural district u_k and criterion 1.

$\alpha_{st} \leqslant \alpha_{MAX}$. In our case, $\bar{\alpha} = 185$, $\alpha_{st} = 222$, $\alpha_{MAX} = 290$. Then, it seems reasonable to give each district a nonfuzzy rating d_{k2} depending on α_k, as shown in Fig. 5.

(3) *Transportation costs*. The average distance to be travelled by the inhabitants of a district to reach the hospital they are assigned to is clearly another evaluating criterion. Therefore, for each district u_k constituting solution S_i, solve a transportation problem like the one outlined by (6), (7), (8). The cost coefficients appearing in (6) are distances. Constraints (8) become

$$\sum_{r \in C(u_k)} z_{rs} = \alpha_k l_s, \quad s \in H(u_k),$$

where α_k is given by (26). Let $J(u_k)$ be the minimum cost obtained by solving the district transportation problem and define the average distance

$$\delta_k = J(u_k)/p(u_k), \quad k = 1, \dots, U_v.$$

Here again, assign district u_k a nonfuzzy rating $\bar{d}_{k3}$ according to the diagram shown in Fig. 6. δ_{MAX} should be the greatest distance appearing in $\mathfrak{U}$. Since, in general, not all districts are determined by the computing procedure, an estimate of δ_{MAX} can be derived by examining subsets $\mathfrak{U}_y$.

(4) *Quality of health-care services*. Since in our health-care districting problem we have assumed the presence of pre-existent hospitals of given characteristics, some indicators are needed to assess the qualities of these hospitals and then of the service offer within each district. Several indicators may be chosen to quantify these qualities (ratio of the number of physicians to the number of bed units; ratio of para-medical to medical personnel, etc.). From the specialized literature, it can be seen that a hospital 'dimension' expressed as the number of bed-units may characterize its efficiency according to the diagram of Fig. 7, where a nonfuzzy rating h_j of hospital j is shown as a function of number l_j. Then, for a given district u_k, we define a nonfuzzy weighted rating

$$\bar{d}_{k4} = \sum_{j \in H(u_k)} l_j h_j \Big/ \sum_{j \in H(u_k)} l_j, \quad k = 1, \dots, U_v.$$

As to the fuzzy weights (18), the comments about the generation of μ_{Dk_j} can be repeated for the m.f. μ_{W_j} corresponding to the four criteria described above. A possible shape for m.f. μ_{W_j}, used to solve the HCD problem for the province of Savona, is shown in Fig. 8, where $\bar{w}_1 = 0.4$, $\bar{w}_2 = 0.5$, $\bar{w}_3 = 0.8$, $\bar{w}_4 = 0.9$ and $\Delta w_j = 0.2$, $j = 1, 2, 3, 4$.

For this case study, the subset $\mathfrak{S}_y$ with $y = 1.2\bar{\alpha}$ has been considered. As can be seen from Table 1, 8 regional partitions constitute the subset. In deriving m.f. μ_{R_i}, triangular m.f. μ_{Dk_j} with the same support $\Delta d_{k_j} = 0.2$ have been chosen (see Fig. 3).

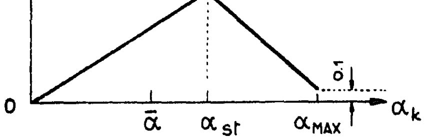

Fig. 5. Nonfuzzy rating for district u_k and criterion 2.

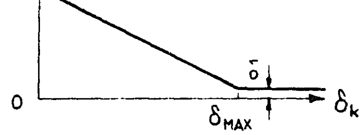

Fig. 6. Nonfuzzy rating for district u_k and criterion 3.

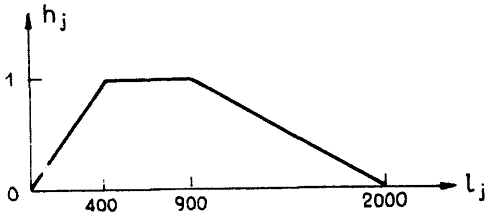

Fig. 7. Nonfuzzy rating for hospital j.

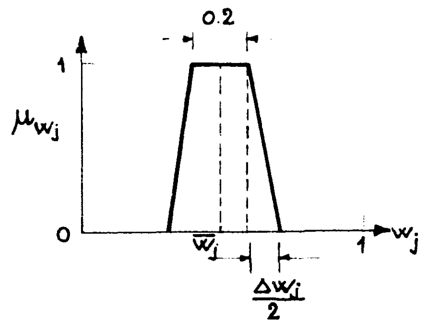

Fig. 8. Shape of the membership functions for the fuzzy weights of the four criteria.

After a preliminary inspection of all districts obtained for $y = 1.2\bar{\alpha}$, the value $\delta_{MAX} = 12$ km per person has been established to define the nonfuzzy ratings $\bar{d}_{k3}$ (see Fig. 6 and the final comment on the transportation cost criterion).

Two examples of m.f. μ_{R_i}, obtained by means of (19), (20), (21), are shown in Fig. 9. The interpretation of these examples in ranking the solutions S_i of the HCD problem will be discussed in the next section.

6. Final remarks on the procedure for ranking regional partitions

Once the fuzzy ratings $\{r_i, \mu_{R_i}\}$ of the regional partitions S_i have been determined, the final ques-

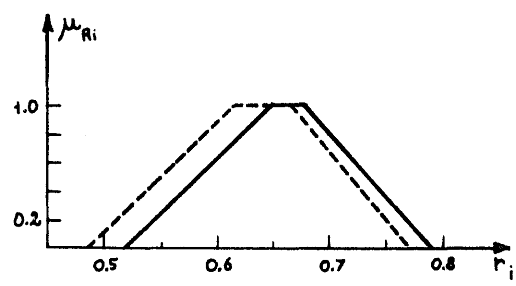

Fig. 9. Examples of membership functions of the ratings of regional partitions.

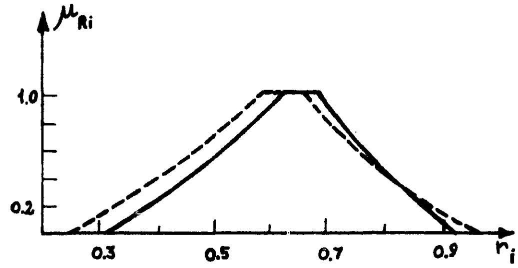

Fig. 10. Examples of membership functions of the ratings of regional partitions.

tions are to be answered as to how to compare these ratings (i.e., how to define the preference ordering $\succcurlyeq$) and as to how to select the preferred regional partition. Of course, it is worth noting that the preference ordering is not the only aim of the fuzzy sets decision analysis, in that the shapes of the m.f. μ_{R_i} themselves contain significant information.

Different criteria may be suggested to compare alternatives by using m.f. For instance, one can decide to prefer the partition having an absolute maximum point of $\mu_{R_i}(r_i)$ to the right of all the absolute maximum points of the other partitions. In some cases, the ranking procedure turns out to be a very simple task, and there is no need of defining a specific comparison criterion. This is the case of the HCD results shown in Fig. 9, where the two m.f. μ_{R_i} have approximately the same shape and support width, so that their positions give an unambiguous preference ordering of the corresponding partitions in quite a natural way.

Of course, more complex situations may arise, like the one shown in Fig. 10, which has been obtained for triangular m.f. μ_{Dkj}, with $\Delta d_{kj} = 0.8$ ($j = 1, 4$), $\Delta d_{kj} = 0.08$ ($j = 2, 3$) and for m.f. μ_{W_j} with $\Delta w_j = 0.8$ for $j = 1, 2, 4$ and $\Delta w_3 = 0.08$ (the values of $\bar{w}_j$ are the same as in the example of Fig. 9). Fig. 10 shows the m.f. μ_{R_i} of the two regional partitions which are candidates to give the best solution for $y = 1.2\bar{\alpha}$. From a cursory inspection of the two functions, it is not straightforward to decide which is the preferred partition.

A possible procedure to obtain a clearer indication may be the following [1]. Consider the number

$$p_i = r_i - \frac{1}{m-1} \sum_{j \in J, j \neq i} r_j, \quad i \in J, \tag{27}$$

where J is a set of integers denoting m partitions S_i to be compared. If ratings r_i were nonfuzzy, p_i

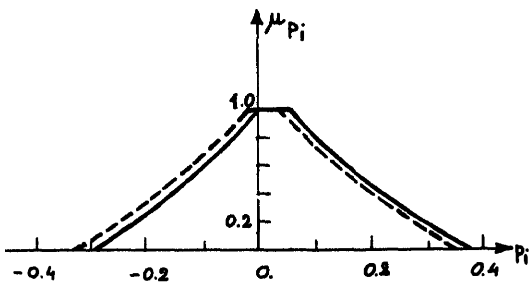

Fig. 11. Preferability membership functions.

would be a measure of preferability of partition S_i over the other partitions. Since ratings r_i are fuzzy, (27) specifies a mapping inducing a fuzzy set $\{p_i, \mu_{P_i}\}$, where the m.f. $\mu_{P_i}(p_i)$ can be determined in the usual way (see [1] for computational details). μ_{P_i} may then be used to judge the preferability of S_i over the other partitions. The preferability m.f. of the two regional partitions considered in Fig. 10 are shown in Fig. 11. Clearly, one of the two partitions may well be considered as the preferred one.

According to the above procedure, it has been found that the partitions obtained for $y = 1.2\bar{\alpha}$ are preferable to those obtained for lower values of y. For the sake of brevity, we do not give here the corresponding results. Therefore, one can infer that increasing the value of α, which has been used as the parametrizing variable y to generate subset $\tilde{S}_y$ (see Section 3), appears to be promising for obtaining 'better' regional partitions.

Clearly, from the preceding discussion, one cannot conclude that the use of preferability m.f. or the introduction of other fuzzy sets ranking criteria can enable one to avoid any kind of ambiguity in the final evaluation. Actually, it must not be forgotten that the main scope of our analysis is not to establish 'mechanistic' methods for a definite ranking of solutions, but to handle a multiple objective decision problem under uncertainty. It is deemed that the proposed method does constitute an efficient tool for achieving, in a natural way, qualitative but rational evaluations of the possible alternatives.

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