Nonconvex Equilibrium Models for Gas Market Analysis: Failure of Standard Techniques and Alternative Modeling Approaches

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Abstract. This paper provides a first approach to assess gas market interaction on a network with nonconvex flow models. In the simplest possible setup that adequately reflects gas transport and market interaction, we elaborate on the relation of the solution of a simultaneous competitive gas market game, its corresponding mixed nonlinear complementarity problem (MNCP), and a first-best benchmark. We provide conditions under which the solution of the simultaneous game is also the solution of the corresponding MNCP. However, equilibria cannot be determined by the MNCP as the transmission system operator's (TSO's) first-order conditions are insufficient, which goes back to nonconvexities of the gas flow model. This also implies that the welfare maximization problem may have multiple solutions that sometimes do not even coincide with any of the market equilibria. Our analysis shows that, even in the absence of strategic firms, market interaction fails to implement desirable outcomes from a welfare perspective due to the TSO's incentive structure. We conclude that the technical environment calls for a market design that commits the TSO to a welfare objective through regulation and propose a design where the market solution corresponds to a welfare maximum and vice versa.

1. Introduction

Gas is nowadays one of the most important energy carriers worldwide; see, e.g., [24, 25]. Ever since, gas is traded internationally and transported over long distances. Consequently, gas market research has focused for a long time on issues of market power and political economy, largely abstracting from the interdependence of markets and the transport infrastructure, e.g., [5, 8, 26, 44]. Recent developments like energy sector coupling or the increasing supply uncertainty at electricity markets due to the massive rollout of renewables, potentially increase the interdependence of gas trade with other energy sectors and, thus, the complexity at gas markets. Gas is, moreover, considered the main transitory technology on the way to sustainable energy supply; for recent discussions see, e.g., [35, 51]. All this may lead to an increasing number of traders, more frequent trade, as well as more entry and exit points in national gas networks. Therefore, an analysis of the interplay of gas trade and gas transmission becomes more and more important.

The analysis of markets in network-based industries has a long tradition in electricity market research. Typically, competitive electricity markets are described by mixed complementarity systems that arise from the optimality conditions of all players on the market, including the transmission system operator (TSO), jointly with additional constraints like a market clearing condition; see, e.g., [13, 21–23, 50]. Under simplifying economic and technical assumptions on the flow restrictions taken

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into account upon spot-market trading, it is known that these mixed complementarity systems are equivalent to the optimality conditions of a single-level optimization problem. The optimization goal of the latter is typically to maximize the agents' joint social welfare. Building on these fundamental results, a large literature has developed that thoroughly analyzes electricity market designs used around the world in (multilevel) electricity market models; cf. [16, 17, 31]. For gas markets, no similar models have been developed in the past, mainly due to the high complexity of network flow models that adequately reflect gas transport.

In gas market research, however, the long term objective is to understand incentives and consequences of different gas market designs as, e.g., the entry-exit system in Central Europe, or the US design based on long-term, point-to-point commercial capacity rights. By now, the related literature typically discusses potential shortcomings of those market regimes based on illustrative examples; cf., e.g., [1, 14, 20, 49] for Europe, [3, 27, 37–39] for the US market, and [32, 48] for China. Whenever those discussions include a quantitative analysis this is typically based on drastically simplified network representations. Equilibrium frameworks that do include a relatively detailed representation of the network as suggested above have not yet been developed in the literature. The main reason is that even stationary gas flow models are inherently nonlinear and nonconvex. Moreover, gas transport involves discrete decisions to be taken in order to control the network. These are the reasons why a straightforward application of techniques for modeling electric power markets is not possible in models that consider the interplay of natural gas markets and gas transport.

This paper is the first step on the way to provide adequate tools for the assessment of gas market designs. The natural first step is to specify an appropriate gas market game (analogously to the standard electricity market setup) and elaborate on the relation of the corresponding mixed nonlinear complementarity problem (MNCP) and the first-best model of a gas market. In order to deliver a first assessment of the scope for gas market modeling on a network, we choose the simplest possible model specification that adequately reflects gas transport restrictions. In particular, we draw on the gas flow representation first formalized in [47]. Here, the pressure drop between network nodes introduces nonconvexities to the models of the underlying equilibrium problems, which rules out a straightforward application of standard approaches from electricity market modeling. As the appropriate technical model of gas physics induces significant challenges, we consider a very basic economic model of the short-run natural gas market, assuming three profit-maximizing types of agents—non-strategic gas-suppliers with convex marginal cost,[1] gas consumers with decreasing demand, and the TSO who operates the transmission network—together with a market clearing condition.

For this model, we show that, under mild conditions, the solution of a simultaneous market game among profit-maximizing but non-strategic agents is also the solution of the corresponding MNCP. However, as it turns out, the converse is not necessarily true. We demonstrate that there exist solutions to the MNCP that do not correspond to market equilibria. Hence, although all market equilibria correspond to solutions of the MNCP, these equilibria cannot be determined by the MNCP due to the insufficiency of the TSO's first-order conditions that goes back to the nonconvexity of the gas flow model. As a second step, we consider the welfare maximization problem that typically serves as a first-best benchmark and under appropriate assumptions—which are not satisfied in our framework—corresponds to the outcome of the competitive equilibrium; cf. the fundamental theorems of welfare economics as

[1] Note that, for constant marginal production costs, multiplicities could occur even for linear network constraints, see [29].

established in [2]. We demonstrate that the nonconvexities originating from gas flow constraints also imply that the welfare maximization problem may have multiple solutions even for simple gas networks. Afterward, we analyze the correspondence between welfare maxima and competitive equilibria and relate our findings to the fundamental theorems of welfare economics. We show that restrictive assumptions can be determined that guarantee that equilibria of the simultaneous game are welfare maximizing. In essence, this applies if no network congestion occurs for the respective supply and demand situation. However, in the general case, there is no 1-1 correspondence between equilibria and welfare-maximal solutions. In particular, there even exist situations, where not a single welfare-maximal solution can be implemented as an equilibrium of the simultaneous game.

Our analysis clearly suggests that market interaction cannot yield desirable outcomes from a welfare perspective due to the TSO's incentive structure. We therefore propose an alternative market environment that induces optimality of market outcomes by regulating the player, whose decisions are responsible for the divergence of equilibrium outcomes and welfare maxima, i.e., the TSO. In particular, we propose a three-level gas market model where the TSO is committed to a welfare objective and moves first by setting the nodal prices that govern gas trade among suppliers and consumers at the second level. At the third level, the TSO operates the network based on the market outcome at level two. We prove that the market solutions correspond to welfare maxima and vice versa, and discuss critical assumptions with respect to the information that players, in particular the TSO, possess.

Our work directly relates to the classical contributions in [46] and [2] that establish the fundamental theorems of welfare economics; for a comprehensive survey see, e.g., [33]. The theorems establish—under suitably chosen assumptions—existence, uniqueness, and Pareto-optimality of competitive equilibria, i.e., a 1-1 correspondence of market equilibria and welfare optima. The fundamental setup that we analyze can clearly be interpreted accordingly if gas at different nodes of the network is considered as a different good. In that context, the TSO is a specific producer that, according to its production technology, can convert input at certain nodes of the network into output at other nodes of the network. However, for the case of gas flow models based on pressure gradients, the network model exhibits nonconvexities that imply that the corresponding market game, which we analyze, does not satisfy the classical assumptions behind the fundamental welfare theorems. Consequently, our framework is not directly covered by the classical contributions on welfare economics. One strand of literature that aims at overcoming the problem of nonconvexities due to network flow models uses linear approximations; see, e.g., [11, 34, 43]. However, the resulting flow models necessarily neglect important characteristics of gas transport. Therefore, in our analysis, the focus is on the relation of market outcomes and welfare maxima in environments with nonconvex flow models. Our results thus provide a first step for the analysis of gas markets accounting appropriately for network constraints.

Our paper is organized as follows. Section 2 is devoted to the basic notation and some mathematical preliminaries. Section 3 introduces a gas market model analogous to the approach used in the electricity market literature and provides results on the correspondence between its equilibria and the associated MNCP. Section 4 provides an analysis of the welfare-maximal benchmark and shows under which conditions the fundamental welfare theorems extend to our environment and when they fail to extend. Section 5 finally proposes an alternative model of gas markets that restores the correspondence of welfare maxima and market solutions. Section 6 concludes.

2. Basic Notation and Mathematical Preliminaries

In this section, we present the basic notation that is used throughout the paper and collect the required mathematical background.

2.1. **Basic Notation.** The gas transport network is modeled as a directed and connected graph $G = (V, A)$ with node set V and arc set A. The set of nodes is further distinguished into the set $V_+ \subseteq V$ of nodes at which supply customers are located and the set $V_- \subseteq V$ of nodes at which discharging customers are located. The set of supply and discharging customers are assumed to be disjoint, i.e., $V_+ \cap V_- = \emptyset$, and $V_+ \cup V_- = V$ holds. The set of arcs represents the pipes. Our main motivation in this paper is to analyze the effects of including the nonconvex physics of gas flow on the market outcome. Thus, we do not regard active elements, e.g., (control) valves or compressors, that would add further nonconvexities of discrete type.

The economic framework is as follows. The function $c_u(\cdot)$ models variable costs of gas supply. From an economic point of view, variable costs of a supplier located at a node $u \in V_+$ are non-negative, $c_u(\cdot) \geq 0$, and zero in case of no supply, $c_u(0) = 0$. Moreover, the functions $c_u(\cdot)$ are monotonically increasing, strictly convex, and continuously differentiable.

The willingness to pay of a consumer located at a node $u \in V_-$ is described by the inverse demand function $P_u(\cdot)$. Each inverse demand function $P_u(\cdot)$ is assumed to be continuous and strictly decreasing.

2.2. **Mathematical Preliminaries.** In this paper, we often make use of the Karush–Kuhn–Tucker (KKT) conditions. For a general optimization problem of the form

$$\max_x \; f(x) \quad \text{s.t.} \quad g_i(x) \leq 0, \; i \in I, \quad h_j(x) = 0, \; j \in J, \tag{1}$$

they are given by

$$\nabla f(x) - \sum_{i \in I} \mu_i \nabla g_i(x) - \sum_{j \in J} \lambda_i \nabla h_j(x) = 0, \tag{2a}$$

$$g_i(x) \leq 0, \quad i \in I, \tag{2b}$$

$$h_j(x) = 0, \quad j \in J, \tag{2c}$$

$$\mu_i \geq 0, \quad i \in I, \tag{2d}$$

$$\mu_i g_i(x) = 0, \quad i \in I. \tag{2e}$$

In this context, we always assume that f, g_i, and h_j are continuously differentiable for all $i \in I$ and $j \in J$. Under a suitable constraint qualification, the KKT conditions are known to be necessary for optimal solutions. In the case of convex optimization problems, the KKT conditions are both necessary and sufficient for global optimality if a suitable constraint qualification like, e.g., Slater's condition, holds; see [7]. In case of nonconvex optimization problems, other constraint qualifications are needed. In our analysis, we mainly use the linear independence constraint qualification (LICQ).

Definition 2.1 (LICQ)**.** Let x be a feasible solution of (1) and let $I^=(x) := \{i \in I : g_i(x) = 0\}$. Then, x fulfills the LICQ if the gradients $\nabla h_j(x)$, $j \in J$, and $\nabla g_i(x)$, $i \in I^=(x)$, are linearly independent.

The LICQ implies the following: For an optimal solution $\hat{x}$ of (1) that satisfies the LICQ, there exists a vector of Lagrange multipliers $(\hat{\mu}, \hat{\lambda})$ such that the KKT conditions are fulfilled by $(\hat{x}, \hat{\mu}, \hat{\lambda})$. Such points are called KKT points. Furthermore, the LICQ implies unique Lagrange multipliers. Thus, the LICQ is also of interest for analyzing convex optimization problems because the Lagrange multipliers correspond to dual variables in this setting. To guarantee sufficiency of KKT conditions in case of nonconvexity, an additional second-order condition has to hold; see, e.g., [36] and the references therein.

3. Nonconvex Gas Market Equilibrium Modeling: The Standard Approach

We now consider a simultaneous game for a nodal-pricing-type gas market under perfect competition. In such a game, all players decide simultaneously on their actions. The formulation chosen here is closely related to the formulation of the electricity market model presented in [21]. In what follows, we first state the simultaneous game and afterward, we show that—due to the nonconvexity of gas physics—the solutions of this game cannot be obtained by a reformulation as a mixed nonlinear complementarity system.

3.1. The Simultaneous Game.

The main players are the supplying customers, the discharging customers, and the transmission system operator (TSO). As we consider the case of perfect competition, all players act as price takers, i.e., they do not take into account the impact of their actions on market prices. Each supplying customer bids a supply function in dependence of the given market price with the resulting profit as payoff. In turn, each discharging customer bids a demand function that relates the requested quantity to the market price. The goal here is to maximize gross consumer surplus less costs. We assume that there is no strategic interaction among suppliers or among consumers. The same holds for the TSO who ensures feasibility of the operation of the transmission network. His payoff depends on price differences between nodes. The simultaneous game is composed of the optimization problems of each player together with market clearing conditions. We start by describing the gas suppliers.

3.1.1. *Gas Suppliers.* Every gas supplying customer located at a node $u \in V_+$ maximizes his profit by solving the concave maximization problem

$$\max_{y_u \geq 0} \quad \pi_u y_u - c_u(y_u), \tag{3}$$

where y_u denotes the amount of gas supplied by the player. The function $c_u(\cdot)$ models variable costs of gas supply and π_u is the given market price at which the gas is sold. Moreover, the amount y_u of supplied gas is bounded to be non-negative. The KKT conditions of Problem (3) can be used to obtain the complementarity system

$$\pi_u - c_u'(y_u) + \eta_u^- = 0, \quad 0 \leq y_u \perp \eta_u^- \geq 0, \tag{4}$$

which characterizes global optima of Problem (3).

In what follows, we prove that the reaction of a gas supplier to a given nodal price is unique. Furthermore, we specify the unique primal solution of the suppliers' profit maximization problem.

Lemma 3.1. *Let π_u be the nodal price at node $u \in V_+$. Then, for the supplier located at this node, the primal solution of Problem (3) is unique and given by*

$$y_u = \begin{cases} 0, & c_u'(0) > \pi_u, \\ (c_u')^{-1}(\pi_u), & c_u'(0) \leq \pi_u. \end{cases}$$

In addition, the dual solution is unique.

Proof. The uniqueness of the primal solution follows from strict convexity of Problem (3). As $c_u(\cdot)$ is strictly convex as well as continuously differentiable, $c_u'(\cdot)$ is strictly increasing, continuous, and thus invertible. In case of $c_u'(0) > \pi_u$,

$\eta_u^- = c_u'(y_u) - \pi_u \geq c_u'(0) - \pi_u > 0$ and $y_u = 0$ follows directly from the KKT conditions (4). In case of $c_u'(0) \leq \pi_u$, $y_u = (c_u')^{-1}(\pi_u + \eta_u^-) \geq (c_u')^{-1}(c_u'(0) + \eta_u^-)$ also follows from the KKT conditions (4). If $\eta_u^- > 0$, then, $y_u > (c_u')^{-1}(c_u'(0)) = 0$, which contradicts (4). Thus, $\eta_u^- = 0$ and $y_u = (c_u')^{-1}(\pi_u)$.

Uniqueness of the dual solution follows directly from the LICQ. □

3.1.2. *Gas Consumers.* The optimization problem of a consumer located at a node $u \in V_-$ reads

$$\max_{d_u \geq 0} \quad \int_0^{d_u} P_u(x)\,\mathrm{d}x - \pi_u d_u, \tag{5}$$

where $P_u(\cdot)$ is the continuous and strictly decreasing inverse demand function of the player. Moreover, the amount d_u of discharged gas is bounded to be non-negative.

Problem (5) is also a concave maximization problem. In analogy to the complementarity problem (4) above, we obtain the again necessary and sufficient system

$$P_u(d_u) - \pi_u + \beta_u^- = 0, \quad 0 \leq d_u \perp \beta_u^- \geq 0. \tag{6}$$

In what follows, we show that the decisions of the discharging customers are unique w.r.t. a given nodal price. Moreover, the unique reaction can be explicitly stated for each consumer.

Lemma 3.2. *Let π_u be the nodal price at node $u \in V_-$. Then, for the consumer located at this node, the primal solution of Problem (5) is unique and given by*

$$d_u = \begin{cases} 0, & P_u(0) < \pi_u, \\ P_u^{-1}(\pi_u), & P_u(0) \geq \pi_u. \end{cases}$$

In addition, the dual solution is unique.

Proof. Analogous to Lemma 3.1. □

3.1.3. *Market Clearing Conditions.* To clear the market, supply must equal demand. This is achieved by imposing the nodal balance equations

$$\sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = q_u, \quad u \in V, \tag{7}$$

as market clearing conditions. Here, $q_u = d_u$ holds for $u \in V_-$ and $q_u = -y_u$ holds for $u \in V_+$. Thus, the gas flow out of node $u \in V$ and the gas flow into that node have to be balanced w.r.t. supply or demand at that node.

3.1.4. *The Transmission System Operator.* Finally, we model the TSO that is responsible to operate the transmission network. We mainly follow [21] and model this player using the problem

$$\max_{q,p} \quad \sum_{a=(u,v) \in A} (\pi_v - \pi_u) q_a \tag{8a}$$

$$\text{s.t.} \quad p_u^2 - p_v^2 = \Lambda_a q_a |q_a|, \quad a = (u,v) \in A, \tag{8b}$$

$$p_u^- \leq p_u \leq p_u^+, \quad u \in V, \tag{8c}$$

with physical arc flows $q = (q_a)_{a \in A}$ and nodal pressures $p = (p_u)_{u \in V}$. The market prices π_u are given, i.e., the TSO is also a price-taker. The goal of the TSO is to take advantage of existing price differences between adjacent nodes by routing as much gas as possible from low-price to high-price nodes.[2] The stationary gas flow through pipes is described by the so-called Weymouth constraint (8b); cf.,

[2]If the TSO would be completely indifferent w.r.t. his routing decisions, i.e., Problem (8) reduces to a pure feasibility problem, then, each physically feasible solution would correspond to a market equilibrium in case of appropriate nodal prices. Such a formulation would not be economically meaningful. Thus, neither the case of a welfare-maximizing but price-taking TSO nor the case

e.g., Chapter [12] in [28]. The change in quadratic nodal pressures over an arc is expressed by the pipe's specific pressure drop coefficient $\Lambda_a \geq 0$ and the quadratic flow term $q_a|q_a|$ that takes into account the flow direction. Thus, the resulting model is nonlinear and nonconvex. The pressure drop coefficient itself is a constant depending on the pipe's diameter, length, and cross-sectional area as well as on the gas temperature, the compressibility factor, and on the specific gas constant; see [12] for the details. By using the Weymouth equation in (8b), we decided to use the most simple representation that is appropriate to model gas physics. Moreover, it is a very common model in the literature on gas network optimization; see, e.g., the recent book [28] or the survey article [41] as well as the references therein. In order to choose the most simple model that allows to analyze the impact of nonconvex physics in market models, we refrain from including more difficult elements like compressor stations and thus also refrain from including transport costs.

Finally, Constraint (8c) bounds the nodal pressures p_u from below by p_u^- and from above by p_u^+ to ensure technical requirements like maximum pressures in pipes or contractual pressure limits. If appropriate, Model (8) can be extended by arc-wise flow bounds that can be derived from the Weymouth equation (8b) and the pressure bounds (8c).

The KKT conditions of the TSO's nonlinear problem (8) are given by

$$2\Lambda_a \gamma_a |q_a| + \pi_v - \pi_u = 0, \quad a = (u,v) \in A, \tag{9a}$$

$$2p_u \left(\sum_{a \in \delta^{\text{in}}(u)} \gamma_a - \sum_{a \in \delta^{\text{out}}(u)} \gamma_a \right) + \zeta_u^- - \zeta_u^+ = 0, \quad u \in V, \tag{9b}$$

$$p_v^2 - p_u^2 + \Lambda_a q_a |q_a| = 0, \quad a = (u,v) \in A, \tag{9c}$$

$$0 \leq \zeta_u^- \perp (p_u - p_u^-) \geq 0, \quad u \in V, \tag{9d}$$

$$0 \leq \zeta_u^+ \perp (p_u^+ - p_u) \geq 0, \quad u \in V. \tag{9e}$$

To guarantee that the KKT conditions are necessary and sufficient for optimality, a constraint qualification as well as a second-order sufficient condition have to be valid for a given solution of Problem (8). The question of necessity and sufficiency of the TSO's KKT conditions is addressed later in this section, where we show that LICQ holds under certain mild assumptions but that second-order conditions may fail to hold.

We now show that—under consideration of the market clearing conditions (7)—the TSO's actions are unique.

Theorem 3.3. *Let $(\pi_u)_{u \in V}$ be nodal prices such that*

$$\sum_{u \in V_+} y_u = \sum_{u \in V_-} d_u$$

holds for y_u being the solution of Problem (3), $u \in V_+$, and d_u being the solution of Problem (5), $u \in V_-$. In addition, let $x = (q,p)$ and $\hat{x} = (\hat{q}, \hat{p})$ be solutions of Problem (8) that fulfill the flow conservation constraints (7). Then, $q = \hat{q}$ holds.

Proof. As nodal prices are given, all supply $(y_u)_{u \in V_+}$ and all demand $(d_u)_{u \in V_-}$ quantities are—in accordance with Lemma 3.1 and Lemma 3.2—unique and thus fixed. The assertion then follows from Theorem 1 in [42]. □

Besides flows, pressures are unique except for constant shifts, i.e., there exists a constant $c \in \mathbb{R}$ such that $p_u = \hat{p}_u + c$ for all $u \in V$ also yield a solution. Note that

of Constraints (7), (8b), and (8c) as market clearing conditions would yield a reasonable market equilibrium model.

this multiplicity of the pressure variables does not influence the market outcomes, which is why we do not focus on this multiplicity in the following analyses.

3.2. **Equilibrium Analysis.** So far, we described the players' strategies by studying their individual maximization problems. Especially we proved that, given nodal prices such that supply equals demand, all players' actions are unique. However, these results are not enough to guarantee a market equilibrium. In equilibrium the actions of all players imply market clearing and furthermore, no player has an incentive to deviate from his decision. The latter is is achieved in case of optimal decisions for each player. Therefore, solving the optimization problems of all market players together, i.e., solving the simultaneous game (SG) that is defined by

$$\begin{aligned} \text{Suppliers:} \quad & (3) \text{ for all } u \in V_+, \\ \text{Consumers:} \quad & (5) \text{ for all } u \in V_-, \\ \text{Market Clearing Conditions:} \quad & (7), \\ \text{TSO:} \quad & (8), \end{aligned} \tag{SG}$$

yields a market equilibrium if it exists. To obtain solutions of the simultaneous game, it is typically reformulated by replacing the players' optimization problems using the corresponding KKT conditions; cf., e.g., [13, 21].

In our application, combining the complementarity systems of all market players yields the mixed nonlinear complementarity system

$$\begin{aligned} \text{Suppliers:} \quad & (4) \text{ for all } u \in V_+, \\ \text{Consumers:} \quad & (6) \text{ for all } u \in V_-, \\ \text{Market Clearing Conditions:} \quad & (7), \\ \text{TSO:} \quad & (9). \end{aligned} \tag{MNCP}$$

However, this reformulation is only meaningful if and only if a constraint qualification and a second-order sufficient condition hold for the TSO's problem. Only then, the corresponding KKT system represents the optimal decisions of the TSO. In what follows, we address the question if

$$\text{(SG)} \iff \text{(MNCP)}$$

holds in our special case. We next show that the implication "(SG) ⇒ (MNCP)" holds true if the following assumptions hold.

Assumption 1. *All lower pressure bounds p_u^-, $u \in V$, are positive and all pressure bounds p_u^- and p_u^+, $u \in V$, are pairwise distinct.*

Assumption 2. *Let $x = (q, p)$ be a solution of Problem (8). Then for each cycle $C = (V(C), A(C))$ in G, there exists an arc $a \in A(C)$ with $q_a \neq 0$.*

Using these assumptions, we show that the TSO's KKT conditions are necessary for optimality. The proof is rather technical and given in Appendix A.

Lemma 3.4. *Let $x = (q, p)$ be a solution of Problem (8) that fulfills Assumption 2. Furthermore, suppose that Assumption 1 holds. Then, there exists a unique vector of Lagrange multipliers $\lambda = (\gamma, \zeta^-, \zeta^+)$ such that (x, λ) satisfies Problem (9).*

As the KKT conditions of the gas supplying and discharging customers are also necessary, the implication "(SG) ⇒ (MNCP)" holds as long as there is no optimal solution of Problem (8) with zero flows on one of the cycles of the network. Thus, each market equilibrium of (SG) is also a solution of the (MNCP).

Theorem 3.5. *Let $x = (y, d, q, p)$ be a solution of the simultaneous game (SG) such that (q, p) fulfills Assumption 2. Furthermore, suppose that Assumption 1*

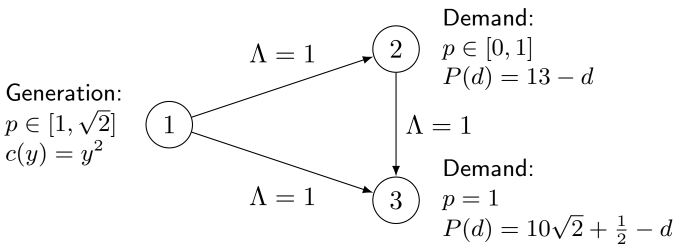

FIGURE 1. Three-node network of Example 3.7

holds. Then, there exists a unique vector $\lambda = (\eta^-, \beta^-, \gamma, \zeta^-, \zeta^+)$ *such that* (x, λ) *is a solution of the* (MNCP)*.*

Proof. Follows from Lemma 3.1, Lemma 3.2, and Theorem 3.4. ☐

It remains to consider the implication "(SG) ⇐ (MNCP)". It turns out that this implication does not hold in general. There may exist solutions to the (MNCP) that do not correspond to market equilibria. Hence, although all market equilibria correspond to solutions of the (MNCP), these equilibria cannot be determined by the (MNCP) due to the insufficiency of the TSO's first-order conditions.

Theorem 3.6. *Let* $(x; \lambda) = (y, d, q, p; \eta^-, \beta^-, \gamma, \zeta^-, \zeta^+)$ *be a solution of the* (MNCP) *such that* (q, p) *fulfills Assumption 2. Furthermore, suppose that Assumption 1 holds. Then, there exist instances such that* $x = (y, d, q, p)$ *does not implement a market equilibrium of the simultaneous game* (SG)*.*

Proof. See Appendix B. ☐

The counterexample given in Appendix B is rather technical. Thus, we additionally present an instance that, on the one hand, violates Assumption 1 but, on the other hand, comprises all phenomena that illustrate the failure in Theorem (3.6). In particular, there also exists an (MNCP) solution that does not yield a market equilibrium.

Example 3.7. *The considered graph consists of three nodes that are connected by three pipes. The example is depicted, together with all physical and economical parameters, in Figure 1. For the nodal prices*

$$\pi_1 = 2 + 2\sqrt{2}, \quad \pi_2 = 12 - \sqrt{2}, \quad \pi_3 = 10\sqrt{2} + \frac{1}{2},$$

we obtain the solution

$$\begin{aligned} y_1 &= 1 + \sqrt{2}, & d_2 &= 1 + \sqrt{2}, & d_3 &= 0, \\ q_{12} &= \sqrt{2}, & q_{13} &= 1, & q_{23} &= -1, \\ p_1 &= \sqrt{2}, & p_2 &= 0, & p_3 &= 1, \end{aligned}$$

of the (MNCP)*. The Lagrange multipliers are given by*

$$\begin{aligned} \gamma_{12} &= \frac{3\sqrt{2} - 10}{2\sqrt{2}}, & \gamma_{13} &= -4\sqrt{2} + \frac{3}{4}, & \gamma_{23} &= -\frac{11}{2}\sqrt{2} + \frac{23}{4}, \\ \zeta_3^- &= 19\sqrt{2} - 13, & \zeta_1^+ &= 26 - \frac{9}{2}\sqrt{2}, \end{aligned}$$

and $\eta_1^- = \beta_2^- = \beta_3^- = \zeta_1^- = \zeta_2^- = \zeta_2^+ = \zeta_3^+ = 0$. *The corresponding Problem* (8) *of the TSO then reads*

$$\begin{aligned} \max_{q,p} \quad & \left(10 - 3\sqrt{2}\right) q_{12} + \left(8\sqrt{2} - \frac{3}{2}\right) q_{13} + \left(11\sqrt{2} - \frac{23}{2}\right) q_{23} \\ \text{s.t.} \quad & p_2^2 = p_1^2 - q_{12}|q_{12}|, \quad p_3^2 = p_1^2 - q_{13}|q_{13}|, \quad p_3^2 = p_2^2 - q_{23}|q_{23}|, \\ & 1 \le p_1 \le \sqrt{2}, \quad 0 \le p_2 \le 1, \quad p_3 = 1. \end{aligned}$$

Using pressures $p_1 = \sqrt{2}$ *and* $p_2 = 1$ *with flows* $q_{12} = 1$, $q_{13} = 1$, *and* $q_{23} = 0$ *yields a higher payoff for the TSO than* $p_1 = \sqrt{2}$ *and* $p_2 = 0$ *Thus, the TSO has an incentive to deviate from the stated solution of the* (MNCP) *and, consequently, the* (MNCP) *solution does not correspond to a market equilibrium.*

4. The Welfare Maximization Problem

In this section, we describe a welfare maximization problem that serves as a first-best benchmark for the simultaneous game discussed in the last section. Then, we study whether the welfare-maximal solution is unique. It turns out that, even for simple gas networks, multiple welfare-maximal solutions exist. Afterward, we analyze the correspondence between welfare maxima and competitive equilibria and relate our findings to the fundamental theorems of welfare economics. Here, we will see that restrictive assumptions can be determined that guarantee that equilibria of the simultaneous game are welfare-maximizing. However, none of these equilibria is welfare maximal in the general case.

The welfare maximization problem is derived by considering an integrated planner who simultaneously decides on supply and demand for gas as well as on the physical gas flows by controlling the nodal pressures. The goal of the integrated planner is to maximize social welfare, i.e., gross consumer surplus less variable costs of supply. The resulting welfare maximization problem reads

$$\begin{aligned} \max_{y,d,q,p} \quad & \sum_{u \in V_-} \int_0^{d_u} P_u(x) \, \mathrm{d}x - \sum_{u \in V_+} c_u(y_u) \\ \text{s.t.} \quad & y_u \ge 0, \quad u \in V_+, \\ & d_u \ge 0, \quad u \in V_-, \\ & \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = q_u, \quad u \in V, \\ & p_u^2 - p_v^2 = \Lambda_a q_a |q_a|, \quad a = (u,v) \in A, \\ & p_u^- \le p_u \le p_u^+, \quad u \in V, \end{aligned} \tag{WMP}$$

where q_u equals d_u for all $u \in V_-$ and $-y_u$ for all $u \in V_+$. The non-negative variable y_u describes the amount of gas supplied by the supplier located at node $u \in V_+$. The amount of gas discharged by the customer located at node $u \in V_-$ is denoted by the non-negative variable d_u. Moreover, the physical gas flow over a pipe $a \in A$ is given by the variable q_a and the pressure at node $u \in V$ is modeled by the variable p_u. The last three constraints again implement nodal flow balance and ensure physical and technical feasibility of the resulting gas flow; cf. (8b) and (8c).

4.1. **Multiplicity of Welfare-Maximal Solutions.** In this section, we show that the solution of the welfare maximization problem (WMP) is in general not unique. We prove this multiplicity by an example. To this end, we consider the three-node network of Example 3.7; see, Figure 1. The corresponding welfare maximization

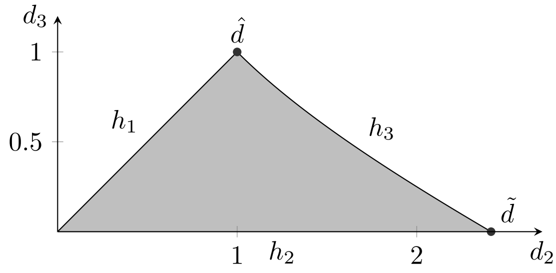

FIGURE 2. Projection of the feasible set of Problem (10) onto the 2d-space of demand variables

problem reads

$$\begin{aligned} \max_{y,d,q,p} \quad & \int_0^{d_2} (13 - x)\,\mathrm{d}x + \int_0^{d_3} \left(10\sqrt{2} + \frac{1}{2} - x\right) \mathrm{d}x - y_1^2 \\ \text{s.t.} \quad & y_1 \geq 0, \quad d_2 \geq 0, \quad d_3 \geq 0, \\ & q_{12} + q_{13} = y_1, \quad q_{12} - q_{23} = d_2, \quad q_{13} + q_{23} = d_3, \\ & p_2^2 = p_1^2 - q_{12}|q_{12}|, \quad p_3^2 = p_1^2 - q_{13}|q_{13}|, \quad p_3^2 = p_2^2 - q_{23}|q_{23}|, \\ & 1 \leq p_1 \leq \sqrt{2}, \quad 0 \leq p_2 \leq 1, \quad p_3 = 1. \end{aligned} \tag{10}$$

As before, we fix the pressure at node 3 and impose pressure bounds at the other nodes such that all flow directions are known because $p_1 \geq p_3 \geq p_2$. Since flow is only possible from higher to lower nodal pressures, it holds $q_{12} \geq 0$, $q_{13} \geq 0$, and $q_{23} \leq 0$. Since the flow over a pipe is proportional to the change in quadratic nodal pressures, $d_2 \geq d_3$ follows directly from $p_1 \geq p_3 \geq p_2$. Using the results of [15], the projection of the feasible set of problem (10) onto the space of withdrawals (d_2, d_3) can be described by the inequality system

$$\begin{gathered} 0 \leq d_3 \leq d_2, \\ 0 \leq \left(2d_2 + d_3 - \sqrt{2(d_2^2 + d_2 d_3)}\right)^2 \leq 2, \\ 0 \leq \left(\sqrt{2(d_2^2 + d_2 d_3)} - d_2\right)^2 \leq 1, \\ -1 \leq \left(\sqrt{2(d_2^2 + d_2 d_3)} - d_2\right)^2 - \left(2d_2 + d_3 - \sqrt{2(d_2^2 + d_2 d_3)}\right)^2 \leq 0. \end{gathered}$$

In Figure 2, the set of pairs (d_2, d_3) that satisfy this system is depicted. The boundaries of the set are defined by the functions

$$\begin{aligned} h_1 &: [0, 1] \to [0, 1], \quad d_2 \mapsto d_2, \\ h_2 &: [0, 1 + \sqrt{2}] \to 0, \quad d_2 \mapsto 0, \\ h_3 &: [1, 1 + \sqrt{2}] \to [0, 1], \quad d_2 \mapsto \frac{1 + 2d_2 - d_2^2}{2d_2}. \end{aligned}$$

In what follows, we show that the points $\hat{d} = (1, 1)$ and $\tilde{d} = (1 + \sqrt{2}, 0)$ both are optimal solutions of the welfare maximization problem (10). Both lie on the graph of h_3. We now consider the welfare function W, cf. the objective of (10), as a

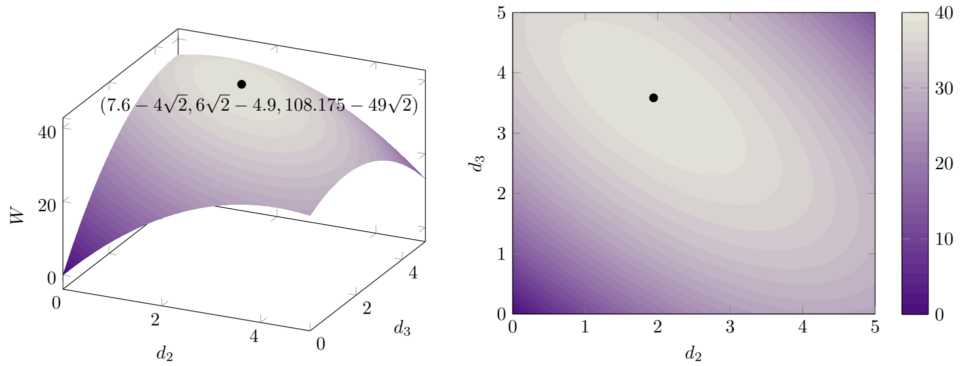

FIGURE 3. The welfare function W (left) and its isolines (right)

function of d_2 and d_3:

$$\begin{aligned} W(d_2, d_3) &= \int_0^{d_2} (13 - x)\,\mathrm{d}x + \int_0^{d_3} \left(10\sqrt{2} + \frac{1}{2} - x\right)\mathrm{d}x - (d_2 + d_3)^2 \\ &= 13d_2 + \left(10\sqrt{2} + \frac{1}{2}\right) d_3 - \frac{3}{2}d_2^2 - \frac{3}{2}d_3^2 - 2d_2d_3. \end{aligned}$$

The unconstrained welfare function W is strictly concave and attains its unique maximum in $(d_2, d_3) = (7.6 - 4\sqrt{2}, 6\sqrt{2} - 4.9)$. In case of no network restrictions, this unique maximum also corresponds to the unique market equilibrium. However, in our example, the network is restrictive. In Figure 3, the welfare objective function and its isolines are depicted. For the region of feasible demands, see Figure 2, it holds that the farther away the isoline of welfare lies from the origin, the better is the corresponding welfare. The two feasible points $\hat{d} = (1, 1)$ and $\tilde{d} = (1 + \sqrt{2}, 0)$ have the same objective value $8.5 + 10\sqrt{2}$. Thus, they are both located on the same isoline of welfare, which is given—in dependence of d_2—by

$$I(d_2) = \frac{1}{6}\left(1 + 20\sqrt{2} - 4d_2 - \sqrt{597 - 200\sqrt{2} + 304d_2 - 160\sqrt{2}d_2 - 20d_2^2}\right).$$

Thus, to prove that both points implement the welfare maximum, it remains to show that, within the interval $[1, 1 + \sqrt{2}]$, I lies above h_3, i.e., $I(d_2) \geq h_3(d_2)$. This can be seen by an easy computation.

We showed that both points $\hat{d}$ and $\tilde{d}$ correspond to optimal solutions of the welfare maximization problem (10). Consequently, the welfare-maximal solution is not unique. Moreover, in Appendix C we show that multiplicities even occur in more simple tree-like networks.

4.2. **Why the Fundamental Welfare Theorems do not extend to Gas Market Models with Network Constraints.** The fundamental theorems of welfare economics state that—under convexity assumptions—every competitive market equilibrium among consumers and producers corresponds to a welfare maximum and that, vice versa, every welfare maximum can be implemented as a competitive market equilibrium; see, e.g., [2] and Chapter 10 of [33]. In the framework of [2], gas at different nodes of the network needs to be considered as a different goods. Thus, $|V|$ goods are traded in a gas network. Furthermore, the TSO in that context is a specific producer that, according to its "production technology", can convert input at certain nodes of the network into output at other nodes of the network. In doing so, the TSO is restricted by the network constraints. One of the main assumptions of the welfare theorems states that the resulting set of production plans is convex, see Assumption I.a in [2]. In our situation, these production plans correspond to transportable supplies and demands as they are depicted, e.g., in Figure 2 for the three-node network of Example 3.7. Due to the nonconvexity of gas physics, the set of transportable supplies and demands faced by the TSO is nonconvex. Therefore, it is not clear if the results of the fundamental welfare theorems can still be established for the case of gas markets accounting for network constraints.

The common way of proving the equivalence of competitive market equilibria and welfare-maximal solutions is by showing that the following relation holds:[3]

$$\text{(SG)} \iff \text{(MNCP)} \iff \text{KKTs of (WMP)} \iff \text{(WMP)}. \tag{11}$$

However, the results of Section 3.2 already imply that the relation fails to hold in general because the first equivalence is typically not given. In this section, we additionally prove the following:

(1) The equivalence

$$\text{(SG)} \iff \text{(WMP)}$$

can be established under certain, more restrictive, assumptions.

(2) For a more general setting, we show that the solution sets of the simultaneous game (SG) and the welfare maximization problem may be disjoint. In other words, there exist situations in which not a single welfare-maximal solution can be implemented as an equilibrium of (SG).

4.2.1. *No Network Congestion: Simultaneous Game Equilibria are Welfare Maxima.* We start with the positive result for which we have to make the assumption that there is no network congestion, i.e., all pressure bounds are inactive:

$$p_u^- < p_u < p_u^+ \quad \text{for all } u \in V.$$

We prove that in this case, a market equilibrium exists, that it is unique, and that it corresponds to the welfare maximum. Moreover, the same market price results for all market participants. Let us emphasize at this point, that we are aware that this result is quite intuitive. In fact, the main focus of the paper is the case where network restrictions do constrain gas market interaction; see Section 4.2.2. At this point, however, we provide a comprehensive analysis of the unconstrained case in order to set the stage for the more complicated setting. We start with analyzing the TSO's problem (8) and the corresponding KKT conditions (9).

Lemma 4.1. *Let $(\pi_u)_{u\in V}$ be given nodal prices and let $x = (q,p)$ be a solution of Problem (8). Furthermore, suppose that there exist Lagrange multipliers $\lambda = (\gamma, \zeta^-, \zeta^+)$ such that (x,λ) is a KKT point of Problem (8). Then, the nodal prices of nodes adjacent to arcs with zero gas flow are equal. That is, $\pi_u = \pi_v$ holds for all $a = (u,v) \in A$ with $q_a = 0$.*

Proof. The vector (x,λ) is a KKT point of Problem (8) if and only if (x,λ) solves (9). The vector (x,λ) solves (9) if (x,λ), in particular, satisfies (9a), i.e., if $2\Lambda_a\gamma_a|q_a| + \pi_v - \pi_u = 0$ for all $a = (u,v) \in A$. This reduces to $\pi_u = \pi_v$ for all $a = (u,v) \in A$ with $q_a = 0$. □

The latter lemma states that zero gas flow on an arc implies the same price at the adjacent nodes.

[3] In the case of electricity markets, the equivalence of welfare-maximal solutions and competitive market equilibria has been shown for different model types, e.g., in Chapter 3 of [13], in [19], or in [6]. Common to all these modeling approaches are convex flow models. Such an equivalence also has been shown in the context of gas markets but on the basis of a linear network flow model; see [40].

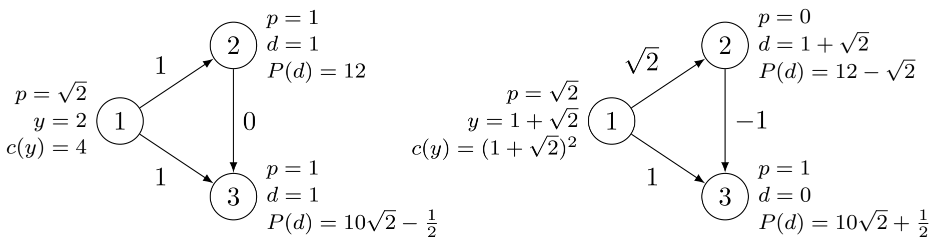

FIGURE 4. Two welfare-maximal solutions derived in Section 4.1; see Figure 1 for the data of this problem

Lemma 4.2. *Let $(\pi_u)_{u \in V}$ be given nodal prices and let $x = (q, p)$ be a solution of Problem (8). Furthermore, suppose that there exist Lagrange multipliers $\lambda = (\gamma, \zeta^-, \zeta^+)$ such that (x, λ) is a KKT point of Problem (8). Then, the signs of the Lagrange multipliers γ related to the Weymouth constraint (8b) are equal to the signs of the nodal price differences. That is,* $\operatorname{sgn}(\gamma_a) = \operatorname{sgn}(\pi_u - \pi_v)$ *holds for all $a = (u, v) \in A$ with $q_a \neq 0$.*

Proof. An argument similar to the one used in the proof of Lemma 4.1 shows that $\gamma_a = (\pi_u - \pi_v)/(2\Lambda_a |q_a|)$ for all $a = (u, v) \in A$ with $q_a \neq 0$. As the denominator is positive, the assertion follows. □

Lemma 4.2 helps us to deal with arcs with nonzero gas flows in the proof of the following theorem, which states that (i) all nodal prices are the same, that (ii) the gas market equilibrium is unique, and that (iii) it is welfare-maximal. The proof is rather technical and therefore given in Appendix D.

Theorem 4.3. *Let $(\pi_u)_{u \in V}$ be given nodal prices and let $x = (y, d, q, p)$ be a solution of the simultaneous game (SG). Suppose further that the nodal pressures are not binding, i.e., $p_u^- < p_u < p_u^+$ for all $u \in V$ and that (q, p) satisfies Assumption 2. Moreover, suppose that Assumption 1 holds. Then, all nodal prices are equal, x is the unique market equilibrium of the simultaneous game (SG), and this unique market equilibrium is welfare-maximal.*

4.2.2. *No Extension of the Fundamental Welfare Theorems to Gas Markets with Network Congestion.* In this section, we show that there are instances in which the sets of (SG) equilibria and welfare-maximal points do not coincide. Furthermore, we prove that there even exist instances where both sets are disjoint.

Theorem 4.4. *There exist instances such that the solution spaces of the simultaneous game (SG) and of the welfare maximization problem (WMP) do not coincide, i.e., the welfare theorems do not hold for gas trade on networks with network congestion.*

Our argument in the proof of Theorem 4.4 is again based on the three-node network of Example 3.7; see Figure 1. We show that the welfare maxima $\hat{x}$ and $\tilde{x}$, see Figure 4, cannot be obtained as simultaneous game equilibria. Since $\hat{x}$ and $\tilde{x}$ are the only welfare-maximal solutions, it follows that there is no equilibrium of (SG) that implements a welfare-maximal solution. Thus, both fundamental welfare theorems cannot be extended to the case of gas markets with binding network constraints in general.

Proof of Theorem 4.4. The nodal prices that implement the welfare maxima in the instance of Example 3.7 are determined by the demand functions in case of demand

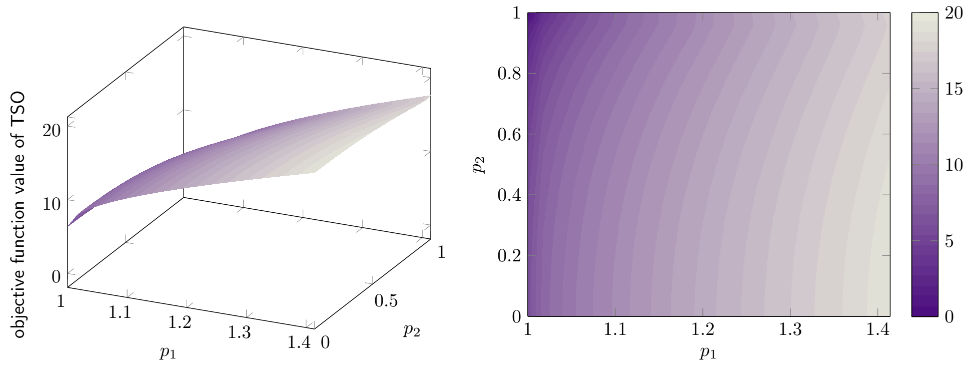

FIGURE 5. The TSO's objective function and its isolines for prices $\hat{\pi}$

nodes and by the marginal costs in case of supply nodes. In case of zero demand, all prices equal to and greater than the prohibitive price are possible. Thus, we have

$$\hat{\pi}_1 = 4, \qquad \hat{\pi}_2 = 12, \qquad \hat{\pi}_3 = 10\sqrt{2} - \frac{1}{2},$$
$$\tilde{\pi}_1 = 2 + 2\sqrt{2}, \qquad \tilde{\pi}_2 = 12 - \sqrt{2}, \qquad \tilde{\pi}_3 \geq 10\sqrt{2} + \frac{1}{2}.$$

We start with the welfare maximum $\hat{x}$, which yields a market solution of (SG) if and only if $(\hat{q}, \hat{p})$ is an optimal solution of Problem (8). Using the data of our example, Problem (8) reads

$$\begin{aligned} \max_{\hat{q},\hat{p}} \quad & 8\hat{q}_{12} + \left(10\sqrt{2} - \frac{9}{2}\right)\hat{q}_{13} + \left(10\sqrt{2} - \frac{25}{2}\right)\hat{q}_{23} \\ \text{s.t.} \quad & \hat{p}_2^2 = \hat{p}_1^2 - \hat{q}_{12}|\hat{q}_{12}|, \quad \hat{p}_3^2 = \hat{p}_1^2 - \hat{q}_{13}|\hat{q}_{13}|, \quad \hat{p}_3^2 = \hat{p}_2^2 - \hat{q}_{23}|\hat{q}_{23}|, \\ & 1 \leq \hat{p}_1 \leq \sqrt{2}, \quad 0 \leq \hat{p}_2 \leq 1, \quad \hat{p}_3 = 1. \end{aligned}$$

As the signs of all flows are predetermined due to the chosen pressure bounds, this problem is equivalent to the optimization problem

$$\begin{aligned} \max_{\hat{p}_1,\hat{p}_2} \quad & 8\sqrt{\hat{p}_1^2 - \hat{p}_2^2} + \left(10\sqrt{2} - \frac{9}{2}\right)\sqrt{\hat{p}_1^2 - 1} - \left(10\sqrt{2} - \frac{25}{2}\right)\sqrt{1 - \hat{p}_2^2} \\ \text{s.t.} \quad & 1 \leq \hat{p}_1 \leq \sqrt{2}, \quad 0 \leq \hat{p}_2 \leq 1. \end{aligned}$$

Figure 5 displays the objective function and its isolines. The optimal solution is $(\hat{p}_1, \hat{p}_2) = (\sqrt{2}, 0)$ and thus differs from $(\sqrt{2}, 1)$; see Figure 4 (left). Consequently, the welfare maximum $\hat{x}$ is not a solution of the simultaneous game (SG).

The same argumentation yields that the TSO's problem (8) for the welfare optimum $\tilde{x}$ is equivalent to the optimization problem

$$\begin{aligned} \max_{\tilde{p}} & \left(10 - 3\sqrt{2}\right)\sqrt{\tilde{p}_1^2 - \tilde{p}_2^2} + \left(\tilde{\pi}_3 - 2 - 2\sqrt{2}\right)\sqrt{\tilde{p}_1^2 - 1} - \left(\tilde{\pi}_3 - 12 + \sqrt{2}\right)\sqrt{1 - \tilde{p}_2^2} \\ \text{s.t.} \ & 1 \leq \tilde{p}_1 \leq \sqrt{2}, \quad 0 \leq \tilde{p}_2 \leq 1, \end{aligned}$$

with $\tilde{\pi}_3 \geq 10\sqrt{2} + 1/2$. From $\tilde{\pi}_3 - 2 - 2\sqrt{2} \geq 0$ and $10 - 3\sqrt{2} \geq 0$ we get $\tilde{p}_1 = \sqrt{2}$. Thus, the TSO's Problem is equivalent to the optimization problem

$$\begin{aligned} \max_{\tilde{p}_2} \quad & \left(10 - 3\sqrt{2}\right)\sqrt{2 - \tilde{p}_2^2} + \tilde{\pi}_3 - 2 - 2\sqrt{2} - \left(\tilde{\pi}_3 - 12 + \sqrt{2}\right)\sqrt{1 - \tilde{p}_2^2} \\ \text{s.t.} \quad & 0 \leq \tilde{p}_2 \leq 1. \end{aligned}$$

Since

$$\begin{aligned} 10 - 3\sqrt{2} &> \left(10 - 3\sqrt{2}\right)\sqrt{2} - \left(10\sqrt{2} + \frac{1}{2} - 12 + \sqrt{2}\right) \\ &\geq \left(10 - 3\sqrt{2}\right)\sqrt{2} - \left(\tilde{\pi}_3 - 12 + \sqrt{2}\right) \end{aligned}$$

holds, the TSO's objective function value with $\tilde{p}_2 = 1$ is larger than the TSO's objective function value with $\tilde{p}_2 = 0$ for all $\tilde{\pi}_3 \geq 10\sqrt{2} + 1/2$. Thus, the TSO prefers $\tilde{p}_2 = 1$. Consequently, the welfare maximum $\tilde{x}$ is again not a solution of the simultaneous game (SG). □

In conclusion, this shows that there exist market situations for which no welfare maximum corresponds to a market equilibrium and vice versa. Thus, the fundamental theorems of welfare economics cannot be extended to gas markets accounting for network constraints, which especially means that market equilibria cannot be identified by solving the corresponding welfare maximization problem.

To strengthen our point, we show in Appendix C that the same results might also be obtained on simple tree networks. Thus, we have shown in this section that even in the most promising case (convex marginal cost of suppliers, strictly decreasing demand, and perfect competition) we cannot establish the extension of the fundamental theorems of welfare economics to gas markets even for very simple networks. We clearly identified the nonconvex "production technology" of the TSO as the origin of this failure. We conjecture that an appropriate regulatory regime might resolve the problem. In the following section we show that indeed, committing the TSO to a welfare objective and, at the same time, giving him discretion to determine the nodal prices can effectively establish an efficient gas market framework. Of course, the analysis considers an idealistic world. Major obvious qualifications of the analysis and our results are discussed in the conclusion.

5. Nonconvex Gas Market Equilibrium Modeling: A Multilevel Approach

We have shown so far that in the case of a profit-maximizing TSO, competitive market equilibria of the wholesale short-run natural gas market do not correspond to welfare-maximal solutions in general and thus may lead to market inefficiency. We conjecture that the problem can be addressed by appropriate regulation of the TSO and thus, analyze a respective market structure in the following. In particular, in the proposed setup, the TSO is committed to a welfare objective and moves first by setting the nodal prices. For the resulting trilevel game, we prove that each market solution corresponds to a welfare-maximal solution and vice versa.

The proposed hierarchical game is structured as follows. The TSO is regulated to operate such that overall social welfare is maximized. Therefore, the TSO decides at the first level about welfare-maximizing market prices. Afterward, the market prices are forwarded to the suppliers and consumers, who decide simultaneously about the quantities of gas that they supply or discharge. Thus, supply and demand is determined at the second level. At the third level, the TSO controls the physical gas flows based on the given supply and demand quantities. An overview of the resulting multilevel problem is depicted in Figure 6.

Thus, the TSO solves the following optimization problem at the first level:

$$\max_{\pi} \quad \sum_{u \in V_-} \int_0^{d_u} P_u(x) \,\mathrm{d}x - \sum_{u \in V_+} c_u(y_u). \tag{12}$$

Here and in what follows, $\pi = (\pi_u)_{u \in V}$ denotes the vector of nodal prices.

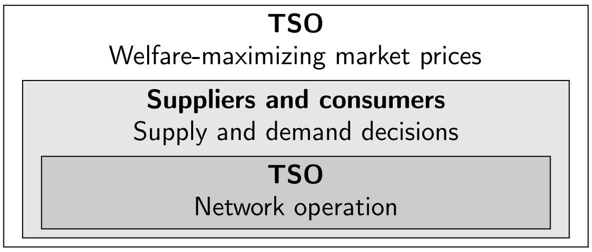

FIGURE 6. Structure of the multilevel gas market model

At the second level, all suppliers and consumers face the welfare-maximizing market prices that have been chosen by the TSO. Note that in this framework, the assumption of non-strategic firms is not necessary since, at prices fixed by the TSO, it is in the firms' best interest to take supply and demand decisions optimally. The problem of incomplete information on the part of the regulator (or here, the regulated TSO) has been extensively addressed in the literature. For seminal contributions see [4] or [30]. For a recent survey see [45]. In the context of energy markets, however, information on cost parameters is often supposed to be available, as recent advances of regulators regarding price control based on verified cost demonstrate; see, e.g., [9].

In this setup, the optimization problem

$$\max_{y_u \geq 0} \quad \pi_u y_u - c_u(y_u), \tag{13}$$

arises for each supplier located at node $u \in V_+$. For each consumer located at node $u \in V_-$, we have

$$\max_{d_u \geq 0} \quad \int_0^{d_u} P_u(x)\,\mathrm{d}x - \pi_u d_u. \tag{14}$$

Note that Problem (13) and (14) correspond to Problem (3) and (5), respectively.

Finally, the TSO determines a feasible operation of the transmission network at the third level. As long as active elements, e.g., (control) valves or compressors, are not considered, no transportation costs arise for the TSO. This leads to the following feasibility problem at the third level:

$$\begin{aligned} p_v^2 &= p_u^2 - \Lambda_a q_a |q_a|, \quad a = (u,v) \in A, \\ p_u^- &\leq p_u \leq p_u^+, \quad u \in V, \\ \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a &= q_u, \quad u \in V, \end{aligned} \tag{15}$$

with

$$q_u = \begin{cases} d_u, & u \in V_-, \\ -y_u, & u \in V_+. \end{cases} \tag{16}$$

Since all supply and demand quantities have already been determined at the second level and are therefore fixed at the third level, uniqueness of the third-level solution w.r.t. the flows follows from the results in [42]. The pressures are also unique except for constant shifts. However, this multiplicity does not influence the market outcome. Uniqueness of the second level solution follows from Lemma 3.1 and 3.2.

In what follows, we prove that market solutions of the hierarchical game correspond to welfare-maximal solutions. Thus, unlike in the case of a profit-maximizing TSO, we can establish an analogue of the fundamental welfare theorems. We start by showing the 1-1 correspondence of the multilevel optimization problem and an optimization problem with equilibrium constraints (MPEC). Afterward, we prove the 1-1 correspondence of the MPEC and the welfare maximization problem.

The MPEC's upper level consists of the optimization problem

$$\max_{q,p,\pi} \quad \sum_{u \in V_-} \int_0^{q_u} P_u(x) \, \mathrm{d}x - \sum_{u \in V_+} c_u(-q_u) \quad \text{s.t.} \quad (15), \ (16). \tag{17}$$

The second level of the hierarchical game, i.e., Problem (13) and (14), corresponds to the MPEC's equilibrium problem. Thus, the MPEC originates from the hierarchical game by aggregating level one and three. We now show that this aggregation does not influence the space of optimal solutions.

Lemma 5.1. *Let $(\pi; d, y; q, p)$ be a market solution of the hierarchical game. Then, $(q, p, \pi; d, y)$ is a solution of the MPEC.*

Proof. As one can easily see, $(q, p, \pi; d, y)$ is feasible for the MPEC. We now prove optimality by contradiction. Assume that $(q, p, \pi; d, y)$ is not optimal for the MPEC. Then, there exists $(\tilde{q}, \tilde{p}, \tilde{\pi}; \tilde{d}, \tilde{y})$ such that for the profit of the leader, i.e., the TSO,

$$\sum_{u \in V_-} \int_0^{\tilde{q}_u} P_u(x) \, \mathrm{d}x - \sum_{u \in V_+} c_u(-\tilde{q}_u) > \sum_{u \in V_-} \int_0^{q_u} P_u(x) \, \mathrm{d}x - \sum_{u \in V_+} c_u(-q_u)$$

holds. If the TSO would set $\tilde{\pi}$ as market prices at the first level of the trilevel problem, the followers, i.e., the suppliers and consumers, would react with supply $\tilde{y}$ and demand $\tilde{d}$. This is the case as the second level of the MPEC and the second level of the hierarchical game coincide. In turn, the TSO would react with flows $\tilde{q}$ and pressures $\tilde{p}$ as the reaction of the TSO is unique with regard to fixed demands and supplies; cf. [42]. Therefore, $(\tilde{\pi}; \tilde{d}, \tilde{y}; \tilde{q}, \tilde{p})$ is feasible for the hierarchical game but yields a higher outcome for the TSO than $(\pi; d, y; q, p)$. Since the TSO is also the leader of the hierarchical game, this contradicts the assumption that $(\pi; d, y; q, p)$ is a market solution of the hierarchical game. Thus, $(q, p, \pi; d, y)$ must be optimal for the MPEC. □

Lemma 5.2. *Let $(q, p, \pi; d, y)$ be a solution of the MPEC. Then, $(\pi; d, y; q, p)$ is a market solution of the hierarchical game.*

Proof. As one can easily see, $(\pi; d, y; q, p)$ is feasible for the hierarchical game. Again, we prove optimality by contradiction. Assume that $(\pi; d, y; q, p)$ is not optimal for the hierarchical game. Then, there exists $(\tilde{\pi}; \tilde{d}, \tilde{y}; \tilde{q}, \tilde{p})$ such that for the profit of the leader, i.e., the TSO,

$$\sum_{u \in V_-} \int_0^{\tilde{d}_u} P_u(x) \, \mathrm{d}x - \sum_{u \in V_+} c_u(\tilde{y}_u) > \sum_{u \in V_-} \int_0^{d_u} P_u(x) \, \mathrm{d}x - \sum_{u \in V_+} c_u(y_u)$$

holds. The vector $(\tilde{q}, \tilde{p}, \tilde{\pi}; \tilde{d}, \tilde{y})$ is feasible for the MPEC but yields a higher outcome for the TSO than $(q, p, \pi; d, y)$. Since the TSO is also the leader of the MPEC, this contradicts the assumption that $(q, p, \pi; d, y)$ is a solution of the MPEC. Thus, $(\pi; d, y; q, p)$ must be optimal for the hierarchical game. □

Thus, we have established a 1-1 correspondence of the hierarchical game and the MPEC. Finally, we also prove the 1-1 correspondence of the MPEC and the welfare maximization problem.

Lemma 5.3. *There is a 1-1 correspondence of the MPEC and the welfare maximization problem* (WMP)*, i.e., the solution spaces of both problems coincide.*

Proof. The unique primal solution of the second level of the MPEC is given by Lemma 3.1 and Lemma 3.2. Thus, the MPEC is equivalent to the following single-level optimization problem:

$$\max_{q,p,\pi} \quad \sum_{u \in V_-} \int_0^{q_u} P_u(x)\,\mathrm{d}x - \sum_{u \in V_+} c_u(q_u) \tag{18a}$$

$$\text{s.t.} \quad \text{Physical feasibility: (15)}, \tag{18b}$$

$$q_u = \begin{cases} 0, & c_u'(0) > \pi_u, \\ -(c_u')^{-1}(\pi_u), & c_u'(0) \le \pi_u, \end{cases} \quad u \in V_+, \tag{18c}$$

$$q_u = \begin{cases} 0, & P_u(0) < \pi_u, \\ P_u^{-1}(\pi_u), & P_u(0) \ge \pi_u, \end{cases} \quad u \in V_-. \tag{18d}$$

Since (18c) and (18d) describe the optimal reactions of suppliers and consumers to market prices, i.e., $y_u = -q_u$ for all $u \in V_+$ and $d_u = q_u$ for all $u \in V_-$ are solutions of the MPEC's lower level equilibrium problem. It follows that $0 \le y_u$ for all $u \in V_+$ and $0 \le d_u$ for all $u \in V_-$. Thus, if (q, p, π) is a feasible solution of Problem (18), then, (d, y, q, p) is a feasible solution of the welfare maximization problem (WMP).

In turn, let (d, y, q, p) be a solution of the welfare maximization problem (WMP). Let π be defined by $\pi_u = c_u'(y_u)$ for all $u \in V_+$ and $\pi_u = P_u(d_u)$ for all $u \in V_-$. Then, in the case $y_u = 0$, $\pi_u = c_u'(0)$ holds and we obtain $q_u = -\left(c_u'\right)^{-1}\left(c_u'(0)\right) = 0$; cf. Equation (18c). In the other case $y_u > 0$, $\pi_u = c_u'(y_u)$ holds and we get $q_u = -\left(c_u'\right)^{-1}(\pi_u) = -\left(c_u'\right)^{-1}\left(c_u'(y_u)\right) = -y_u$, because $c_u'(0) < \pi_u = c_u'(y_u)$ follows from the fact that the marginal costs $c_u'(\cdot)$ are strictly increasing, continuous, and thus invertible. Hence, the prices π_u yield $-y_u$ in Equation (18c) for all $u \in V_+$. By an analogous argumentation, it can be shown that the prices π_u also yield d_u in Equation (18d) for all $u \in V_-$. Thus, (q, p, π) is a feasible solution of Problem (18).

As the set of feasible solutions and the objective functions of Problem (18) and the welfare maximization problem (WMP) coincide, the assertion follows. □

In summary, we have shown the 1-1 correspondence of the hierarchical game and the welfare maximization problem.

Theorem 5.4. *There is a 1-1 correspondence of the solutions of the hierarchical game and the welfare maximization problem* (WMP)*, i.e., the solution spaces of both problems coincide.*

Consequently, all welfare-maximal solutions correspond to market solutions of the proposed hierarchical game and vice versa. The fundamental welfare theorems are therefore re-established for the wholesale short-run natural gas market under perfect competition in case of a welfare-maximizing TSO.

6. Conclusion

This paper lays the foundation for an analysis of the interplay of gas markets and gas transport in partial equilibrium models. To set the stage for an in-depth understanding of the particular restrictions of standard equilibrium analysis in this context, we first analyze a standard approach from electricity market modeling; see, e.g., [13, 21, 23]. We demonstrate that standard results from welfare economics [2], which typically make the analysis tractable in electricity market modeling, do not extend to our framework due to nonconvexities of the network constraints that obtain even for the simplest possible gas transport model. We investigate in detail the correspondence between equilibria of a gas market game and the corresponding MNCP, as well as the relation between market equilibria and welfare maxima of the analyzed gas market environment. As we show, the respective solutions fall apart even for tree-structured networks if flow restrictions are binding such that standard techniques to determine market outcomes necessarily fail. We propose an alternative gas market model that involves a regulated TSO focused on welfare maximization. For the corresponding three-level model we show that market outcomes coincide with welfare-maximal solutions.

The contribution of our paper is twofold. First, we provide detailed insights on when and why standard approaches to model market interaction on networks fail for the case of gas markets. Second, we pin down the origin of the missing correspondence of market equilibria and welfare maxima and propose a model, where regulation of the affected players re-establishes a well-functioning market environment. In fact, various energy market designs around the world seem to follow this idea in principle, although in quite different manners. In the US, electricity market designs sometimes involve a regulated TSO who has some discretion to set market prices; see, e.g., [10]. This seems to accomodate the insight that nodal pricing in electricity market bears similar problems as those we point to here if real network constraints are taken into consideration. In Europe, gas market regulation decouples network management from gas trade through the entry-exit system; see, e.g., [14]. This procedure necessarily leaves network capacity unused and therefore decreases overall welfare as compared to the optimal design described in this paper. Assessing the size of the welfare gap is left for future research. A first approach to model the entry-exit system is provided in [18], where also discrete controls for the network's operation are addressed.

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Appendix A. Proof of Lemma 3.4

We now prove Lemma 3.4. Note that Assumption 1 is needed in the proof to guarantee that in the case of no gas flows on a path $P = (V(P), A(P))$ in G, at most one of the nodes in $V(P)$ has an active pressure bound.

Lemma A.1. *Let $x = (q, p)$ be a solution of Problem (8). In addition, let $P = (V(P), A(P))$ be a path in G with $q_a = 0$ for all $a \in A(P)$. Furthermore, suppose that Assumption 1 holds. Then, there exists at most one node $u \in V(P)$ with active an pressure bound.*

Proof. Since $q_a = 0$ for all $a = (u, v) \in A(P)$, $p_u = p_v$ follows from constraint (8b). Consequently, all nodal pressures are equal along the path P. As all pressure bounds are pairwise distinct by Assumption 1, the assertion follows. □

Within the following proof, we denote for a given matrix $M \in \mathbb{R}^{m \times n}$ by $M_{R,C}$ the sub-matrix with all rows corresponding to the elements of the index set R and all columns corresponding to the elements of the index set C. Analogously, we denote for a given vector $m \in \mathbb{R}^m$ by m_R the subvector with all entries corresponding to the elements of the index set R.

Proof of Lemma 3.4. We show that under the given assumptions, the LICQ holds in x. To this end, we use the notation

$$V^{\text{lo}} = \{u \in V : p_u = p_u^-\}, \qquad A^{=0} = \{a \in A : q_a = 0\},$$
$$V^{\text{up}} = \{u \in V : p_u = p_u^+\}, \qquad A^{\neq 0} = \{a \in A : q_a \neq 0\},$$
$$V^{\text{in}} = \{u \in V : p_u^- < p_u < p_u^+\}.$$

We further introduce the function

$$f_a : \mathbb{R} \to \mathbb{R}, \quad q_a \mapsto \Lambda_a q_a |q_a|.$$

for all $a \in A$ and denote the vector of squared pressures $(p_u^2)_{u\in V}$ by $\bar{p}$, the vector of lower pressure bounds $(p_u^-)_{u\in V}$ by p^-, and the vector of upper pressure bounds $(p_u^+)_{u\in V}$ by p^+. In addition, let $f(q)$ denote $(f_a(q_a))_{a\in A}$ and let f' denote $(f_a'(q_a))_{a\in A}$ with

$$f_a' : \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad q_a \mapsto 2\Lambda_a |q_a|. \tag{19}$$

Let $B \in \{0, \pm 1\}^{|V|\times|A|}$ denote the node-arc-incidence-matrix of the graph G and let I_n be the identity matrix of dimension $n \times n$. The Jacobian of the constraint set of Problem (8) is then given by

$$\begin{bmatrix} \nabla_q \left(B^T \bar{p} - f(q)\right) & \nabla_p \left(B^T \bar{p} - f(q)\right) \\ \nabla_q (p^- - p) & \nabla_p (p^- - p) \\ \nabla_q (p - p^+) & \nabla_p (p - p^+) \end{bmatrix}.$$

Without loss of generality, we assume that the flow vector is ordered as $q^\top = (q_{A^{=0}}^\top, q_{A^{\neq 0}}^\top)$ and that the pressure vector is ordered as $p^\top = (p_{V^{\text{lo}}}^\top, p_{V^{\text{up}}}^\top, p_{V^{\text{in}}}^\top)$. In the following, we consider the corresponding rectangular and homogeneous system

$$J\lambda = \begin{bmatrix} -\operatorname{diag}(f'_{A^{=0}}) & 0 & 0 & 0 \\ 0 & -\operatorname{diag}(f'_{A^{\neq 0}}) & 0 & 0 \\ \hat{B}_{V^{\text{lo}},A^{=0}} & \hat{B}_{V^{\text{lo}},A^{\neq 0}} & -I_{|V^{\text{lo}}|} & 0 \\ \hat{B}_{V^{\text{up}},A^{=0}} & \hat{B}_{V^{\text{up}},A^{\neq 0}} & 0 & I_{|V^{\text{up}}|} \\ \hat{B}_{V^{\text{in}},A^{=0}} & \hat{B}_{V^{\text{in}},A^{\neq 0}} & 0 & 0 \end{bmatrix} \begin{pmatrix} \lambda_{A^{=0}} \\ \lambda_{A^{\neq 0}} \\ \lambda_{V^{\text{lo}}} \\ \lambda_{V^{\text{up}}} \end{pmatrix} = 0.$$

Here, $\hat{B} \in \mathbb{R}^{|V|\times|A|}$ is defined by

$$\hat{B}_{u,a} = \begin{cases} 2p_u, & u = v, \\ -2p_u, & u = w, \\ 0, & \text{otherwise}, \end{cases}$$

for all $u \in V$ and $a = (v, w) \in A$.

Since $f_a'(q_a) = 0$ if and only if $q_a = 0$ for all $a \in A$, we have $f'_{A^{\neq 0}} \neq 0$ and therefore $\lambda_{A^{\neq 0}} = 0$. If $A^{=0} \neq \emptyset$ holds, the first block row is zero and can be ignored. Thus, the system reduces to

$$\begin{bmatrix} \hat{B}_{V^{\text{lo}},A^{=0}} & -I_{|V^{\text{lo}}|} & 0 \\ \hat{B}_{V^{\text{up}},A^{=0}} & 0 & I_{|V^{\text{up}}|} \\ \hat{B}_{V^{\text{in}},A^{=0}} & 0 & 0 \end{bmatrix} \begin{pmatrix} \lambda_{A^{=0}} \\ \lambda_{V^{\text{lo}}} \\ \lambda_{V^{\text{up}}} \end{pmatrix} = 0$$

Thus, LICQ holds if this reduced square linear system has full rank, i.e., if the only solution is the zero solution.

Since all pressures are positive by Assumption 1, this system is a network flow problem on the subgraph $G_{A^{=0}} = \left(V(A^{=0}), A^{=0}\right)$. We define $\hat{\lambda}_u := \lambda_u/(2p_u)$ for all $u \in V^{\text{lo}} \cup V^{\text{up}}$ and obtain the flow problem

$$\begin{aligned} B_{V^{\text{lo}},A^{=0}} \lambda_{A^{=0}} &= \hat{\lambda}_{V^{\text{lo}}}, \\ B_{V^{\text{up}},A^{=0}} \lambda_{A^{=0}} &= -\hat{\lambda}_{V^{\text{up}}}, \\ B_{V^{\text{in}},A^{=0}} \lambda_{A^{=0}} &= 0. \end{aligned} \tag{20}$$

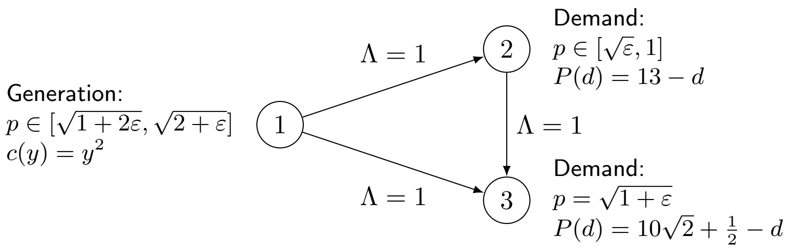

FIGURE 7. Perturbed three-node network

As there is no cycle in $G_{A=0}$ by Assumption 2, each connected component of $G_{A=0}$ corresponds to a path $P = (V(P), A(P))$ in G with $q_a = 0$ for all $a \in A(P)$. From Lemma A.1, it follows for each connected component of $G_{A=0}$ that at most one pressure bound is active and therefore at most one entry of the right-hand side is variable. Consequently, Problem (20) is only solved by the zero vector. Therefore, LICQ holds. □

APPENDIX B. PROOF OF THEOREM 3.6

Proof of Theorem 3.6. We consider an instance $G = (V, A)$ with three nodes that are connected by three pipes. The instance is depicted, together with all physical and economical parameters, in Figure 7. It is similar to the one of Example 3.7 but with slightly perturbed physical parameters such that Assumption 1 is fulfilled. For the nodal prices

$$\pi_1 = 2 + 2\sqrt{2}, \quad \pi_2 = 12 - \sqrt{2}, \quad \pi_3 = 10\sqrt{2} + \frac{1}{2},$$

we obtain the solution

$$\begin{aligned} y_1 &= 1 + \sqrt{2}, & d_2 &= 1 + \sqrt{2}, & d_3 &= 0, \\ q_{12} &= \sqrt{2}, & q_{13} &= 1, & q_{23} &= -1, \\ p_1 &= \sqrt{2+\varepsilon}, & p_2 &= \sqrt{\varepsilon}, & p_3 &= \sqrt{1+\varepsilon}, \end{aligned}$$

of the (MNCP). The Lagrange multipliers are given by

$$\begin{aligned} \gamma_{12} &= \frac{3\sqrt{2} - 10}{2\sqrt{2}}, & \zeta_1^+ &= \sqrt{2+\varepsilon}\left(13\sqrt{2} - \frac{9}{2}\right), \\ \gamma_{13} &= -4\sqrt{2} + \frac{3}{4}, & \zeta_2^- &= \sqrt{\varepsilon}\left(\frac{17}{2} - 6\sqrt{2}\right), \\ \gamma_{23} &= -\frac{11}{2}\sqrt{2} + \frac{23}{4}, & \zeta_3^- &= \sqrt{1+\varepsilon}\left(19\sqrt{2} - 13\right), \end{aligned}$$

and $\eta_1^- = \beta_2^- = \beta_3^- = \zeta_1^- = \zeta_2^+ = \zeta_3^+ = 0$. For the TSO's problem (8), the LICQ holds in (q, p); cf. Lemma 3.4. Thus, being a solution of (MNCP) is a necessary condition for being a market equilibrium. The corresponding problem of the TSO reads

$$\begin{aligned} \max_{q,p} \quad & \left(10 - 3\sqrt{2}\right) q_{12} + \left(8\sqrt{2} - \frac{3}{2}\right) q_{13} + \left(11\sqrt{2} - \frac{23}{2}\right) q_{23} \\ \text{s.t.} \quad & p_2^2 = p_1^2 - q_{12}|q_{12}|, \quad p_3^2 = p_1^2 - q_{13}|q_{13}|, \quad p_3^2 = p_2^2 - q_{23}|q_{23}|, \\ & \sqrt{1+2\varepsilon} \le p_1 \le \sqrt{2+\varepsilon}, \quad \sqrt{\varepsilon} \le p_2 \le 1, \quad p_3 = \sqrt{1+\varepsilon}. \end{aligned}$$

As, for small epsilon $\varepsilon > 0$, the pressures $p_1 = \sqrt{2+\varepsilon}$ and $p_2 = 1$ with flows $q_{12} = \sqrt{1+\varepsilon}$, $q_{13} = 1$, and $q_{23} = -\sqrt{\varepsilon}$ yield a higher payoff for the TSO than

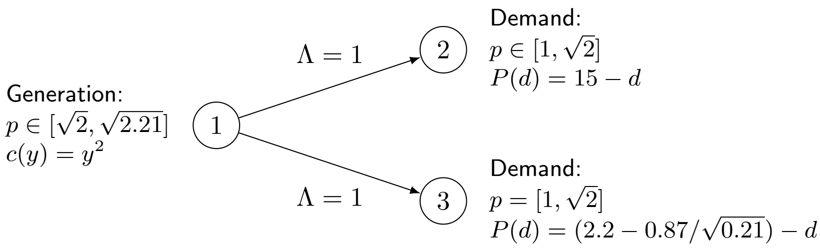

FIGURE 8. Three-node tree network

$p_1 = \sqrt{2+\varepsilon}$ and $p_2 = \sqrt{\varepsilon}$, the TSO has an incentive to deviate from the stated solution of the (MNCP). Thus, the (MNCP) solution does not correspond to a market equilibrium due to the insufficiency of the TSO's first-order conditions. □

APPENDIX C. THE WELFARE MAXIMIZATION PROBLEM ON TREES

In this section we show that the welfare maximization problem possesses multiple solutions even on tree-like network structures. Afterward we show that the welfare theorems also cannot be extended to gas networks that are trees.

C.1. **Multiplicity of Welfare-Maximal Solutions.** In what follows, we show that even on simple tree-like networks, there exist multiple solutions of the welfare maximization problem (WMP). To this end, we consider a graph that consists of three nodes connected by two pipes. An overview of all physical and economical parameters is given in Figure 8. The corresponding welfare maximization problem reads

$$\max_{y,d,q,p} \quad \int_0^{d_2} (15 - x)\,\mathrm{d}x + \int_0^{d_3} (2.2 - 0.87/\sqrt{0.21} - x)\,\mathrm{d}x - y_1^2 \tag{21a}$$

$$\text{s.t.} \quad y_1 \geq 0, \quad d_2 \geq 0, \quad d_3 \geq 0, \tag{21b}$$

$$q_{12} + q_{13} = y_1, \quad q_{12} = d_2, \quad q_{13} = d_3, \tag{21c}$$

$$p_2^2 = p_1^2 - q_{12}|q_{12}|, \quad p_3^2 = p_1^2 - q_{13}|q_{13}|, \tag{21d}$$

$$\sqrt{2} \leq p_1 \leq \sqrt{2.21}, \quad 1 \leq p_2 \leq \sqrt{2}, \quad 1 \leq p_3 \leq \sqrt{2}. \tag{21e}$$

An argument similar to the one in Section 4.1 yields $q_{12} \geq 0$ and $q_{13} \geq 0$. The projection of the feasible set onto the space of withdrawals (d_2, d_3) is described by the inequality system

$$0 \leq d_2^2 \leq 1.21, \quad 0 \leq d_3^2 \leq 1.21, \quad -1 \leq d_3^2 - d_2^2 \leq 1,$$

cf. [15]. The set of demands (d_2, d_3) that satisfy this system is depicted in Figure 9. The important part of the boundary of the feasible set is defined by the function

$$h : [1, 1.1] \to [0, \sqrt{0.21}], \quad d_2 \mapsto \sqrt{d_2^2 - 1}.$$

Again, we show that the points $\hat{d} = (1, 0)$ and $\tilde{d} = (1.1, \sqrt{0.21})$ both implement welfare-maximal solutions. Both lie on the graph h and on the same isoline of welfare. The latter—in dependence of d_2—is given by

$$I(d_2) = \frac{1}{3}\left(2.2 - \frac{0.87}{\sqrt{0.21}} - 2d_2\right) + \frac{1}{3}\sqrt{\left(2.2 - \frac{0.87}{\sqrt{0.21}} - 2d_2\right)^2 + 6 \cdot \left(15d_2 - \frac{3}{2}d_2^2 - 13.5\right)}.$$

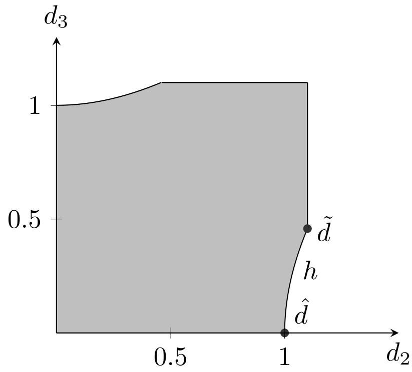

FIGURE 9. Projection of the feasible set of Problem (21) onto the 2d-space of demand variables

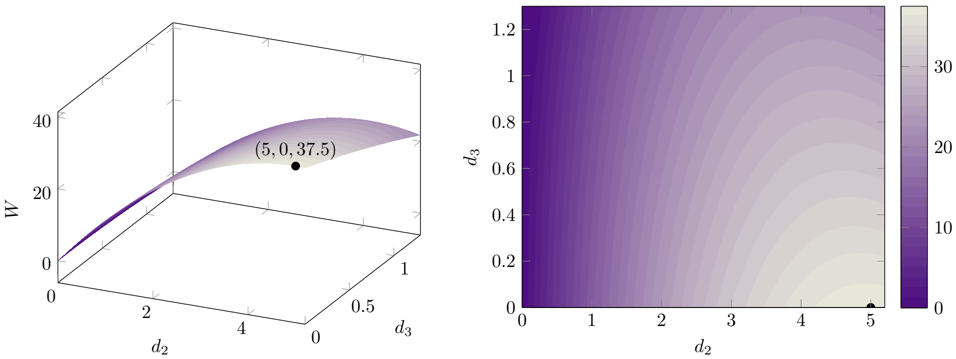

FIGURE 10. The welfare function (left) and its isolines (right)

In Figure 10, the unconstrained welfare function and its isolines are depicted as functions of d_2 and d_3. The unique maximum is attained in $(d_2, d_3) = (5, 0)$. For the region of feasible demands, see Figure 9, it holds again that the farther away the isoline of welfare is from the origin, the better is the corresponding welfare. It again remains to show that, within the interval $[1, 1.1]$, $h(d_2) - I(d_2) \geq 0$ holds. This can be shown by an easy computation. We thus proved that both points $\hat{d}$ and $\tilde{d}$ correspond to optimal solutions of the welfare maximization problem (21). Hence, even on simple tree networks, the welfare-maximal solution is not unique.

C.2. **Welfare Theorems do not Extend to Gas Markets on Trees.** In what follows, we show that—also for our three-node tree network—the equilibria of (SG) and the welfare-maximal points are disjoint; cf. Section 4.2.2. Again, we show that the welfare maxima $\hat{x}$ and $\tilde{x}$, see Figure 11, cannot be obtained as simultaneous game equilibria. Since $\hat{x}$ and $\tilde{x}$ are the only welfare-maximal solutions, it follows that there is no equilibrium of (SG) that implements a welfare-maximal solution. Thus, both fundamental welfare theorems cannot be extended to the case of gas markets accounting for physical network constraints even on simple tree networks.

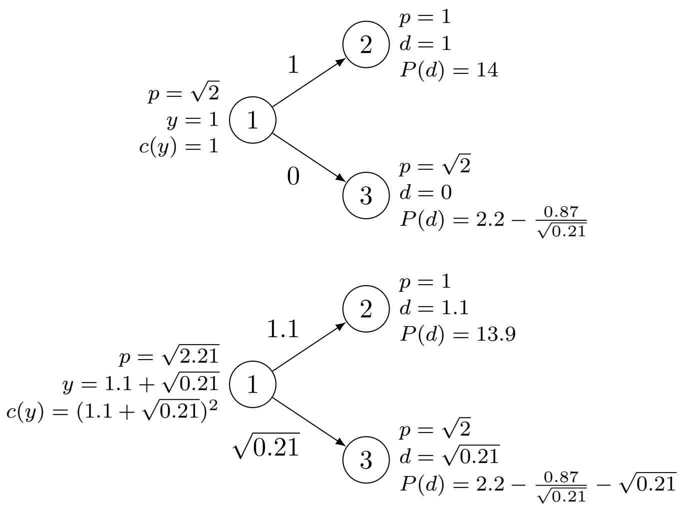

FIGURE 11. Two welfare-maximal solutions derived in Appendix C; see Figure 8 for the data of this problem

For our example, the following nodal prices implement the welfare maxima $\hat{x}$ and $\tilde{x}$:

$$\hat{\pi}_1 = 2, \qquad \hat{\pi}_2 = 14, \qquad \hat{\pi}_3 \geq \left(2.2 - \frac{0.87}{\sqrt{0.21}}\right),$$
$$\tilde{\pi}_1 = 2 \cdot (1.1 + \sqrt{0.21}), \qquad \tilde{\pi}_2 = 13.9, \qquad \tilde{\pi}_3 = \left(2.2 - \frac{0.87}{\sqrt{0.21}}\right) - \sqrt{0.21}.$$

We start with the welfare maximum $\hat{x}$, which yields a market solution of (SG) if and only if $(\hat{q}, \hat{p})$ is an optimal solution of Problem (8). As, again, the signs of all flows are predetermined due to the chosen pressure bounds, Problem (8) is equivalent to the optimization problem

$$\begin{aligned} \max_{\hat{p}} \quad & 12\sqrt{\hat{p}_1^2 - \hat{p}_2^2} + (\hat{\pi}_3 - 2)\sqrt{\hat{p}_1^2 - \hat{p}_3^2} \\ \text{s.t.} \quad & \sqrt{2} \leq \hat{p}_1 \leq \sqrt{2.21}, \quad 1 \leq \hat{p}_2 \leq \sqrt{2}, \quad 1 \leq \hat{p}_3 \leq \sqrt{2}, \end{aligned}$$

with $\hat{\pi}_3 \geq 2.2 - 0.87/\sqrt{0.21}$. Since

$$12\sqrt{2.21 - 1} + (\hat{\pi}_3 - 2)\sqrt{2.21 - 2} \geq 12$$

holds, the TSO's objective function value with $\hat{p}_1 = \sqrt{2.21}$, $\hat{p}_2 = 1$, and $\hat{p}_3 = \sqrt{2}$ is larger than the TSO's objective function value with $\hat{p}_1 = \sqrt{2}$, $\hat{p}_2 = 1$, and $\hat{p}_3 = \sqrt{2}$ for all $\hat{\pi}_3 \geq 2.2 - 0.87/\sqrt{0.21}$. Thus, the TSO has an incentive to deviate from the welfare-maximal strategy. Consequently, the welfare maximum $\hat{x}$ is not a solution of the simultaneous game (SG).

The TSO's problem (8) for the nodal prices $\tilde{\pi}$ is equivalent to the optimization problem

$$\begin{aligned} \max_{\tilde{p}} \quad & \left(11.7 - 2\sqrt{0.21}\right)\sqrt{\tilde{p}_1^2 - \tilde{p}_2^2} - \left(\frac{0.87}{\sqrt{0.21}} + 3\sqrt{0.21}\right)\sqrt{\tilde{p}_1^2 - \tilde{p}_3^2} \\ \text{s.t.} \quad & \sqrt{2} \leq \tilde{p}_1 \leq \sqrt{2.21}, \quad 1 \leq \tilde{p}_2 \leq \sqrt{2}, \quad 1 \leq \tilde{p}_3 \leq \sqrt{2}. \end{aligned}$$

The optimal solution is $\tilde{p} = (\sqrt{2}, 1, \sqrt{2})$ and thus differs from $(\sqrt{2.21}, 1, \sqrt{2})$, see Figure 11 (below). Consequently, the welfare maximum $\tilde{x}$ is not a solution of the simultaneous game (SG).

In summary, it follows that the fundamental theorems of welfare economics cannot be extended to gas markets accounting for network constraints even on simple tree networks.

Appendix D. Proof of Theorem 4.3

Proof of Theorem 4.3. Since x is a solution of (SG), (q, p) is a solution of (8). According to Theorem 3.4, there exist Lagrange multipliers $(\gamma, \zeta^-, \zeta^+)$ such that $(q, p, \gamma, \zeta^-, \zeta^+)$ solves (9).

We first show that $\pi_u = \pi_v$ holds for all $a = (u, v) \in A$. Since $\pi_u = \pi_v$ holds for all arcs $a \in A$ with $q_a = 0$, cf. Lemma 4.1, it is sufficient to prove $\pi_u = \pi_v$ for all $a \in A$ with $q_a \neq 0$. In turn, this is equivalent to show $\gamma_a = 0$ for all $a \in A$ with $q_a \neq 0$, cf. Equation (9a).

Because $0 < p_u^- < p_u < p_u^+$ for all $u \in V$, Equation (9b) reduces to

$$\sum_{a\in\delta^{\text{in}}(u)} \gamma_a - \sum_{a\in\delta^{\text{out}}(u)} \gamma_a = 0, \quad u \in V. \tag{22}$$

Thus, $\gamma = (\gamma_a)_{a\in A}$ is a circulation in G. We now consider a cycle $C = (V(C), A(C))$ of G. By Assumption 2, there exists an arc $a \in A(C)$ with $q_a \neq 0$. Without loss of generality, we assume that all arcs $a = (u, v) \in A$ are directed such that $\pi_u \leq \pi_v$ holds. As then $\operatorname{sgn}(\gamma_a) = \operatorname{sgn}(\pi_u - \pi_v) \leq 0$ holds for all arcs of the cycle with $q_a \neq 0$, cf. Lemma 4.2, these arcs must have the same direction to allow for circulations. If this is not the case, we have $\gamma_a = 0$ for all $a \in A(C)$ with $q_a \neq 0$ and we are done. Hence, we assume that all arcs with $q_a \neq 0$ have the same direction. From Equation (9a) and Lemma 4.1, it follows

$$\sum_{a\in A(C)\colon q_a\neq 0} \gamma_a 2\Lambda_a |q_a| = \sum_{a=(u,v)\in A(C)\colon q_a\neq 0} (\pi_u - \pi_v) = \sum_{a=(u,v)\in A(C)} (\pi_u - \pi_v) = 0.$$

As $\gamma_a \leq 0$ for all $a \in A(C)$ with $q_a \neq 0$, we have $\gamma_a = 0$ for all $a \in A(C)$ with $q_a \neq 0$. Thus, $\pi_u = \pi_v$ follows for all $a = (u, v) \in A$.

Since variable costs of gas supply are strictly convex, continuously differentiable, and increasing, the same holds for the aggregated market supply. An analogous argument yields that the aggregated inverse market demand is continuous and strictly decreasing. Hence, the point of intersection of the overall market supply and the overall inverse market demand function is unique. Consequently, there exists only one solution to (SG) that fulfills $\pi_u = \pi_v$ for all $a = (u, v) \in A$. Therefore, x is the unique market equilibrium of (SG).

In the absence of physical network restrictions, the unique welfare maximum is implemented in case of a uniform market price; see Chapter 10.E of [33]. Thus, the unique market equilibrium of (SG) corresponds to the unique welfare maximum of (WMP). □

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