Decomposable Robust Two-Stage Optimization: An Application to Gas Network Operations Under Uncertainty

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November 20, 2017

We study two-stage robust optimization problems with a special decomposable structure. By exploiting this special structure, it is shown how the two-stage problem can be reformulated to a standard single-stage optimization problem where the right hand side of the inequality system are pre-computed by solving a series of optimization problems. As a consequence of the decomposable structure, we develop an aggregated model which leads to a drastic reduction of the number of subproblems. The ideas developed here are then applied to a real-world gas network problem with active elements under uncertainty. The approach is able to deal with both uncertainties in the demand as well as uncertainties in the physical parameters. After having shown how this problem fits into the setting described here, we use piecewise relaxations and preprocessing techniques for gas networks that have been developed previously and extend them such that they are able to incorporate uncertain input data. The practical feasibility and the effectiveness of the new robust optimization approach is demonstrated using a series of benchmarks on realistic gas network instances.

1 Introduction

In this paper, we study two-stage robust optimization problems with a special decomposable structure. The aim of robust optimization is to solve optimization tasks that are parameterized by a so-called uncertainty set $\mathcal{U}$. In the easiest setting, solving a robust problem amounts to specifying a solution for the problem variables that is feasible for all possible realizations of the uncertainty. If this is too conservative or even impossible, e.g., when the problem has unique and differing solutions for each realization of the uncertainty set, one may use a two-stage approach. The problem variables are split into *here-and-now*

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variables which have to be decided without knowledge of the uncertainty and *wait-and-see* or *adjustable* variables whose value can be chosen after the uncertainty is revealed. A two-stage approach is a natural model choice for gas network optimization tasks with remotely controllable elements ("active elements") with an uncertain demand situation or uncertainties in the physical parameters. The network operator has to decide the configuration of the controllable network elements before knowing the realization of the uncertainty whereas the physical state of the network adjusts itself once the uncertainty is revealed.

This two-stage model has two special properties: first, the physical state of the network which is described by the adjustable variables is uniquely determined by the uncertain parameter and second, there is no coupling between first and second stage variables. We show how this structure can be exploited to reformulated the two-stage problem as a single stage problem whose right hand side comprises the optimal values of a series of subproblems arising from this transformation. By further exploiting the problem structure, we develop an aggregation method which can be used to drastically reduce the number of subproblems to solve. We also show that solving the subproblems to global optimality is not absolutely necessary as relaxations can be used instead to obtain a more conservative—but still robust feasible—solution.

In order to apply these ideas to the nonlinear, non-convex gas network problem, we extend the well-known piecewise linearization technique for the nonlinear pressure drop constraints to incorporate uncertain parameters. As our approach requires solving a potentially large number of subproblems where each one is very similar to the nominal problem, it is crucial to reduce the problem size as much as possible for the overall solution time to stay within an acceptable time frame. To this end, we describe a variety of preprocessing techniques which are adapted from the literature and generalized to handle uncertain parameters.

This work is structured as follows: In section 2, the problem is introduced as a general two-stage robust optimization task. After specifying the special structure, the transformation to single-stage problem and the aggregation idea are explained.
In section 3, we show how this transformation can be applied to gas network problems under uncertainty. We first give a brief introduction of the gas network problem with active elements under uncertainty in section 3.1. Next it is shown in section 3.2 how the previously developed ideas can be apply to the gas network problem. How to construct relaxations of arising nonlinear problem is described in section 3.2.1. Lastly, section 3.3 presents several generalizations of known preprocessing techniques to incorporate uncertain parameters.
In section 4, the presented methods are benchmarked on a variety of freely available gas network instances to show their practical feasibility.
This work closes with a summary in section 5.

2 Two-Stage Robust Optimization Problems with Disconnected First Stage and Unique Second Stage

Robust optimization deals with optimization problems under uncertainty. A robust optimization task is given by a family of optimization problems which are parameterized by a so-called uncertainty set $\mathcal{U} \subseteq \mathbb{R}^{n_u}$:

$$\left\{ \min_{z \in \mathbb{R}^n} \{ f(z) : g(z,u) = 0,\, h(z,u) \leq 0 \} \right\}_{u \in \mathcal{U}}. \tag{1}$$

In addition to the problem variables $z \in \mathbb{R}^n$, the constraint functions $g\colon \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^{m_1}$ and $h\colon \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^{m_2}$ accept an additional *data* vector $u \in \mathcal{U}$. Without loss of generality, we assume a certain objective function $f\colon \mathbb{R}^n \to \mathbb{R}$, see [2]. The goal of robust optimization is to find solutions for (1) which are immunized against all possible realizations $u \in \mathcal{U}$. In the most basic setting, all problem variables have to be fixed before the uncertainty becomes known. This leads to the *robust counterpart*

$$\min_{z \in \mathbb{R}^n} \{ f(z) \,|\, g(z,u) = 0,\, h(z,u) \leq 0 \ \forall u \in \mathcal{U} \} \tag{RC}$$

where the feasible region is the set of all z which are feasible for all possible realizations of the uncertainty.

If we drop the requirement of fixing z *before* the uncertainty is revealed, we arrive at a two-stage robust optimization problem. Here, the problem variables are partitioned into *first* and *second* stage variables: $z = (x,y) \in \mathbb{R}^n = \mathbb{R}^{n_1+n_2}$. First stage or *here-and-now* variables $x \in \mathbb{R}^{n_1}$ have to be fixed before the uncertainty becomes known whereas second stage or *wait-and-see* variables $y \in \mathbb{R}^{n_2}$ can be decided with knowledge of the revealed uncertainty. Since the second stage variables thus depend on the uncertain parameter, they are also called *adjustable variables*. The notion of different actions happening at different points in time directly leads to the *adjustable robust counterpart*

$$\min_{x} \{ f(x) \,|\, \exists x \in \mathbb{R}^{n_1} \ \forall u \in U \ \exists y \in \mathbb{R}^{n_2} \text{ with } g(x,y,u) = 0,\, h(x,y,u) \leq 0 \}. \tag{ARC}$$

Again, without loss of generality, the objective only depends on here-and-now variables.

In this article we consider two-stage problems with a special constraint structure:

Assumption 1. *a. The equation system $g(x,y,u) = 0$ does not depend on x and admits a unique solution $y^*(u)$ for all $u \in \mathcal{U}$.*

b. The inequality constraints $h(x,y,u) \leq 0$ are separable: $h(x,y,u) = s(x) + t(y,u)$.

Under this assumption, the two-stage problem can be reformulated as a single stage problem:

Lemma 2.1. *Under assumption 1, the set of feasible first stage decisions x of the adjustable robust counterpart (ARC) is given by*

$$\mathcal{X} = \{ x \in \mathbb{R}^{n_1} \,|\, s(x) \leq b \} \tag{2}$$

where $b \in \mathbb{R}^{m_1}$ with

$$b_i = -\max_{u \in \mathcal{U}} \{ t_i(y,u) \,|\, g(y,u) = 0,\, y \in \mathbb{R}^{n_2} \}. \tag{3}$$

Proof. Due to assumption 1, the equality constraints of (ARC) only depend on y and u: $g(x,y,u) \equiv g(y,u)$ and the inequality constraints are separable: $h(x,y,u) = s(x)+t(y,u)$. Therefore, the set of feasible first stage decisions x of (ARC) can be written as

$$\{x \,|\, \exists x \in \mathbb{R}^{n_1} \ \forall u \in U \ \exists y \in \mathbb{R}^{n_2} \text{ with } g(y,u) = 0,\ s(x) + t(y,u) \leq 0\}\,. \tag{4}$$

From assumption 1 it follows that there is a function $y^*(u)$ which maps values u of the uncertainty set $\mathcal{U}$ to solutions of $g(y,u) = 0$. This function exists and is well defined since solutions of the equality system exist for all $u \in \mathcal{U}$ and are unique. With this in mind, the $\exists y$ quantor and the equality constraints can be eliminated from (4) and the feasible region of a single stage robust optimization problem of the form (RC) is obtained:

$$\{x \in \mathbb{R}^{n_1} \,|\, s(x) + t(y^*(u),u) \leq 0 \ \forall u \in U\}\,. \tag{5}$$

To eliminate the "for all" condition, we maximize the left hand side of the inequality:

$$s(x) + t(y^*(u),u) \leq 0 \ \forall u \in \mathcal{U} \quad \Longleftrightarrow \quad \max_{u\in\mathcal{U}} \left(s(x) + t(y^*(u),u)\right) \leq 0 \tag{6}$$

$$\Longleftrightarrow \quad s(x) + \max_{u\in\mathcal{U}} \left(t(y^*(u),u)\right) \leq 0. \tag{7}$$

After rewriting the solution function y^* in terms of g and letting

$$b_i := -\max_{u\in\mathcal{U}} \{t_i(y,u) \,|\, g(y,u) = 0,\ y \in \mathbb{R}^{n_2}\} \qquad \text{for all} = 1,\ldots,m_1, \tag{8}$$

we obtain

$$s_i(x) + \max_{u\in\mathcal{U}} \left(t_i(y^*(u),u)\right) = s_i(x) - b_i \leq 0 \tag{9}$$

and thus set of feasible first stage decisions x of (ARC) is equivalent to

$$\{x \in \mathbb{R}^{n_1} \,|\, s(x) \leq b\}\,. \tag{10}$$

□

We want to highlight two properties of (2) which may be beneficial for solving problems of this type in practice.

Remark 2.2. Calculating b_i involves solving an optimization task to global optimality. In situations where this is not possible or where it can only be done with great effort, relaxations of the problem can be used instead. We propose to use convex or mixed-integer linear relaxations as they lead to optimization tasks which can be solved to global optimality. Consider the definition of b_i:

$$b_i = -\underbrace{\max_{u\in\mathcal{U}} \{t_i(y,u) \,|\, g(y,u) = 0,\ y \in \mathbb{R}^{n_2}\}}_{(\star)}\,. \tag{11}$$

Replacing $(\star)$ by a relaxed optimization problem leads to some $b_i' \leq b_i$. Plugging b' into (10) yields a smaller feasible region for the first-stage decision variables x:

$$\{x \in \mathbb{R}^{n_1} \,|\, s(x) \leq b'\} \subseteq \{x \in \mathbb{R}^{n_1} \,|\, s(x) \leq b\}\,. \tag{12}$$

This leads to more conservative solutions (i.e., larger objective) of (10) while preserving robust feasibility.

Remark 2.3. Suppose there are two constraints $h_i(x, y, u) = s_i(x) + t_i(y, u) \leq 0$ for $i = 1, 2$ which are replaced by $s_1(x) \leq b_1$ and $s_2(x) \leq b_2$ due to lemma 2.1. If the functions s_1 and s_2 are identical, both constraints can be aggregated into a single constraint by setting

$$s_1(x) = s_2(x) \leq \min(b_1, b_2). \tag{13}$$

For ease of explanation, the implications of this observation will be explained using these two constraint as an example. However, it is easy to see how to generalize this notion to any finite number of constraints h_i.

We want to emphasize that this situation may not be that uncommon in practice. For example, all variable bounds $y_i \leq \overline{y}_i$ are independent of x and are therefore reduced to a constraint of the form $s(x) = 0 \leq c$ for some $c \in \mathbb{R}$.

In order to implement reduction (13), one still needs to solve two optimization tasks of the form (8) to obtain b_i. Using the definition of b_i, it is possible to obtain $\min(b_1, b_2)$ using a single optimization task:

$$\min(b_1, b_2) \overset{(8)}{=} \min_{i=1,2} \left(- \max_{u \in \mathcal{U}} \{ t_i(y, u) \,|\, g(y, u) = 0,\ y \in \mathbb{R}^{n_2} \} \right) \tag{14}$$

$$= \min_{i=1,2} \left(\min_{u \in \mathcal{U}} \{ -t_i(y, u) \,|\, g(y, u) = 0,\ y \in \mathbb{R}^{n_2} \} \right) \tag{15}$$

$$= \min_{u \in \mathcal{U}} \left\{ \min_{i=1,2} \left(-t_i(y, u) \right) \,\middle|\, g(y, u) = 0,\ y \in \mathbb{R}^{n_2} \right\}. \tag{16}$$

Compared to (8), the aggregated optimization problem (16) has the same feasible set but a different objective.

3 Stationary Gas Network Operations with Simplified Compressor Models Under Uncertainty

First, the gas network problem under uncertainty is modeled. Next, we show how to reformulate the model to satisfy the assumptions of section 2. Finally, relaxations for the nonlinear constraints as well as preprocessing methods for enhancing model performance are presented.

3.1 The Stationary Gas Network Transport Problem with Linear Compressors

A gas network comprises a number of different components. Apart from pipes which allow gas transport, there are compressors and control valves which increase and decrease the pressure, valves which can prohibit flow along an arc, or resistors which hinder the gas flow and decrease pressure. In this article, we only consider networks with pipes and simple compressor models. Let $G = (V^+, A)$ be a digraph with $|V^+| = |\{0, \ldots, n+1\}| = n + 1$ nodes and $|A| = m \geq n$ arcs. Without loss of generality, we assume G to be weakly connected, i.e., the corresponding undirected graph is connected. If G is not weakly connected, all presented results can be applied to each connected component separately.

Gas can be inserted or withdrawn at the graph's nodes. Let $q_v^{\text{nom}+} \in \mathbb{R}$ denote the demand at node $v \in V$ where inflowing gas is indicated by a negative sign and outflowing gas by a positive sign. In gas network operations, the vector $q^{\text{nom}+} \in \mathbb{R}^{|V^+|}$ is also called the network's *nomination*. As is true for any network flow problem, this vector has to be balanced: $\sum_{v \in V^+} q_v^{\text{nom}+} = 0$.

Let $q_a \in \mathbb{R}$ denote the gas flow over arc $a \in A$. Flow in arc direction is indicated by a positive sign of q_a whereas flow in reverse arc direction is indicated by a negative sign of q_a. Just like linear network flow problems, the gas flow has to satisfy flow conservation at each node $v \in V^+$:

$$\sum_{a=(v,w)\in A} q_a - \sum_{a=(w,v)\in A} q_a = q_v^{\text{nom}+}. \tag{17}$$

Furthermore, we introduce variables $p_v \in \mathbb{R}_{\geq 0}$ to model the pressure at node $v \in V$. Since pressure variables always appear squared in our setting, a squared pressure variable $\pi_v = p_v^2$ is introduced to simplify the model and linearize all p_v^2.

The pressure at two adjacent nodes $(v, w) = a \in A$ is linked through one or more constraints induced by the present network component on arc a. Let $A = A_{\text{pi}} \cup A_{\text{cs}}$ with $A_{\text{pi}} \cap A_{\text{cs}} = \emptyset$ be a partition of the network's arcs into a set A_{pi} of pipes and a set A_{cs} of compressors.

Gas traveling along a pipe $a = (v, w) \in A_{\text{pi}}$ experiences a pressure drop:

$$\pi_w - \pi_v = -\phi_a q_a \left|q_a\right|. \tag{18}$$

The magnitude of this pressure loss depends on the amount q_a of gas and the *pressure drop coefficient* $\phi_a \in \mathbb{R}_{>0}$. The factor ϕ_a is computed from a number of physical properties of the pipe and the quality of the gas mixture. Since many of these parameters can be affected by uncertainty, this parameter lends itself for a robust treatment.

Compressors can be used to increase the pressure of the gas. This is needed e.g., when gas is inserted into a higher pressure network, or in order to compensate for pressure loss when gas is transported over a long distance. There are several compressor models available, ranging from very simple to highly complex see blah . We use a simple linear compressor model where the increase in pressure is independent of the flow through the compressor. The squared pressure difference may be increased linearly by some nonnegative amount $\Delta_a \in [\underline{\Delta}_a, \overline{\Delta}_a] \subseteq \mathbb{R}_{\geq 0}$:

$$\pi_w - \pi_v = \Delta_a, \quad (v, w) = a \in A_{\text{cs}}. \tag{19}$$

Let $\Delta \in \mathbb{R}_{\geq 0}^{|A_{\text{cs}}|}$ be the vector of all compressor "power levels" within in the network. We model the costs of compressor operation (e.g., due to fuel consumption) by a linear cost function

$$w^T \Delta \tag{20}$$

where $w \in \mathbb{R}_{\geq 0}^{|A_{\text{cs}}|}$ is a cost vector associated with the given compressors.

Let $\mathcal{A}^+ \in \mathbb{R}^{n\times m}$ be the node-arc-incidence-matrix of G, that is $(\mathcal{A}^+)_{av} = -1$ and $(\mathcal{A}^+)_{aw} = +1$ for $a = (v, w) \in A$. With this matrix, the gas transport problem can be expressed in a very concise fashion. Flow conservation (17) can be written as

$$\mathcal{A}^+ q = q^{\text{nom}+}. \tag{21}$$

Let $F\colon \mathbb{R}^{|A_{\text{pi}}|} \times \mathbb{R}^{|A|} \times \mathbb{R}^{|A_{\text{cs}}|} \to \mathbb{R}$ be an aggregation of pipe (18) and compressor (19) models:

$$F_a(\phi, q, \Delta) = \begin{cases} -\phi_a q_a |q_a| & \text{if } a \in A_{\text{pi}}, \\ \Delta_a & \text{if } a \in A_{\text{cs}}. \end{cases} \tag{22}$$

Hence, the constraint for the network component on arc $a = (v, w) \in A = A_{\text{pi}} \cup A_{\text{cs}}$ is given by

$$\pi_w - \pi_v = F_a(\phi, q, \Delta). \tag{23}$$

Using the node-arc-incidence-matrix $\mathcal{A}^+$, constraint (23) for all arcs amounts to

$$\mathcal{A}^{+T} \pi = F(\phi, q, \Delta). \tag{24}$$

Combining flow conservation (21), pressure constraints (24), and objective function (20) yields the gas transport problem with linear compressors:

$$\min\ w^T \Delta \tag{25}$$
$$\mathcal{A}^+ q = q^{\text{nom}+}, \tag{26}$$
$$\mathcal{A}^{+T} \pi = F(\phi, q, \Delta), \tag{27}$$
$$\Delta \in [\underline{\Delta}, \overline{\Delta}] \subseteq \mathbb{R}^{|A_{\text{cs}}|}_{\geq 0} \tag{28}$$
$$\pi \in [\underline{\pi}, \overline{\pi}] \subseteq \mathbb{R}^{|V|}_{\geq 0}, \tag{29}$$
$$q \in \mathbb{R}^{|A|}. \tag{30}$$

3.1.1 The Gas Network Problem Under Uncertainty

There are two main sources of uncertainty which are studied in this article: fluctuations in the demand vector $q^{\text{nom}+}$ as well as uncertainties in the pressure drop coefficient ϕ. We define two polyhedral uncertainty sets, $\mathcal{P} \subseteq \mathbb{R}^{|A_{\text{pi}}|}_{>0}$ for uncertain pressure drop coefficients and $\mathcal{D} \subseteq \mathbb{R}^{|V|}$ for uncertain demands. When not explicitly stated otherwise, the considered problems may be affected by both uncertainty sets simultaneously. Thus we typically use a combined uncertainty set $\mathcal{U} = \mathcal{P} \times \mathcal{D}$.

Uncertain pressure loss coefficients This is an *arc-wise* uncertainty and influences the pressure drop between two incident nodes. For all arcs $a \in A_{\text{pi}}$, let $\underline{\phi}_a$, $\overline{\phi}_a \in \mathbb{R}_{>0}$ with $0 < \underline{\phi}_a, \leq \overline{\phi}_a$. Let the uncertainty set be given by

$$\mathcal{P} = \left\{ \phi \in \mathbb{R}^{|A_{\text{pi}}|}_{>0} \,\middle|\, \underline{\phi}_a \leq \phi_a \leq \overline{\phi}_a, \quad a \in A_{\text{pi}} \right\}. \tag{31}$$

This type of uncertainty is also known as “box uncertainty”.

Uncertain demand This *node-wise* uncertainty has an impact on the solution space of the linear network flow problem. For all nodes $v \in V$, let $\underline{q}_v^{\text{nom}+}, \overline{q}_v^{\text{nom}+} \in \mathbb{R}$ with $\underline{q}_v^{\text{nom}+} \leq \overline{q}_v^{\text{nom}+}$. As the overall demand always has to be balanced, the uncertainty set includes a balancing constraint.

$$\mathcal{D} = \left\{ q^{\text{nom}+} \in \mathbb{R}^{|V|} \;\middle|\; \begin{array}{rl} \underline{q}_v^{\text{nom}+} \leq & q_v^{\text{nom}+} \leq \overline{q}_v^{\text{nom}+}, \quad v \in V \\ & \mathbb{1}^T q^{\text{nom}+} = 0 \end{array} \right\}. \tag{32}$$

3.1.2 Robust Counterpart

Robust treatment of the gas network problem (25)–(30) requires a two-stage model. Assume for a moment that single stage (also known as *strict*) robustness is applied to the problem at hand. In the single stage setting, a solution for flow q and squared pressure π has to be found which is valid for all possible realizations of the uncertainty set. Considering flowing gas as a physical system, such a solution is very unlikely to exists as one would expect the flow and pressure situation within the network to be highly dependent upon overall demand and the pressure loss coefficients. Therefore, a two-stage robust approach is appropriate.

In two-stage robustness, the problem variables are partitioned into first-stage or *here-and-now* and second-stage or *wait-and-see* variables. First-stage variables have to be decided without knowledge of the uncertain parameters whereas second-stage variables can adjust to the revealed uncertainty. For the gas network problem, compressor power Δ are first-stage variables whereas flow q and squared pressure π are adjustable second-stage variables. Thus, the two-stage problem can be formulated as

$$\min \left\{ w^T \Delta \;\middle|\; \exists\, \Delta \text{ such that } \forall u = (\phi, q^{\text{nom}+}) \in \mathcal{U}\ \exists q, \pi \text{ that satisfy (26)–(30)} \right\}. \tag{33}$$

This is also known as the *(fully) adjustable robust counterpart*.

3.2 A Separable Reformulation of the Gas Transport Problem

We now show that problem (25)–(30) can be reformulated so that assumption 1 is satisfied. This is possible as long as no compressors are part of a cycle in the graph:

Assumption 2. *Let a gas network problem with compressors over graph G be given. Then no compressors are part of a cycle in the undirected counterpart of G.*

Furthermore, without loss of generality we assume G to be weakly connected, i.e., the corresponding undirected graph is connected. This is no restriction since all presented transformations can be applied to each weakly connected component separately.

In [6], the authors show how all squared pressure variables and $|V|-1$ flow variables can be eliminated from the model. This is done by expressing the flows within the network as a combination of flows on a spanning tree together with flows on the remaining cycles. The pressure at each node is expressed relative to an arbitrary chosen root node by defining an aggregated pressure drop function between root node and each node in G. Similar to Kirchhoff's loop rule in electrical circuits, another set of constraints forces the

aggregated pressure drop on every fundamental cycle to be zero. The next paragraph introduces the required concepts used in the reformulation.

Since we assume G to be weakly connected, its node-arc-incidence-matrix $\mathcal{A}^+$ has rank $n-1$ and an arbitrary row can be removed while preserving the solution space. We discard the row corresponding to node 1 and obtain a full-rank matrix $\mathcal{A}$ from $\mathcal{A}^+$ in this way. After fixing a basis B of $\mathcal{A}$, matrix $\mathcal{A}$ is partitioned into a basic submatrix $\mathcal{A}_B \in \mathbb{R}^{(n-1)\times(n-1)}$ and a non-basic submatrix $\mathcal{A}_N \in \mathbb{R}^{(n-1)\times(m-n+1)}$. Moreover, let (F_B, F_N) and (q_B, q_N) be the respective partitions of F and q. In the reduced model only non-basic flows $q_N \in \mathbb{R}^{|N|}$ remain. Since every non-basic element corresponds to a fundamental cycle in G, we call the remaining flow variables $q_N \in \mathbb{R}^{|N|}$ *cycle flows*.

It is well known from linear algebra that the solution space of the linear equation system $\mathcal{A}q = q^{\text{nom}}$ can be parameterized by the non-basis variables q_N. We define a function $q(\cdot, q^{\text{nom}})$ that maps non-basic flows to flows in the whole network:

$$q\colon \mathbb{R}^{|N|} \times \mathbb{R}^{|V|} \to \mathbb{R}^{|A|}, \tag{34}$$

$$q_a(q_N, q^{\text{nom}}) = \begin{cases} \left(\mathcal{A}_B^{-1}\left(q^{\text{nom}} - \mathcal{A}_N q_N\right)\right)_a, & \text{if } a \in B, \\ (q_N)_a, & \text{if } a \in N. \end{cases} \tag{35}$$

Next we define an aggregated pressure loss function:

$$\tilde{g}(\phi, q, \Delta) = \left(\mathcal{A}_B^T\right)^{-1} F_B\left(\phi, q, \Delta\right) \tag{36}$$

Notice how each entry F_a only depends on either ϕ_a, q_a or Δ_a. Thus, the arguments of F_B can be restricted to their basic parts ϕ_B, q_B and Δ_B. Next, with (34) we rewrite $q_B = q_B(q_N, q^{\text{nom}})$ such that the new function $\bar{g}$ depends on q_N and q^{nom} instead of q_B:

$$\bar{g}(\phi_B, q^{\text{nom}}, q_N, \Delta_B) = \left(\mathcal{A}_B^T\right)^{-1} F_B\left(\phi_B, q_B(q_N, q^{\text{nom}}), \Delta_B\right). \tag{37}$$

We extend $\bar{g}$ to incorporate the root node and obtain the final definition of the aggregated pressure drop function g:

$$\begin{aligned} &g\colon \mathbb{R}_{>0}^{|A|} \times \mathbb{R}^{|V|} \times \mathbb{R}^{|N|} \times \mathbb{R}_{\geq 0}^{|B|} \to \mathbb{R}^{|V|}, \\ &g_v(\phi_B, q^{\text{nom}}, q_N, \Delta_B) = \begin{cases} 0, & \text{if } v = 1, \\ \bar{g}_v(\phi_B, q^{\text{nom}}, q_N, \Delta_B), & \text{if } v \in V \setminus \{1\}. \end{cases} \qquad (v \in V) \end{aligned} \tag{38}$$

By construction, $\bar{g}_v(\phi_B, q^{\text{nom}}, q_N, \Delta_B)$ is the sum of all pressure loss values on the unique path between root node 1 and node $v \in V$ on the spanning tree as induced by B. This includes pressure drop due to pipes as well as pressure changes due to compressors.

Elimination of Variables and Equations The next theorem establishes an equivalent formulation of the gas network problem over networks without compressors. In the absence of compressors, we omit the compressor variables and abbreviate $g_v(\phi_B, q^{\text{nom}}, q_N) \equiv g_v(\phi_B, q^{\text{nom}}, q_N, \Delta_B)$.

Theorem 3.1 ([6]). *Assume that $G = (V, A)$ is weakly connected and contains only pipes, i.e., $A = A_{\text{pi}}$ and $A_{\text{cs}} = \emptyset$. Let $\mathcal{A}$ be the node-arc-incidence-matrix of G, with a partition $(\mathcal{A}_B, \mathcal{A}_N)$ into basis and non-basis as described above. Let (F_B, F_N) and (q_B, q_N) be the corresponding partitions of F and q, respectively. Let g be the aggregate pressure drop function as defined in (38).*

Then (26)–(30) has a feasible solution if and only if the following reduced system in variables q_N has a solution. Moreover, any solution of the reduced system can be expanded to a solution of the original system.

$$\mathcal{A}_N^T g(\phi_B, q^{\text{nom}}, q_N) = F_N(\phi_N, q^{\text{nom}}, q_N), \tag{39}$$

$$g_w(\phi_B, q^{\text{nom}}, q_N) - g_v(\phi_B, q^{\text{nom}}, q_N) \leq \overline{\pi}_v - \underline{\pi}_w, \quad (v, w \in V) \tag{40}$$

$$q_N \in \mathbb{R}^{|N|}. \tag{41}$$

The parameters $\underline{\pi}_v$, $\overline{\pi}_v$ denote the squared pressure bounds at node v.
If a feasible q_N for (39) and (40) exists, the remaining original variables q_B, π can be recovered through $q_B = \mathcal{A}_B^{-1}(q^{\text{nom}} - \mathcal{A}_N q_N)$ and $\pi_v = \pi_0 - g_v(\phi_B, q^{\text{nom}}, q_N)$ $(v \in V)$. The value of π_0 is an arbitrary given element of

$$\left[\max_{v \in V}[\underline{\pi}_v + g_v(\phi_B, q^{\text{nom}}, q_N)],\ \min_{v \in V}[\overline{\pi}_v + g_v(\phi_B, q^{\text{nom}}, q_N)]\right].$$

Conversely, a vector q_N that was extracted from a solution q^*, π^* of (26)–(30) is feasible for (39)–(41).

Proof. See [6]. □

Corollary 3.2. *The theorem still holds if a network with compressors is considered, i.e., $A_{\text{cs}} \neq \emptyset$.*

Proof. The original result of [6] is established for pipe-only networks, i.e., $A_{\text{cs}} = \emptyset$. However, the pressure drop law is never used explicitly in the proof. Therefore, we conclude that their result must still hold for networks with compressors or other pressure constraints of the form

$$\pi_w - \pi_v = \alpha_a(q_a), \quad a = (v, w) \in A \tag{42}$$

where α_a is some scalar-valued function of the flow q_a. □

Existence and Uniqueness of Flow Consider a connected gas network without pressure bounds and without compressors. Due to [4, 12], it is known that in this setting a feasible flow always exists.

Theorem 3.3 ([4]). *Consider a connected gas network without pressure bounds and compressors. Then the set*

$$\mathcal{X} = \{(q, \pi) \mid (q, \pi) \text{ satisfy (26)–(28) and (30)}\} \tag{43}$$

of feasible solutions has the following properties:

1. *A flow solution always exists and is unique, i.e.,* $|\{q \mid \exists \pi \colon (q, \pi) \in \mathcal{X}\}| = 1$.

2. *Given a flow solution* q^**, the set* $\mathcal{X}$ *of feasible solutions has the form*

$$\mathcal{X} = \left\{ (q^*, \pi) \,\middle|\, \pi = \pi^* + \eta(1, \dots, 1)^T,\ \eta \in \mathbb{R} \right\}.$$

A feasible π^* *can be computed by first fixing the pressure* π^* *at the root node to an arbitrary value and then computing the remaining squared pressures via* $\pi_v = \pi_0 - g_v(\phi_B, q^{\text{nom}}, q_N^*)$ $(v \in V)$

Corollary 3.4. *Consider a gas network with compressors where assumption 2 is fulfilled, i.e., no compressors are part of a cycle. Fix any compressor power* Δ^**. Then theorem 3.3 still holds.*

Proof. 1. The solution space of the linear flow problem can be parameterized by the non-basic flows q_N (see (34)). A non-basic flow q_N^* is feasible for the gas network problem, if it satisfies the equation system (39):

$$\mathcal{A}_N^T g(\phi_B, q^{\text{nom}}, q_N, \Delta_B^*) = F_N(\phi_N, q^{\text{nom}}, q_N, \Delta_N^*). \tag{44}$$

However, due to assumption 2, this system does not depend on Δ^*. Therefore, the existence and uniqueness of a q_N^* is implied by theorem 3.3 by considering an equivalent gas network $\tilde{G}$ which consists of all the cycles in the original network G.

2. Since the compressor power Δ^* is fixed and is linked to the adjacent node's squared pressures via

$$\pi_w - \pi_v = \Delta_a^*, \quad (v, w) = a \in A_{\text{cs}}, \tag{45}$$

any feasible π^* remains feasible after adding $\eta(1, \dots, 1)^T$ $(\eta \in \mathbb{R})$.

□

The Reduced Gas Network Problem as a Two-Stage Problem In this paragraph, we show how the gas network problem can be reformulated to fit assumption 1 of section 2.

Consider a two-stage robust version of the gas transport problem with compressors. Compressor variables Δ are first stage decisions whereas squared pressure π and flow variables q are second stage variables.

$$\min \left\{ w^T \Delta \,\middle|\, \exists\, \Delta \text{ such that } \forall u = (\phi, q^{\text{nom}+}) \in \mathcal{U}\ \exists q, \pi \text{ that satisfy (26)–(30)} \right\}. \tag{46}$$

With corollary 3.2, the squared pressure variables and some equations can be eliminated from the model to obtain an equivalent formulation:

$$\min \left\{ w^T \Delta \,\middle|\, \exists\, \Delta \text{ such that } \forall u = (\phi, q^{\text{nom}+}) \in \mathcal{U}\ \exists q_N \text{ that satisfies (39) and (40)} \right\}. \tag{47}$$

Theorem 3.5. *Suppose that assumption 2 holds, i.e., no compressors are part of a cycle. Then the constraints of problem (47) satisfy assumption 2.*

Proof. Consider the equation system (39):

$$\mathcal{A}_N^T g(\phi_B, q^{\text{nom}}, q_N, \Delta) = F_N(\phi_N, q^{\text{nom}}, q_N, \Delta) \tag{48}$$

Assumption 2 implies that this system is independent of the first stage variables Δ. Moreover, uniqueness and existence of a solution q_N^* holds due to corollary 3.4. Thus the constraint system satisfies the first part of assumption 1.

With the assumption's second part in mind, we take a closer look at the inequality system (40):

$$g_w(\phi_B, q^{\text{nom}}, q_N, \Delta_B) - g_v(\phi_B, q^{\text{nom}}, q_N, \Delta_B) \leq \overline{\pi}_v - \underline{\pi}_w, \quad (v, w \in V). \tag{49}$$

If $v = 0$ is the root node, then $g_0(\phi_B, q^{\text{nom}}, q_N, \Delta_B) = 0$. Suppose $v \neq 0$, then

$$g_v(\phi_B, q^{\text{nom}}, q_N, \Delta_B) = \left[\left(\mathcal{A}_B^T\right)^{-1} F_B\left(\phi_B, q_B(q_N, q^{\text{nom}}), \Delta_B\right) \right]_v, \tag{50}$$

$$= \left(\mathcal{A}_B^T\right)^{-1}_{v\cdot} F_B\left(\phi_B, q_B(q_N, q^{\text{nom}}), \Delta_B\right), \tag{51}$$

where $\left(\mathcal{A}_B^T\right)^{-1}_{v\cdot}$ denotes the v-th line of $\left(\mathcal{A}_B^T\right)^{-1}$. From this definition it is evident that g_v is a linear combination of the entries of F_B. Every entry in F_B either concerns a pipe and depends on q_N or concerns a compressors and depends on Δ. Therefore the inequalities are separable:

$$\begin{aligned}\left(\mathcal{A}_B^T\right)^{-1}_{w\cdot} F_B\left(\phi_B, q_B(q_N, q^{\text{nom}}), \Delta_B\right) - \left(\mathcal{A}_B^T\right)^{-1}_{v\cdot} F_B\left(\phi_B, q_B(q_N, q^{\text{nom}}), \Delta_B\right) - \overline{\pi}_v + \underline{\pi}_w \\ = s_{vw}(\Delta) + t_{vw}(\phi_B, q_B(q_N, q^{\text{nom}})) \leq 0, \quad (v, w \in V),\end{aligned} \tag{52}$$

for suitable functions s_{vw} and t_{vw}. □

Next, we apply lemma 2.1 in order to solve the two-stage robust gas transport problem as a single stage linear program

$$\begin{aligned}&\min_{\Delta \in [\underline{\Delta}, \overline{\Delta}]} w^T \Delta, \\ &s_{vw}(\Delta) \leq b_{vw}, \quad (v, w \in V),\end{aligned} \tag{53}$$

where

$$b_{vw} = -\max \left\{ t_{vw}(\phi_B, q_B(q_N, q^{\text{nom}})) \;\middle|\; \begin{array}{c} \mathcal{A}_N^T g(\phi_B, q^{\text{nom}}, q_N) = F_N(\phi_N, q^{\text{nom}}, q_N) \\ (\phi, q^{\text{nom}}) \in \mathcal{U} \\ q_N \in \mathbb{R}^{|N|}. \end{array} \right\}. \tag{54}$$

3.2.1 Piecewise Linear Relaxations of the Gas Transport Problem

In order to solve the gas network problem via the linear problem (53), one first needs to compute b_{vw} by solving a series of nonlinear and non-convex optimization problems to global optimality. Since this is a difficult task in general, we first replace all nonlinear terms by piecewise linear relaxations and use this surrogate model to compute the right hand side b. As was discussed in section 2, using relaxations for computing b leads to a smaller feasible region but preserves robust feasibility of the obtained solution.

For ease of explanations, we transform problem (54) back into its non-reduced form. We observe that the objective of (54) is a linear combinations of entries in F, i.e., pressure drops. Let

$$c_{vw} = \left(\mathcal{A}_B^T\right)^{-1}_{w\cdot} - \left(\mathcal{A}_B^T\right)^{-1}_{v\cdot} \tag{55}$$

be the coefficients of the linear objective and let $\lambda = F(\phi, q, \Delta)$ be a variable modeling the pressure drops over each arc. With this in mind, the objective of (54) can be written as $c^T\lambda$. Next, we split the pressure drop constraints ${\mathcal{A}^+}^T\pi = F(\phi, q, \Delta) = \lambda$ into one set for pipes and one set for compressors. The pressure drop in the pipes is rewritten by introducing the equivalent set of feasible pressure drops $\mathcal{L}^a_{\phi_a}$ on pipe $a \in A$:

$$\pi_w - \pi_v = \lambda_a = -\phi_a q_a \left|q_a\right| \tag{56}$$

$$\iff (q_a, \lambda_a) \in \mathcal{L}^a_{\phi_a} = \left\{(q_a, \lambda_a) \,\middle|\, \lambda_a = -\phi_a \left|q_a\right| q_a,\ q_a \in \mathbb{R}\right\}. \tag{57}$$

We set $\Delta = 0$ so that compressors do not influence the maximum pressure drop over the remaining pipes and fix the pressure at an arbitrary node to an arbitrary value: $\pi_0 = 0$.

$$b_{vw} = -\max\ {c_{vw}}^T\lambda - \overline{\pi}_v + \underline{\pi}_w \tag{58a}$$

$$\mathcal{A}^+ q = q^{\text{nom}+}, \tag{58b}$$

$${\mathcal{A}^+_{A_{\text{cs}}}}^T \pi = 0, \tag{58c}$$

$${\mathcal{A}^+_{A_{\text{pi}}}}^T \pi = \lambda, \tag{58d}$$

$$(q_a, \lambda_a) \in \mathcal{L}^a_{\phi_a} \quad (a \in A_{\text{pi}}), \tag{58e}$$

$$\pi_0 = 0, \tag{58f}$$

$$q \in \mathbb{R}^{|A|}. \tag{58g}$$

Our aim is to solve problem (58) or a relaxation thereof to global optimality. To this end, we present several relaxations of the nonlinear and non-convex set $\mathcal{L}^a_{\phi_a}$ that can be used for that purpose. Since global optimal solutions are required to ensure robustness, we develop piecewise linear relaxations of $\mathcal{L}^a_{\phi_a}$ which can then be used in a MIP. Of course other relaxations like linear (see section 3.3) or SDP relaxations arising from polynomial programming (see [1]) are also conceivable, however, we restrict ourselves to piecewise linear relaxations as they provide us with easy a-priori error bounds. In the remainder of this section, we drop the arc-specific indices of $\mathcal{L}^a_{\phi_a}$. Furthermore, we assume the flow variables to be in a finite interval, i.e., $q_a \in [\underline{q}_a, \overline{q}_a]$. This is no restriction as the pressure

and flow variables of problem (58) are always bounded, see section 3.3 for more details regarding their computation.

Relaxations with a-priori error bounds are of particular interest as they allow us to compute solutions with arbitrary precision by reducing the error ε.

Definition 3.1 (ε-exact relaxation)**.** Let $\mathcal{L} = \{(x, y) \in [\underline{x}, \overline{x}] \times \mathbb{R} \,|\, y = f(x)\} \subseteq \mathbb{R}^2$ be the function graph of a function $f : \mathbb{R} \to \mathbb{R}$ over a finite interval $[\underline{x}, \overline{x}]$ and let $\varepsilon > 0$ be a given error. We call $\tilde{\mathcal{L}} \subseteq [\underline{x}, \overline{x}] \times \mathbb{R}$ an *ε-exact relaxation* of $\mathcal{L}$, if

1. $\mathcal{L} \subseteq \tilde{\mathcal{L}}$ and
2. $|y - \tilde{y}| \leq \varepsilon$ for all $x \in [\underline{x}, \overline{x}]$ with $(x, y) \in \mathcal{L}$ and $(x, \tilde{y}) \in \tilde{\mathcal{L}}$.

As all presented relaxations are based around piecewise linear functions, we quickly restate how to express piecewise linear functions in MIPs using the delta method. Let $(x_i, y_i)_{i=1,\dots,k}$ a series of points in $\mathbb{R}^2$ with $x_1 < x_2 < \ldots < x_k$. Then the graph of the piecewise linear function with sampling points (x_i, y_i) is described given by the following mixed-integer constraints:

$$x = x_1 + \sum_{i=1,\dots,k-1} (x_{i+1} - x_i)\delta_i, \tag{59a}$$

$$y = y_1 + \sum_{i=1,\dots,k-1} (y_{i+1} - y_i)\delta_i, \tag{59b}$$

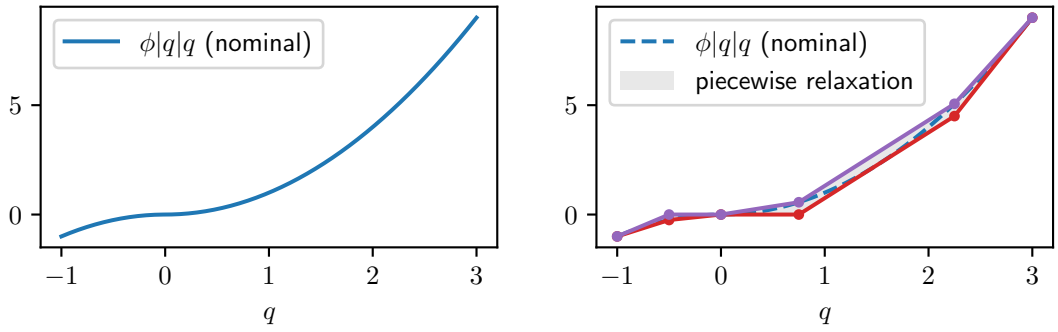
$$\delta_1 \geq z_1 \geq \delta_2 \geq z_2 \geq \ldots \geq z_{k-2} \geq \delta_{k-1}, \tag{59c}$$

$$\delta_i \in [0, 1] \qquad (i = 1, \dots, k-1), \tag{59d}$$

$$z_i \in \{0, 1\} \qquad (i = 1, \dots, k-2). \tag{59e}$$

Finding approximations for $\mathcal{L}$, i.e., the function graph of $cx|x|$ for a fixed $c > 0$ is straightforward. After eliminating the absolute value by splitting the function graph into negative and positive parts, only square functions cx^2 need to be treated. We show briefly how the error of approximating cx^2 by a (piecewise) linear function only depends on the parameter c and the distance between two adjacent sampling points but not on the position of the chosen sampling points. Pick any two points (x_1, y_1), (x_2, y_2) on the graph of $f^{\text{sq}}(x) = cx^2$ with $x_1 < x_2$ and let $f^{\text{lin}}(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$ be the line connecting both points. To calculate the maximum deviation $\varepsilon = \max_x f^{\text{lin}}(x) - f^{\text{sq}}(x)$, observe that $f^{\text{lin}}(x) - f^{\text{sq}}(x)$ is a degree two polynomial and thus attains its extreme value between its two roots x_1 and x_2 at $x^* = \frac{1}{2}(x_1 + x_2)$. A short calculation shows the maximum error to be given by $\varepsilon = f^{\text{lin}}(x^*) - f^{\text{sq}}(x^*) = \frac{c}{2}(x_2 - x_1)^2$. Since the approximation error only depends on c and the distance $x_2 - x_1$ between two sampling points, we conclude that cx^2 can be approximated by a piecewise linear function with a given error ε by equidistant sampling points. See fig. 1 for an example.

Two different cases need to be distinguished when building relaxations for $\mathcal{L}^a_{\phi_a}$, depending on whether ϕ_a is constant or affected by uncertainty. If the coefficient ϕ_a is constant, the standard relaxation from literature with equidistant sampling points can be applied (see BLAH) to obtain a ε-exact relaxation $\tilde{\mathcal{L}}$. For the second case, we assume the



(a) Nominal pressure drop

(b) Nominal piecewise linear relaxation

Figure 1: Nominal pressure drop and piecewise linear relaxation for constant pressure drop coefficient $\phi = 1$ and approximation quality $\varepsilon = 1.0$.

pressure drop coefficient ϕ_a to be affected by uncertainty. In general, this would require a relaxation $\mathcal{L}_{\phi_a}$ which is parameterized by ϕ_a so that it can adjust to the different realizations of ϕ_a to preserve the ε-approximation quality. However, due to uncorrelated pressure drop coefficients (31), a simplification can be applied. Since the realization of pressure drop coefficient ϕ_a at arc a is independent of all other uncertainties, it is sufficient to construct a ε-exact relaxation of the union

$$\bigcup_{\phi_a \in [\underline{\phi}_a, \overline{\phi}_a]} \mathcal{L}^a_{\phi_a}. \tag{60}$$

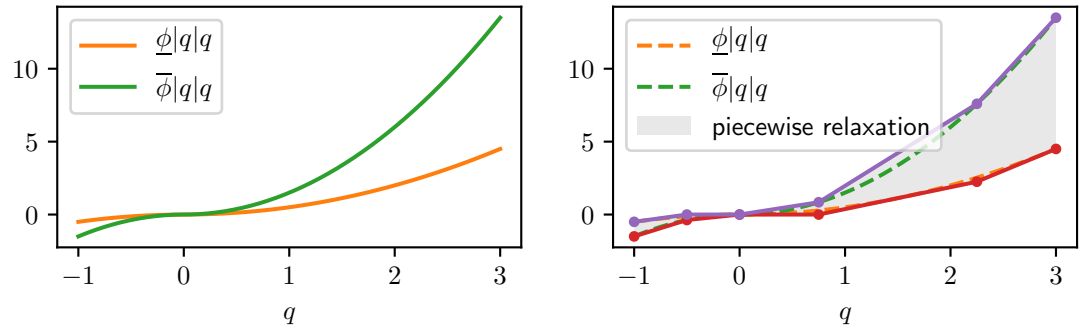
Due to continuity and monotonicity of the function value $\phi_a q_a |q_a|$ in ϕ_a, the union $\cup_{\phi_a \in [\underline{\phi}_a, \overline{\phi}_a]} \mathcal{L}^a_{\phi_a}$ has no holes and its boundary can be described by piecewise $\phi_a q_a |q_a|$ functions:

$$\bigcup_{\phi_a \in [\underline{\phi}_a, \overline{\phi}_a]} \mathcal{L}^a_{\phi_a} = \left\{ (q_a, \lambda_a) \;\middle|\; \begin{array}{ll} -\overline{\phi}_a q_a |q_a| \le \lambda_a \le -\underline{\phi}_a q_a |q_a|, & \text{if } q_a \ge 0, \\ -\underline{\phi}_a q_a |q_a| \le \lambda_a \le -\overline{\phi}_a q_a |q_a|, & \text{if } q_a \le 0. \end{array} \right\} \tag{61}$$

Finally, the relaxations for functions with constant ϕ_a can be applied to the border-defining functions of (61) in order to obtain a ε-exact relaxation. We remark that this leads to a ε-exact relaxation for the whole continuum of pressure drop constraints on an arc and not to a ε-exact relaxation of a single pressure drop constraint where the coefficient ϕ_a was fixed. However, this is no restriction since the uncertainty set for the pressure drop coefficients is uncorrelated and thus we can always chose some ϕ'_a whose pressure drop function graph is at most ε away from the solution within the relaxation. See fig. 2 for an example.

3.3 Reducing Model Size by Preprocessing

Binary variables are used for the construction of the presented piecewise linear relaxations. The overall complexity of solving a MIP typically heavily depends on the number of



(a) Uncertain pressure drop

(b) Piecewise linear relaxation under uncertainty

Figure 2: Pressure drop and piecewise linear relaxation for uncertain pressure drop coefficient $\phi \in [0.5, 1.5]$ and approximation quality $\varepsilon = 1.0$.

binary or integer variables. Our settings requires solving not a single but a series of MIPs to determine the right hand sides b_i. It is therefore very desirable to speed up the solution process as much as possible. Improving the solution time entails other advantages like being able to use a smaller approximation factor ε or being able to solving larger instances in the same amount of time.

The number of binary variables in our setting depends on the flow bounds $[\underline{q}_a, \overline{q}_a]$ and the approximation error ε. Since the approximation error is given, we decrease the number of required binary variables by providing strong bounds for q_a. To this end, our methods comprise two trivial bounds and an optimization method using LP relaxations. Similar preprocessing ideas can be found in [5], including more complex procedures like pressure and flow propagation heuristics. However, we cannot use most the mentioned ideas as is since they are tailored towards nominal problems without uncertainty. Due to the nature of our problem setting where we optimize over the uncertainty set, our preprocessing methods must preserve the full range of states in the network depending on the uncertainty. All methods are presented with the full range of uncertainty in mind, i.e., uncertain demand and uncertain pressure drop coefficients. For problems where only one or no uncertainty is given, the presented methods can often be simplified considerably.

Trivial bounds We present two trivial flow bounds, one resulting from the maximum overall total demand and one resulting from the decomposition of the linear flow solution space into tree and cycle flows. The problem is assumed to be affected by uncertain demand and uncertain pressure drop coefficient.

A trivial flow bound can be derived by calculating the maximum possible positive demand:

$$d^{\text{total}} = \frac{1}{2} \max \left\{ \left\| q^{\text{nom}+} \right\|_1 \,\middle|\, q^{\text{nom}+} \in \mathcal{D} \right\} \tag{62}$$

In general this problem is NP-hard and has to be reformulated as a MIP. Typically, the preprocessing should to very be fast (compared to solving the actual problem MIPs), so

we use an easy to calculate upper bound on d^{total} instead. Consider a demand uncertainty set (32). After omitting the balancing hyperplane, every demand parameter $q_v^{\text{nom}+}$ is only affected by upper and lower bounds:

$$\underline{q}_v^{\text{nom}+} \leq q_v^{\text{nom}+} \leq \overline{q}_v^{\text{nom}+}, \quad v \in V. \tag{63}$$

With this in mind, we first estimate the maximum total gas injection and withdrawal. A simple bound can be obtained from these quantities by taking the minimum of their absolute values:

$$d^+ = \sum_{v \in V} \max\left(0, \overline{q}_v^{\text{nom}+}\right), \quad d^- = \sum_{v \in V} \min\left(0, \underline{q}_v^{\text{nom}+}\right), \tag{64}$$

$$d^{\text{relax}} = \min(d^+, -d^-). \tag{65}$$

The gas flow over each arc can never exceed the total injection, thus

$$q_a \in [-d^{\text{relax}}, d^{\text{relax}}] \tag{66}$$

is a feasible bound for all arcs $a \in A$.

The previous bound can be improved considerably for certain arcs if the structure of the linear network flow solution space is exploited. Recall from (34) that any feasible flow q can be written as

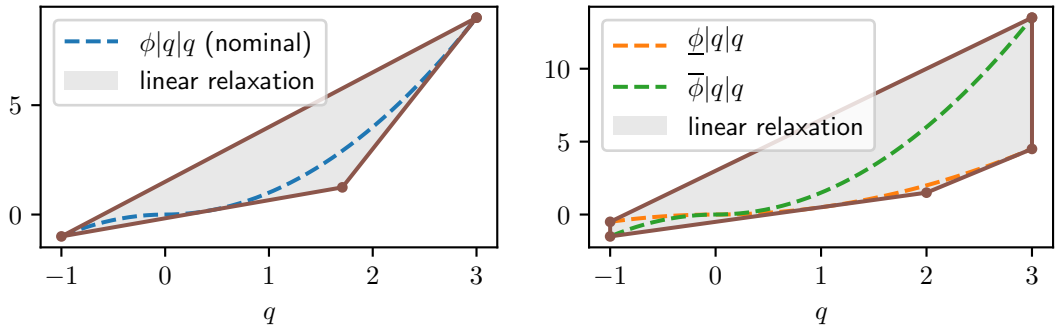
$$q = \mathcal{A}_B^{-1}\left(q^{\text{nom}} - \mathcal{A}_N q_N\right), \tag{67}$$

where $q_N \in \mathbb{R}^{|N|}$ is a free parameter. It is known from linear flow theory that due to (67), the flow over all arcs which are not part of a cycle is independent of q_N, i.e., can be written as $q_a = (\mathcal{A}_B^{-1} q^{\text{nom}})_a$. This allows us to find tight bounds for q_a over non-cycle arcs a by optimizing over the demand uncertainty set:

$$q_a \in \left[\min_{q^{\text{nom}} \in \mathcal{D}} (\mathcal{A}_B^{-1} q^{\text{nom}})_a, \max_{q^{\text{nom}} \in \mathcal{D}} (\mathcal{A}_B^{-1} q^{\text{nom}})_a\right]. \tag{68}$$

If it is not desirable to solve an optimization task, lower and upper bounds can be found with a similar approach as (64). We remark that for problems without demand uncertainty, there is nothing to optimize and the exact, constant flow q_a can be evaluated by calculating $(\mathcal{A}_B^{-1} q^{\text{nom}})_a$. In this case, the nonlinear pressure drop equation can be removed by evaluating the signed square function with q_a.

Bounds due to linear relaxations The previously derived flow bounds have a serious drawback as they cannot be used for cycle arcs since each fundamental cycle introduces a free parameter into the description of the flow solution. Since the linear flow model is not sufficient to derive bounds for these variables on its own, we add a coarse surrogate model for the pressure drop constraints. This was done previously for the nominal case in [5] by defining a convex hull of the pressure drop constraint's graph through linear inequalities. Compared to the piecewise linearization approach in section 3.2.1, the defined set is still a relaxation of the original constraint but does not guarantee a ε-approximation. We



(a) Nominal linear relaxation

(b) Linear relaxation under uncertainty

Figure 3: Linear convex hull of nominal ($\phi = 1.0$) and uncertain pressure drop coefficient ($\phi \in [0.5, 1.5]$).

generalize the linear model to incorporate variable pressure drop coefficients ϕ, i.e., build a convex hull $\hat{\mathcal{L}}^a$ of the union over all possible pressure drops given an interval of pressure drop coefficients:

$$\hat{\mathcal{L}}^a \supset \bigcup_{\phi_a \in [\underline{\phi}_a, \overline{\phi}_a]} \mathcal{L}^a_{\phi_a}. \tag{69}$$

See fig. 3 for a linear convex hull for nominal pressure drop as well as pressure drop under uncertainty.

Given relaxations $\hat{\mathcal{L}}^a$ for each pipe $a \in A_{\text{pi}}$, a very similar problem as (58) is introduced by replacing the nonlinear sets (58e) by their corresponding relaxation. Let $\hat{\mathcal{X}}$ be the feasible region of the resulting problem. Then bounds $q_a \in [q^*_{a,\text{lb}}, q^*_{a,\text{ub}}]$ for the flow along each arc $a \in A$ can be derived by minimizing and maximizing q_a over $\hat{\mathcal{X}}$:

$$q^*_{a,\text{lb}} = \min_{q,\pi \in \hat{\mathcal{X}}} q_a, \qquad q^*_{a,\text{ub}} = \max_{q,\pi \in \hat{\mathcal{X}}} q_a. \tag{70}$$

We remark that in case of demand uncertainty, the problems (70) also contains a description the uncertainty set. For pressure drop uncertainty, the uncertainty set is incorporated into the relaxations of the pressure drop equations and therefore does not show up in the linear programs.

Preprocessing strategy In our implementation, all three bounds—trivial total demand, linear flows on non-cycle arcs and LP relaxation based bounds—are combined into an iterative bound tightening procedure. Initially, trivial flow bounds (66) are derived for each arc to obtain finite bounds. Next, the flow bounds of all non-cycle arcs are tightened with (68). The procedure then enters a loop where the linear programs (70) are solved repeatedly for all remaining arcs. In one iteration step, model (70) is build only once and then reused with different objectives for each flow variable. At the end of each iteration step, all flow bounds are updated with the newly calculated bound information.

The algorithm terminates if either a maximum number of iterations is reached or if the euclidean norm of the difference between the bounds of two subsequent iterations is smaller than a specified cutoff value. In our computations, we use a maximum number of fifteen iterations and a cutoff value of 1.0.

4 Numerical Experiments

In this section, the performance of the developed methods is evaluated on a family of gas network instances. We first examine the influence of the preprocessing routines and of the aggregation idea to determine the best possible combination of both. Using this a basis for further study, problem run times are studied in more detail under different aspects like approximation quality and size of uncertainty set.

Instances and setup The studied problems are taken from GasLib [13], a freely available collection of gas network instances comprising of topology and nomination data. We used networks GasLib-11, GasLib-24, and GasLib-40 with their supplemented demand and pressure nominations. The employed instances were modified to fit the context of this article: Any element which is no compressor and no pipe is replaced by a so-called *short pipe*—an special type of pipe with zero pressure drop coefficient that can be traversed freely by gas without experiencing a pressure drop. In order to satisfy assumption 2, the compressor of GasLib-40 which is part of a cycle was replaced by a short pipes as well. Furthermore, the demand values of GasLib-40's nomination were halved as manual experiments revealed the provided nomination to quickly result in infeasible instances when adding uncertainties. Table 1 gives an overview over the features of the used instances.

Table 1: Instances for numerical experiments.

	#nodes	#pipes	#compressors	#control valves	#short pipes
GasLib-11	11	8	2	0	1
GasLib-24	24	19	3	1	2
GasLib-40	40	39	5	0	1

Each instance can be affected by uncertainty. We use relative perturbations around the nominal demand values or pressure drop coefficients as uncertainty sets. A unified naming schema of the defined uncertainty sets is utilized for both demand and pressure drop uncertainty. The network's demand or pressure drop coefficients may be affected independently by four levels of uncertainty: *nominal* (i.e., no uncertainty), *small*, *medium*, and *large* uncertainty. Any combination of the provided levels defines an uncertainty set for the numerical experiments, ranging from no uncertainty ("nominal demand and nominal pressure drop uncertainty") to the combination of large demand uncertainty with large pressure drop uncertainty. The chosen uncertainty level is then applied to

all affected elements, e.g., demands or pressure drop coefficients. Thus, there are 16 uncertainty sets in total. The concrete definitions can be found in table 2.

Table 2: Any combination of demand and pressure drop uncertainty level defines an uncertainty set used in the numerical study,

	demand	pressure drop coefficient
nominal	$\{q^{\text{nom}+}\}$	$\{\phi\}$
small	$[0.95 \cdot q^{\text{nom}+},\ 1.05 \cdot q^{\text{nom}+}]$	$[\phi,\ 1.10 \cdot \phi]$
medium	$[0.90 \cdot q^{\text{nom}+},\ 1.10 \cdot q^{\text{nom}+}]$	$[\phi,\ 1.50 \cdot \phi]$
large	$[0.80 \cdot q^{\text{nom}+},\ 1.20 \cdot q^{\text{nom}+}]$	$[\phi,\ 2.00 \cdot \phi]$

The linearization error for the piecewise linear relaxation of the pressure drop equations was chosen as $\varepsilon \in \{0.01, 0.1, 1.0\}$.

All experiments were carried out on a notebook with four Intel i7-4810MQ cores running at 2.80GHz each and 16 GB of RAM. The linear and mixed-integer linear problems were solved using Gurobi 7.5 [8] using 4 threads.

Run time improvements due to preprocessing and aggregation We compare preprocessing strategies on GasLib-11 and GasLib-24 since GasLib-40 is already too large to be solved in an acceptable timespan without preprocessing. In order to cover a wide range of problems, we derive average run times of instance groups where each group contains all possible combinations of the remaining parameters like approximation quality ε and uncertainty set.

First we take a look at the average run times depending on the employed preprocessing method, see table 3 and fig. 4a. The columns denote the different preprocessing choices: *trivial* (66), *treeflows* (68), and LP-based bound tightening (70) ("*opt*"). Each entry of table 3 is the averaged run time of 96 instances arising from all combinations of uncertainty set, $\varepsilon \in 0.01, 0.1, 1.0$, and choice of individual or aggregated model. The number in brackets displays the relative speedup compared to the trivial preprocessing bounds. Speedups from treeflows preprocessing is negligible, possibly due to the GasLib-instances containing only few arcs which are not part of cycle. We observe a dramatic speedup of 30–40× when the LP based bound tightening is used.

Table 3: Mean run times when using different preprocessing strategies. The number in brackets denotes the speedup compared to "trivial" preprocessing.

	trivial	treeflows	opt
GasLib-11	29.1 s (1.0×)	23.1 s (1.3×)	0.8 s (38.1×)
GasLib-24	149.2 s (1.0×)	146.0 s (1.0×)	4.7 s (31.9×)

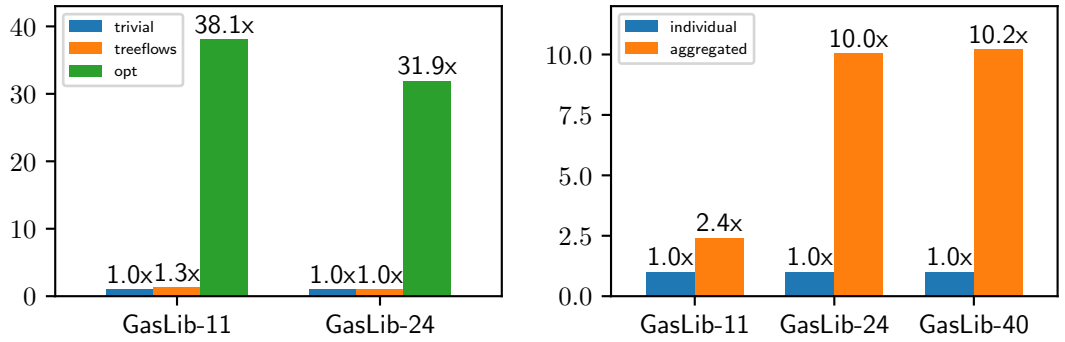
Next, the influence of aggregation is benchmarked. We only consider instances after applying LP-based bound tightening. Recall that in order to calculate the right hand

side of (53) with problem (58), we can either solve $\mathcal{O}(|V|^2)$ problems individually to obtain each b_{vw} or a smaller number problems after applying an aggregation step (13). In table 4 we compare the mean run times on instances GasLib-11, GasLib-24, and GasLib-40 when choosing to solve all problem *individually* or in an *aggregated* fashion. As with the preprocessing strategies, the numbers in brackets denote the relative speedup compared to the slowest method, see also fig. 4b. Each cell of table 4 is an average over all instances with varying approximation quality and uncertainty set. We observe a speedup of about 10× for the larger instances and a smaller speedup of about 2× for the smallest instance when using the aggregated model.

Table 4: Mean run times of individual and aggregated models after LP-based preprocessing. The number in brackets denotes the speedup compared to the individual model.

	individual	aggregated
GasLib-11	1.1 s (1.0×)	0.4 s (2.4×)
GasLib-24	8.5 s (1.0×)	0.8 s (10.0×)
GasLib-40	246.0 s (1.0×)	24.1 s (10.2×)

In total, a combination of LP-based bound tightening and an aggregation of subproblems can yield a mean speedup of about 300× compared to no preprocessing and solving all problems individually (see fig. 4).



(a) Mean speedup due to preprocessing

(b) Mean speedup due to aggregation after LP-based preprocessing

Figure 4: Different speedups due to preprocessing and aggregation. With both techniques, mean speedup can reach a factor of 300–400×.

Run times and number of subproblems The overall run time of the gas network problem mainly consists of run times of the preprocessing LPs, run times of the MIP subproblems, and run time of last LP for deciding a solution of the compressors. In our

setting, solving all occurring LP problems is trivial and can be done the fraction of a second. Therefore, we only focus on the performance of the MIP subproblems after the application of LP-based preprocessing.

Table 5 gives a more detailed summary over the mean run times for the individual and aggregated method. As usual, every cell is the average over all possible combinations of approximation quality and uncertainty set. The columns are partitioned into one group related to solving all problems individually and one group where the aggregated method is applied. In each column group, we list the number of required subproblems together with their mean and total run times. It can be observed that applying the aggregation method drastically reduces the number of subproblems required to solve. For the studied instances, the run times of the aggregated models increases at a smaller rate compared to the reduction of problems. Thus, the increase in complexity of the aggregated models is more than compensated by the reduction of instances.

Table 5: Number of MIP-subproblems for individual and aggregated models, together with their mean and total run times

	individual			aggregated		
	#probs	rt mean	rt total	#probs	rt mean	rt total
GasLib-11	110	0.010 s	1.063 s	7	0.061 s	0.430 s
GasLib-24	552	0.015 s	8.407 s	21	0.036 s	0.754 s
GasLib-40	1560	0.157 s	245.248 s	31	0.755 s	23.391 s

5 Concluding Remarks

We study a two-stage robust optimization problem with a special structure: first and second stage variables are not coupled, and the second stage variables are uniquely determined by the uncertain data. It is shown how this problem can be transformed to a single stage problem whose right hand side is the product of a series of optimization tasks. Even if the subproblems cannot be solved to global optimality, it is nevertheless possible to obtain conservative but robust feasible solutions of the overall problem by solving relaxations of the subproblems instead. Further exploitation of the problem structure leads to a reduction of subproblems by describing an aggregated model.

The developed approaches are applied to gas network problems with active elements under uncertainties in demand or pressure drop coefficients. After reformulating the gas network problem to match the special structure, it is shown how to generalize the previously known piecewise linearization techniques to incorporate uncertain parameters. This allows the formulation of relaxations of the non-convex and nonlinear gas network problem under uncertainty. In order to decrease the computational burden, we develop uncertainty-aware preprocessing by adapting techniques for the nominal case from the literature.

This work concludes with a series of benchmarks on freely available, medium sized

GasLib instances to demonstrate the practical feasibility of the approach. By combining optimization-based preprocessing techniques with the aggregation model, speedups of 300–400× can be achieved when compared to models without preprocessing and aggregation. Solving a robust 40-node instance is possible within the order of minutes, depending on uncertainty and approximation quality.

As an outlook, we remark that the abstract two-stage problem and the developed aggregation technique can be applied to any robust problem that admits this structure. Furthermore, the presented uncertainty-aware preprocessing and linearization techniques could be used on other potential driven network problems like e.g., water networks.

Acknowledgments

The authors thank the DFG for their support within Project B06 in CRC TRR 154. Furthermore, the authors would like to acknowledge the support of the ZISC.

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