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Analysis of mixed integer programming formulations for single machine scheduling problems with sequence dependent setup times and release dates

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Abstract In this article, six different mixed integer programming (MIP) formulations are proposed and analyzed. These formulations are based on the knowledge of four different paradigms for single machine scheduling problems (SMSP) with sequence dependent setup times and release dates. Each formulation reflects a specific concept on how the variables and parameters are defined, requiring particular settings and definitions. A thorough historical overview of a variety of formulations for this family of problems is provided. All MIP formulations studied are implemented and tested, considering, as objective functions, the weighted completion time and the total weighted tardiness. For the Sousa and Wolsey based formulation, a set of constraints to improve its lower bound value is adapted and evaluated. Extensive computational experiments are performed considering a variety of instances to capture several aspects of practical situations. Based on the results, recommendations are made for the best adaptation of the MIP formulation paradigm for the considered problems.

Keywords Mixed integer programming · Single machine scheduling · Sequence dependent setup times · Valid inequalities

Mathematics Subject Classification (2000) 90C11 · 90-02 · 90B35

1 Introduction

Scheduling research is concerned with the allocation of scarce resources to activities over time with the goal of optimizing one or more objectives. This huge family of problems is explicitly or implicitly present in countless applications, from production planning to bioinformatics related problems. Its study goes back to early 1950's

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were, from the point of view of Operations Research, the first problems on industrial applications began to be identified and formulated. This article deals with one of its simplest forms, a single machine environment. However when considering the presence of sequence-dependent setup times with release dates, the problem transforms itself in a very difficult combinatorial optimization problem.

As it can be seen in the literature review, several authors worked on similar problems, see Tables 1 and 2. Based on those previous works, specific formulations are proposed for two scheduling problems with sequence dependent setups and release dates. Each formulation reflects a specific concept on how the variables and parameters are defined, requiring particular changes and definitions.

As it is already well known, the setup consideration may cause a huge impact on the problem complexity. The single machine scheduling problem considering sequence dependent setup times with the objective of minimizing the maximum completion time (makespan) may be treated as the classical traveling salesman problem (TSP). The Öncan et al. (2009) survey that compares some MIP (mixed integer programming) formulations for the TSP problem is highlighted. This study focuses scheduling problems diverging from TSP. It deals with two objective functions: weighted completion time and total weighted tardiness, and with restrictions in scheduling problems from classical TSP formulations.

The problems studied here are denoted as $1|r_j, s_{ij}|\sum_j w_j C_j$ and $1|r_j, s_{ij}|\sum_j w_j T_j$ in accordance with Graham et al. (1979). The MIP formulation paradigms for scheduling problems are classified according to its developers: *(i)* Manne (1959) formulation *(ii)* Potts (1980) formulation *(iii)* Wagner (1959) formulation *(iv)* Sousa and Wolsey (1992) formulation and *(v)* Pessoa et al. (2010) formulation.

Manne formulations are characterized by continuous variables that define the completion time of each job; Potts formulations are defined by binary linear ordering variables that describe precedence relationships among all jobs. In the Wagner formulations, the decision variables are defined based on the notion that each machine has a fixed number of positions into which jobs can be assigned. Sousa and Wolsey formulations are characterized by assigning jobs to time periods belonging to a discretized planning horizon (H). Finally, Pessoa et al. formulations are also characterized by a discrete planning horizon (H), combining assigning jobs to time periods while considering precedence relationships.

A summary of the literature review with these MIP formulations approaches for scheduling problems is presented in Tables 1 and 2. Table 1 depicts research works in alphabetical order, with the same objective functions adopted in this article ($\sum_j w_j C_j$ and $\sum_j w_j T_j$) and Table 2 organizes other close related works.

Although a large number of machine scheduling articles and surveys have already appeared in literature, only a few of them compare several problems and MIP formulations. Queyranne et al. (1994) provides a review, performance analysis and a synthesis of polyhedral approaches to single and parallel machine scheduling problems without considering setup times. Their survey was based on Manne, Wagner, Potts and Sousa and Wolsey formulations. Khowala et al. (2005) compares the computational performance of the same MIP formulations that Queyranne et al. (1994) for the single machine total weighted tardiness problems without considering setup times.

Table 1 Previous specific research works for scheduling problems

MIP Formulations	Problem Parameters	Performance Measures $\sum_j w_j C_j$	Performance Measures $\sum_j w_j T_j$
Manne	no parameters	Keha et al. (2009), Queyranne and Wang (1991)	Keha et al. (2009), Khowala et al. (2005)
	r_j and no s_{ij}	Keha et al. (2009)	Keha et al. (2009)
	with s_{ij}	Queyranne (1993)	Queyranne (1993)
Potts	no parameters	Blazewicz et al. (1991), Chudak and Hochbaum (1999), Keha et al. (2009)	Blazewicz et al. (1991), Keha et al. (2009), Khowala et al. (2005), Keha et al. (2009)
	r_j and no s_{ij}	Dyer and Wolsey (1990), Keha et al. (2009), Queyranne et al. (1994), Unlu and Mason (2010)	
	with s_{ij}		Tanaka and Araki (2013)
Wagner	no parameters	Keha et al. (2009), Khowala et al. (2005), Lasserre and Queyranne (1992), Queyranne et al. (1994)	Keha et al. (2009)
	r_j and no s_{ij}	Keha et al. (2009)	Keha et al. (2009)
Sousa and Wolsey	no parameters	Keha et al. (2009), Khowala et al. (2005)	Bigras et al. (2008a), Keha et al. (2009), Razaq et al. (1990), Sadykov (2006), Sadykov and Vanderbeck (2011), Sourd (2009a), Sousa and Wolsey (1992), Tanaka et al. (2009)
	r_j and no s_{ij}	Avella et al. (2005), Keha et al. (2009), Queyranne et al. (1994)	Keha et al. (2009), Queyranne et al. (1994)
Pessoa et al.	no parameters		Pessoa et al. (2010)

Rocha et al. (2008) analyze the authors analyze parallel scheduling-problem formulations with setup times for Manne and Wagner based formulations; their performance is also compared against a Branch and Bound algorithm. Keha et al. (2009) compares the computational performance of the same MIP formulations of Queyranne et al. (1994) for some single machine scheduling problems without considering setup times. Unlu and Mason (2010) defines and compares computational results for four different MIP formulations based on Wagner, Potts, Sousa and Wolsey and Cakici and Mason, which are presented for various parallel machine scheduling problems without considering setup times. Blazewicz et al. (1991); Allahverdi et al. (1999, 2008) and Pinedo (2008) are also referred.

The purpose of this study is to propose and test six specific MIP formulations for the single machine scheduling problem with sequence dependent setup times and release dates. Furthermore, a new set of inequalities is adapted in order to improve Sousa and Wolsey (1992)'s lower bound in the presence of sequence dependent setup times.

2 Mathematical Formulations

A set J of n jobs is considered, where each job must be exactly processed once in a single machine that can handle one job at a time without preemption. For a given job j, let p_j be its processing time, d_j its due date, w_j its priority or weight, r_j its release date. s_{ij} is also defined as the setup time needed to process the job j immediately after job i, C_j its completion time, C_{max} the maximum completion time (makespan) of a

Table 2 Previous general research works for scheduling problems

MIP Formulations	Problem Parameters	Performance Measures Other Objective Functions
Manne	no parameters	Keha et al. (2009), Manne (1959)
	r_j and no s_{ij}	Balakrishnan et al. (1999), Dyer and Wolsey (1990), Keha et al. (2009), Zhu and Heady (2000)
	with s_{ij}	Ascheuer et al. (2001), Balas (1985), Balas et al. (2008), Ballicu et al. (2002), Choi and Choi (2002), Eijl Van (1995), Eren and Guner (2006), Larsen (1999), Maffioli and Sciomachen (1997), Naderi et al. (2011), Queyranne (1993), Queyranne et al. (1994), Ríos Mercado and Bard (2003), Rocha et al. (2008), Zhu and Heady (2000)
Potts	no parameters	Keha et al. (2009)
	r_j and no s_{ij}	Keha et al. (2009), Nemhauser and Savelsbergh (1992), Sadykov (2006)
Wagner	no parameters	Keha et al. (2009), Wagner (1959)
	r_j and no s_{ij}	Dauzère Pérès and Sevaux (2003), Keha et al. (2009), Soríc (2000), Unlu and Mason (2010)
	with s_{ij}	Jr. and Tseng (2002), Lee and Asllani (2004), Rocha et al. (2008), Shufeng and Yiren (2002)
	Others	Abdekhodaee and Wirth (2002), İşler et al. (2012), Pan et al. (2001), S Sakuraba et al. (2009)
Sousa and Wolsey	no parameters	Anglani et al. (2005), Chen and Luh (2003), Czerwinski and Luh (1994), Hoitomt et al. (1990), Keha et al. (2009), Luh and Hoitomt (1993), Pan and Shi (2007), Sourd (2009a), Sousa and Wolsey (1992), Tanaka and Araki (2008), Tanaka et al. (2009), Wang and Luh (1997)
	r_j and no s_{ij}	Crauwels et al. (2006), Detienne et al. (2010), Jin and Luh (1999), Keha et al. (2009), Liu and Chang (2000), Queyranne et al. (1994), Unlu and Mason (2010), van den Akker et al. (1999)
	with s_{ij}	Buil et al. (2012), Hoitomt et al. (1993), Liu and Chang (2000), Luh and Hoitomt (1993), Luh et al. (1998), Luh et al. (1999), Luh et al. (2000), de Paula et al. (2010), Sun et al. (1999)
Pessoa et al.	With s_{ij}	Abeledo et al. (2010), Abeledo et al. (2013), Battarra et al. (2013), Bigras et al. (2008b)

given solution and T_j the tardiness of job j. In two MIP formulation paradigms, M is used as a very large constant and its value is analyzed for each case. Other specific notation will be discussed and introduced when needed.

As previously mentioned, the problems studied in this article are denoted as $1|r_j, s_{ij}|\sum_j w_jC_j$ and $1|r_j, s_{ij}|\sum_j w_jT_j$ and they are $NP-hard$ (Lenstra et al., 1977; Lawler et al., 1993; Pinedo, 2008). It is important to point out that, with the exception of Wagner and Pessoa et al., all formulations require a setup time data that satisfy the triangle inequality $s_{ij} \leq s_{il} + p_l + s_{lj}$, where i, j and $l \in J$ and $i \neq j \neq l$.

These formulations are going to be directly tested with a commercial solver. The selection of the value of M is essential for the performance of the mathematical models. To set the value of M in the problems, $1|r_j, s_{ij}|\sum_j w_jC_j$ and $1|r_j, s_{ij}|\sum_j w_jT_j$, the propositions 1, 2 and 3 are submitted.

Before presenting these propositions, two important scheduling concepts will be defined: regular criterion and active scheduling. An objective function is said to be regular when its value is a function that is nondecreasing in its completion times, (Pinedo, 2008). This concept ensures the existence of optimal schedules and avoids anomalies like infinite number of preemptions (Baptiste et al. (2009)). Both, $\sum_j w_jC_j$ and $\sum_j w_jT_j$, are regular criteria (see Leung (2004)). Following Baker (1974) and

Pinedo (2008) a schedule S is active if by changing the order of jobs, it is not possible to construct a schedule with at least one job finishing earlier without delaying another job. Therefore, these scheduling problems can be understood as the problem of finding an active schedule, where no job can be shifted to the left to improve the objective function, without making the schedule infeasible (Leung (2004)).

The proposition 1 determines that the optimal solution of the weighted objective function (i.e., $\sum_j w_j C_j$ and $\sum_j w_j T_j$) is not necessarily the smallest makespan. Finally, propositions 2 and 3 define the value of the constant M for the Manne and Wagner formulations.

Proposition 1 *Considering the problems* $1|r_j,s_{ij}|\sum_j w_j C_j$ *and* $1|r_j,s_{ij}|\sum_j w_j T_j$*, let* S^* *and* S *be, respectively, an optimal and a feasible solution for one of the problems. Then, the makespan of* S^**, (*$C_{max}(S^*)$*) is not necessarily smaller than the makespan of* S*, (*$C_{max}(S)$*).*

Proof Considering the problem $1|r_j,s_{ij}|\sum_j w_j C_j$ (this proof has the same logic for $1|r_j,s_{ij}|\sum_j w_j T_j$). Let S^* and S be, respectively, an optimal and a feasible solution; where the only difference between them is the sequence of the two last adjacent jobs i and j. Job i is scheduled immediately before j in S^* and immediately after j in S. Moreover, it is considered that there is a job k scheduled immediately before the last two jobs. Considering $s_{ij} > s_{ji}$ and $s_{ki} = s_{kj}$, it is possible that a $w_i > w_j$ exists such that $w_i C_i^* + w_j C_j^* < w_i C_i + w_j C_j$. Thus, $f(S^*) < f(S)$ with $C_{max}(S^*) > C_{max}(S)$.□

By the proposition 1, the optimal solution can be any active schedule, not necessarily the smallest. Thus, the constants defined as M_{ij} for the Manne formulation (proposition 2) and M'_k for the Wagner formulation (proposition 3) are makespan's upper bound values that should allow it. These M values are presented together with their respective MIP formulations.

The MIP formulation paradigms considered in this work are Manne, Wagner, Sousa and Wolsey and Pessoa et al. There is a fifth paradigm formulation, proposed by Potts (1980), but when a single machine scenario is considered the formulation is equivalent to the Manne's one. For this reason, this approach is not investigated in this article.

2.1 Manne Formulation

This formulation was originally proposed by Manne (1959) for the Jobshop scheduling problem $J||C_{max}$. The adapted formulation is based on previous works by Manne (1959); Queyranne (1993); Queyranne et al. (1994); Ballicu et al. (2002); Khowala et al. (2005); Eren and Guner (2006); Balas et al. (2008); Rocha et al. (2008); Keha et al. (2009) and Unlu and Mason (2010). In this formulation, the binary variables γ_{ij}, are equal to 1 if job i is processed before job j and equal to 0 otherwise. When $\gamma_{ij} = 1$, job i is not necessarily positioned immediately before job j. The constraint set of the MIP formulation is the following:

$$C_j \geq C_i + s_{ij} + p_j - M_{ij}(1 - \gamma_{ij}) \quad \forall i, j \in J,\ i \neq j, \tag{1}$$

$$\gamma_{ij} + \gamma_{ji} = 1 \;\; \forall i, j \in J, i < j, \tag{2}$$

$$C_j \geq r_j + p_j \;\; \forall j \in J, \tag{3}$$

$$C_j \geq 0 \;\; \forall j \in J, \tag{4}$$

$$T_j \geq C_j - d_j \;\; \forall j \in J, \tag{5}$$

$$T_j \geq 0 \;\; \forall j \in J, \tag{6}$$

$$\gamma_{ij} \in \{0,1\} \;\; \forall i, j \in J, i \neq j. \tag{7}$$

The constraint set (1) ensures that the completion time of job j happens only after the completion time of job i plus the setup from job i to job j and the processing time of job j. The constraint set (2) imposes that either job i is positioned before job j or otherwise. The constraint set (3) ensures that completion time of job j is greater than or equal to its release date plus its processing time. The constraint set (4) is the non-negativity constraint. The constraint set (5) implies that the tardiness of job j is greater than or equal to the difference between its completion time and its due date. The constraint set (6) ensures that the tardinesses of job j is positive. The constraint set (7) is the integrality constraint.

Proposition 2 *For all job ordered pairs (i, j), such that $i, j \in J$, $i \neq j$ and i is processed before j, in the Manne formulation, if*

$$M_{ij} = M_i - r_j + s_{ij}, \tag{8}$$

with

$$M_i = \max\{r_i, r^{max}_{l \in J, l \neq i} + \sum_{l \in J, l \neq i} s^{max}_l + \sum_{l \in J, l \neq i} p_l\} + p_i, \tag{9}$$

where $s^{max}_l = \max_{i \in J, i \neq l}\{s_{li}\}$ and $r^{max}_{l \in J, l \neq i} = \max_{l \in J, l \neq i}\{r_l\}$, thus all active schedules are feasible solutions of the mathematical problem.

Proof Let $S = [j_1, \ldots, j_{n-1}, j_n]$ be an active schedule n-jobs, which can be represented by ordered pairs (i, j), such that i is processed before j. For any pair $(i, j) \in S$, the job's j completion time (C_j) is at least the maximum between $C_i + p_j + s_{ij}$ and $r_j + p_j$. Thus, C_{j_n} can be defined for the ordered pair $(j_{n-1}, j_n) \in S$ as

$$C_{j_n} = \max\{r_{j_n}, C_{j_{n-1}} + s_{j_{n-1}, j_n}\} + p_{j_n}. \tag{10}$$

Considering $s^{max}_j = \max_{i \in S, i \neq j}\{s_{ji}\}$ and $r^{max}_{j \in S, j \neq j_n} = \max_{j \in S, j \neq j_n}\{r_j\}$, M_{j_n} can be defined as an upper bound for the makespan, when j_n is the last job to be processed, as

$$M_{j_n} = \max\{r_{j_n}, r^{max}_{j \in S, j \neq j_n} + \sum_{j \in S, j \neq j_n} s^{max}_j + \sum_{j \in S, j \neq j_n} p_j\} + p_{j_n}, \tag{11}$$

and as

$$r^{max}_{j,j\neq j_n} + \sum_{j\in S, j\neq j_n} s^{max}_j + \sum_{j\in S, j\neq j_n} p_j \geq C_{j_{n-1}} + s_{j_{n-1},j_n}, \tag{12}$$

therefore

$$M_{j_n} \geq C_{j_n}. \tag{13}$$

In the Manne formulation, the constraint set (1) is not enough to define the M value as M_j. If $\gamma_{ij} = 0$, the constraint set (2) indicates that the job i is processed after the job j, as $\gamma_{ij} + \gamma_{ji} = 1$. Thus, it is possible that the job i is the last job in the schedule, thereby C_i can be equal to M_i.

$$C_j \geq C_i + s_{ij} + p_j - M_j(1-\gamma_{ij}), \tag{14}$$

therefore, the constraint set (14) can be rewritten as

$$C_j \geq M_i + s_{ij} + p_j - M_j, \tag{15}$$

however, the constraint set (3) defines that $C_j \geq r_j + p_j$, but if,

$$M_i + s_{ij} - M_j > r_j, \tag{16}$$

the constraint may discard feasible solutions. Therefore, it is necessary to define a constant M_{ij} instead of M_j. The value M_{ij} is defined as

$$M_{ij} = M_i - r_j + s_{ij}, \tag{17}$$

it is easy to see that M_{ij} will satisfy the condition imposed by the constraint set (3).□

For the Manne formulation, if the problem is $1|r_j, s_{ij}|\sum_j w_j C_j$, the variables C_j, γ_{ij} and the constraint sets (1), (2), (3), (4) and (7) are required. Therefore, there are $O(n^2)$ decision variables and constraints. The problem $1|r_j, s_{ij}|\sum_j w_j T_j$ needs the variables C_j, γ_{ij}, T_j and constraint sets (1), (2), (3), (4), (5), (6) and (7). Thus, there are $O(n^2)$ decision variables and constraints.

Based on the works of Eijl Van (1995), Maffioli and Sciomachen (1997) and Ascheuer et al. (2001), a different formulation is proposed to reduce the effects of the constant M. These works are focused in the "Asymmetric Travelling Salesman Problem with Time Windows". This modification is based on the variables C_j, that are redefined as C_{ji} with the property that $\gamma_{ij} = 0$ implies $C_{ij} = 0$. If $\gamma_{ij} = 1$ then C_{ij} denotes the time when the processing of job i is completed and indicates that the job j is processed after job i. This modified formulation uses a fictitious job "0" indicating the starting and ending point of the sequence (all parameter values are null for a fictitious job), then J' is defined as $J \bigcup 0$. The formulation will be denominated "Manne Alternative" and its constraint set is the following:

$$\sum_{i\in J, i\neq j} C_{ij} + \sum_{i\in J', i\neq j} (p_j + s_{ij})\gamma_{ij} \leq \sum_{k\in J', k\neq j} C_{jk} \quad \forall j \in J, \tag{18}$$

$$\sum_{j \in J', i \neq j} \gamma_{ij} = 1 \ \ \forall i \in J', \tag{19}$$

$$\sum_{i \in J', i \neq j} \gamma_{ij} = 1 \ \ \forall j \in J', \tag{20}$$

$$\gamma_{ij}(r_i + p_i) \leq C_{ij} \leq \gamma_{ij} M_i \ \ \forall i, j \in J', i \neq j, \tag{21}$$

$$T_j \geq \sum_{i \in J, j \neq i} C_{ji} - d_j \ \ \forall j \in J', \tag{22}$$

$$T_j \geq 0 \ \ \forall j \in J', \tag{23}$$

$$\gamma_{ij} \in \{0, 1\} \ \ \forall i, j \in J', i \neq j. \tag{24}$$

The constraint set (18) has the same meaning as (1). The constraint sets (19) and (20) establish that each job is succeeded and preceded by one job respectively. The constraint set (21) defines the C_{ij} domain, which has M_i as fictitious job i deadline. The constraint sets (22), (23) and (24) have the same meaning as (5), (6) and (7).

In Manne Alternative formulation, if the problem is $1|r_j, s_{ij}|\sum_j w_j C_j$ needs the variables C_{ij}, γ_{ij} and constraint sets (18), (19), (20), (21) and (24). Therefore, there are $O(n^2)$ decision variables and constraints. The problem $1|r_j, s_{ij}|\sum_j w_j T_j$ needs the variables C_{ij}, γ_{ij}, T_j and constraint sets (18), (19), (20), (21), (22), (23) and (24). Thus, there are $O(n^2)$ decision variables and constraints.

2.2 Wagner Formulation

This formulation was initially proposed by Wagner (1959) for the Jobshop scheduling problem $J||C_{max}$. The formulation is adapted from the knowledge acquired from Wagner (1959); Queyranne et al. (1994); Dauzère Pérès and Sevaux (2003); Lee and Asllani (2004); Khowala et al. (2005); Rocha et al. (2008); Keha et al. (2009) and Unlu and Mason (2010). In this formulation a set K of n processing positions is defined ($K = \{1, ..., n\}$) and the binary assignment variables (ν_{jk}) are equal to 1 if job j is assigned to position k and equal to 0 otherwise. The variables β_{ij}^{k} define the assignment of job i to the position k and job j to the position $k+1$, and y_k defines the completion time of the job in the position k. The constraint sets of the MIP formulation are the following:

$$\sum_{k \in K} \nu_{jk} = 1 \ \ \forall j \in J, \tag{25}$$

$$\sum_{j \in J} \nu_{jk} = 1 \ \ \forall k \in K, \tag{26}$$

$$y_k \geq \sum_{j \in J} (r_j + p_j) \nu_{jk} \ \ \forall k \in K, \tag{27}$$

$$\beta_{ij}^{k-1} \geq 1 - (2 - \nu_{i(k-1)} - \nu_{j(k)})$$
$$\forall i, j \in J, i \neq j, k \in \{2, \ldots, n\} \subset K, \tag{28}$$

$$y_k \geq y_{k-1} + \sum_{j \in J} p_j \nu_{jk} + \sum_{i \in J} \sum_{\substack{j \in J, \\ j \neq i}} \beta_{ij}^{k-1} s_{ij} \ \ \forall k \in \{2, \ldots, n\} \subset K, \tag{29}$$

$$C_j \geq y_k - M'_k(1 - \nu_{jk}) \ \ \forall k \in K, j \in J, \tag{30}$$

$$C_j \geq 0 \ \ \forall j \in J, \tag{31}$$

$$T_j \geq y_k - d_j - M'_k(1 - \nu_{jk}) \ \ \forall k \in K, j \in J, \tag{32}$$

$$T_j \geq 0 \ \ \forall j \in J, \tag{33}$$

$$y_k \geq 0 \ \ \forall k \in K, \tag{34}$$

$$\beta_{ij}^{k} \in \{0, 1\} \ \ \forall k \in K, i, j \in J, i \neq j, \tag{35}$$

$$\nu_{jk} \in \{0, 1\} \ \ \forall j \in J, k \in K. \tag{36}$$

The constraint sets (25) and (26) establish that a job is exactly assigned to one position and each position is assigned to one job. The constraint set (27) ensures that the completion time of a job at the position k is greater than or equal to its release date plus its processing time. The constraint set (28) establishes the use of a setup time from job i to job j between the positions $k-1$ and k if job i is at the position $k-1$ and job j is at the position k. The constraint set (29) computes the completion time for the jobs at the positions $2, \ldots, n$. The constraint set (30) ensures the association of the completion time of the job j with its assigned position, and the constraint set (31) ensures that the completion time of job j is non-negative. The constraint set (32) establishes that if the job j is in the position k, the tardiness of the job j is greater than or equal to the difference between its completion time in the position k and its due date. The constraints (33) and (34) are non-negativity constraints. The constraint sets (35) and (36) are integrality constraints.

Proposition 3 *In the Wagner formulation, for each position $k \in K$, if*

$$M'_k = \max_{j \in J}^{1}\{p_j + r_j\} + \max_{j \in J}^{k-1}\{p_j + s_j^{max}\}, \tag{37}$$

where $\max_{j \in J}^{l}$ is the sum of the l larger values of $\{p_j + r_j\}$, for $j \in J$, $\max_{j \in J}^{0} = 0$ and $s_j^{max} = \max_{i \in J, i \neq j}\{s_{ji}\}$, then all active schedules are feasible solutions of the mathematical problem.

Proof For each position $k \in K$, and considering the constraint sets (30 and 32), an upper bound M'_k for the completion time at position k can be defined. At the first position, $k = 1$, no jobs will be completed after

$$M'_1 = \max_{j \in J}^{1}\{p_j + r_j\}. \quad (38)$$

In the second position the limit is

$$M'_2 = \max_{j \in J}^{1}\{p_j + r_j\} + \max_{j \in J}^{1}\{p_j + s_j^{max}\}, \quad (39)$$

where $s_j^{max} = \max_{i \in J, i \neq j}\{s_{ji}\}$. In the third position the limit is

$$M'_3 = \max_{j \in J}^{1}\{p_j + r_j\} + \max_{j \in J}^{2}\{p_j + s_j^{max}\}. \quad (40)$$

Thereby, generalizing for all $k \in K$ positions,

$$M'_k = \max_{j \in J}^{1}\{p_j + r_j\} + \max_{j \in J}^{k-1}\{p_j + s_j^{max}\}, \quad (41)$$

which define the job completion time upper bound at position k, y_k, for all $k \in K$ positions.□

Rocha et al. (2008) defines that $\beta_{ij}^k \in \{0,1\}$, however the proposition 4 ensures that the integrality constraints can be relaxed without compromising the integrality of the problem.

Proposition 4 *In the Wagner formulation, the integrality constraints of variables* $\beta_{ij}^k \in \{0,1\}$*, can be relaxed without interfering with the solution integrality.*

Proof When the integrality conditions of the binary variables β_{ij}^k are relaxed $0 \leq \beta_{ij}^k \leq 1$; By the constraint set (28) it is obtained,

$$\beta_{ij}^k \geq \left\{ \begin{array}{l} 1, \text{ if } v_{i(k-1)} = 1 \text{ and } v_{j(k)} = 1 \\ 0, \text{ if } v_{i(k-1)} = 1 \text{ or } v_{j(k)} = 1 \\ 0, \text{ if } v_{i(k-1)} = 0 \text{ and } v_{j(k)} = 0 \end{array} \right\}. \quad (42)$$

When $\beta_{ij}^k > 0$, the constraint set (29) increases the value of y_k and consequently the objective function. Thus, β_{ij}^k will be 0 or 1. Therefore, β_{ij}^k is defined as:

$$\beta_{ij}^k = \left\{ \begin{array}{l} 1, \text{ if } v_{i(k-1)} = 1 \text{ and } v_{j(k)} = 1 \\ 0, \text{ if } v_{i(k-1)} = 1 \text{ or } v_{j(k)} = 1 \\ 0, \text{ if } v_{i(k-1)} = 0 \text{ and } v_{j(k)} = 0 \end{array} \right\}, \quad (43)$$

thereby, even if $0 \leq \beta_{ij}^k \leq 1$ the integrality of this variable is guaranteed. □

If the problem is $1|r_j, s_{ij}|\sum_j w_j C_j$, the variables v_{jk}, y_k, C_j, β_{ij}^k and the constraint sets (25), (26), (27), (28), (29), (30), (31), (34), (35) and (36) are required. Therefore, there are $O(n^3)$ decision variables and constraints. The problem $1|r_j, s_{ij}|\sum_j w_j T_j$ needs the variables v_{jk}, y_k, T_j, β_{ij}^k and constraint sets (25), (26), (27), (28), (29), (32), (33), (34), (35) and (36). Thus, there are $O(n^3)$ decision variables and constraints.

2.3 Sousa and Wolsey Formulation

This approach has been first proposed by Sousa and Wolsey (1992) for single machine problems $1|r_j|\sum_j w_jC_j$ and $1|r_j|\sum_j w_jT_j$. The formulation is based on studies of Sousa and Wolsey (1992); Queyranne et al. (1994); van den Akker et al. (1999); Avella et al. (2005); Khowala et al. (2005); Pan and Shi (2007); Keha et al. (2009); de Paula et al. (2010) and Unlu and Mason (2010). In the Sousa and Wolsey formulation, the planning horizon is discretized for each job i into the periods $0,\ldots,h_i$ and the h_i constant has the same value as M_i (view proposition 2). The set of the periods is defined as $H = \{0,\ldots,\max_{i\in J}\{h_i\}\}$. The binary time index variables, x_{jt}, are defined. x_{jt} is equal to 1 if job j starts at time t and equal to 0 otherwise. The constraint sets of the MIP formulation are defined as follows:

$$\sum_{t=r_j}^{h_j-p_j+1} x_{jt} = 1 \;\; \forall \in J, \tag{44}$$

$$x_{jt} + \sum_{s=\max\{r_i,t-p_i-s_{ij}+1\}}^{\min\{t+p_j+s_{ji}-1,h_i-p_i+1\}} x_{is} \leq 1$$
$$\forall i,j \in J, i \neq j, t \in \{r_j,\ldots,h_j-p_j+1\} \subset H, \tag{45}$$

$$C_j \geq \sum_{t=r_j}^{h_j-p_j+1} (t+p_j)x_{jt} \;\; \forall j \in J, \tag{46}$$

$$C_j \geq 0 \;\; \forall j \in J, \tag{47}$$

$$T_j \geq \sum_{t=r_j}^{h_j-p_j+1} (t+p_j)x_{jt} - d_j \;\; \forall j \in J, \tag{48}$$

$$T_j \geq 0 \;\; \forall j \in J, \tag{49}$$

$$x_{jt} \in \{0,1\} \;\; \forall j \in J, t \in \{r_j,\ldots,h_j-p_j+1\} \subset H. \tag{50}$$

The constraint set (44) ensures that the processing of each job starts at only one time period in the machine. The constraint set (45) ensures that if the job j is scheduled in the time period t, no other job i ($i \neq j$) can be scheduled between $t-p_i-s_{ij}+1$ and $t+p_j+s_{ji}-1$ periods. The constraint set (46) ensures a completion time greater than or equal to its starting time plus its processing time. The constraint set (47) ensures the non-negativity of variables C_j. The constraint set (48) ensures that the tardiness of job j is greater than or equal to the difference between its completion time and its due date. The constraint set (49) ensures non-negativity constraints. The constraint set (50) ensures the integrality domain of x_{jt}.

If the problem is $1|r_j,s_{ij}|\sum_j w_jC_j$, the variables x_{jt}, C_j and the constraint sets (44), (45), (46), (47) and (50) are required. Therefore, there are $O(nh)$ decision variables and $O(n^2h)$ constraints ($h >>> n$). The problem $1|r_j,s_{ij}|\sum_j w_jT_j$ needs the variables x_{jt}, T_j and constraint sets (44), (45), (48), (49) and (50). Thus, there are $O(nh)$ decision variables and $O(n^2h)$ constraints.

2.3.1 Improved Formulation

Khowala et al. (2005); Keha et al. (2009) and Unlu and Mason (2010) showed that the lower bounds obtained from the formulations using the Sousa and Wolsey formulation were tight, but the LP (linear programming) relaxations were harder to resolve. However, the computational results from de Paula et al. (2010) suggested that when sequence dependent setup times are introduced, the LP relaxation of Sousa and Wolsey (1992) formulation bounds are not as tight. To improve this formulation, one family of valid inequalities was introduced to help improving the lower bounds obtained when considering sequence dependent setup times.

The Sousa and Wolsey formulation presents some negative points: *(i)* the formulation size depends on the length of the planning horizon (h), due to the number of variables and constraints (memory requirements), and *(ii)* the integrality relaxation of constraint set (45) allows that several jobs can be sequenced simultaneously. The later implies in a poor lower bound. This situation is observed in Sun et al. (1999), which discusses a lagrangian relaxation approach for the single machine scheduling problem with sequence dependent setup to minimize total weighted squared tardiness. Similarly, a constraint set is adapted to improve the model's lower bound performance.

The new constraint set (51) ensures that when the integrality of variables x_{it} is relaxed, the number of assigned jobs $i \in J$ between $\max\{t - p_i - SMin_i + 1, r_i\}$ and $\min\{t, h_i - p_i + 1\}$ is at most 1, where $SMin_i$ is the minimum setup time from $i \in J$ for any $j \in J, j \neq i$. If the variables x_{it} are not relaxed the proposed constraints are redundant.

Proposition 5 *For all t in H, the inequality*

$$\sum_{i \in J} \sum_{s=\max\{t-p_i-SMin_i+1, r_i\}}^{\min\{t, h_i-p_i+1\}} x_{is} \leq 1 \quad \forall t \in H \tag{51}$$

is valid.

Proof As $SMin_i = \min_{j \in J}\{s_{ij}\}$ for any pair of jobs $i, j \in J$, i.e., does not depend on the job sequence, then $p_i^{'} = p_i + SMin_i$ can be defined. Therefore, the constraint set (51) can be rewritten as

$$\sum_{i \in J} \sum_{s=\max\{t-p_i^{'}+1, r_i\}}^{\min\{t, h-p_i+1\}} x_{is} \leq 1 \quad \forall t \in H. \tag{52}$$

Sousa and Wolsey (1992) define the constraint set for single machine scheduling problem without setup times as

$$\sum_{i \in J} \sum_{s=\max\{t-p_i+1, r_i\}}^{\min\{t, h_i-p_i+1\}} x_{is} \leq 1 \quad \forall t \in H, \tag{53}$$

which is identical to (52) that is a particular case of (45) when all setup times are null or do not depend on the job sequence. □

This formulation will be hereinafter referred to as “Sousa and Wolsey Improvement” formulation. The constraint set (51) is included in this formulation. The size of its variable and constraint set for single machine scheduling problems $1|r_j,s_{ij}|\sum_j w_jC_j$ ($O(n^2)$ decision variables and $O(n^2)$ constraints) and $1|r_j,s_{ij}|\sum_j w_jT_j$ ($O(n^2)$ decision variables and $O(n^2)$ constraints) are the same as of the Sousa and Wolsey formulation.

2.4 Pessoa et al. Formulation

This formulation is based on Tanaka and Araki (2008), Sourd (2009a) and Pessoa et al. (2010). It is important to mention the works of Fox (1973) and Fox et al. (1980) on the time-dependent traveling salesman problem (TDTSP) and its adaptation for single machine scheduling problems by Bigras et al. (2008b). As in the Sousa and Wolsey formulation, the planning horizon is discretized into the periods $\{0,\ldots,\max_{i\in J}\{h_i\}\}$. It defines a binary arc time indexed variables, x^t_{ij} with $i \neq j$, which indicate that the job j starts at the time t and job i is positioned immediately before j. This formulation uses a fictitious job “0” which is the starting and ending point of the sequence in an adapted formulation (all parameters are null for a fictitious job). The variables x^t_{jj} indicate that the machine was idle from $t-1$ to t and the last processed job was j. Therefore, another change is necessary to define a new parameter $s^{'}_{ij}$, which is p_i+s_{ij} if $i \neq j$ or 1 if $i=j$ (idle from $t-1$ to t). Finally, set J' is defined as $J\bigcup 0$. The constraint sets of the MIP formulation are the following:

$$\sum_{\substack{i\in J',\\ i\neq j}} \sum_{t=\max\{r_i+s^{'}_{ij},r_j\}}^{h_j-p_j} x^t_{ij} = 1 \quad \forall j \in J', \tag{54}$$

$$\sum_{\substack{j\in J',\\ t\geq r_j+s^{'}_{ji}}} x^t_{ji} - \sum_{\substack{j\in J',\\ r_j\leq t+s^{'}_{ij}\leq h_j-p_j}} x^{t+s^{'}_{ij}}_{ij} = 0$$

$$\forall i \in J, t \in \{r_i,\ldots,h_i-p_i\} \subset H, \tag{55}$$

$$C_j \geq \sum_{\substack{i\in J',\\ i\neq j}} \sum_{t=\max\{r_i+s^{'}_{ij},r_j\}}^{h_j-p_j} (t+p_j)x^t_{ij} \quad \forall j \in J, \tag{56}$$

$$C_j \geq 0 \quad \forall j \in J, \tag{57}$$

$$T_j \geq \sum_{\substack{i\in J',\\ i\neq j}} \sum_{t=\max\{r_i+s^{'}_{ij},r_j\}}^{h_j-p_j} (t+p_j)x^t_{ij} - d_j \quad \forall j \in J, \tag{58}$$

$$T_j \geq 0 \quad \forall j \in J, \tag{59}$$

$$x_{ij}^t \in \{0,1\} \quad \forall i,j \in J' \text{ with } i \neq 0 \text{ or } j \neq 0, \\ t \in \{\max\{r_i + s_{ij}^{'}, r_j\}, \ldots, h_j - p_j\} \subset H. \tag{60}$$

The constraint set (54) establishes that every job must be processed; the constraint set (55) ensures that if job i is scheduled in the time period t, the next job in the sequence j ($i \neq j$) or i (idle machine) must be scheduled in the time period $t + s_{ij}^{'}$ or $t + s_{ii}^{'}$ respectively. The constraint set (56) ensures that the completion time is greater than or equal to its starting time plus its processing time. The constraint set (57) indicates the non-negativity domain. The constraint set (58) ensures that the tardiness of job j is greater than or equal to the difference between its completion time and its due date. The constraint sets (59) and (60) are the non-negativity and integrality constraints.

If the problem is $1|r_j, s_{ij}|\sum_j w_j C_j$ the variables x_{ij}^t, C_j and the constraint sets (54), (55), (56), (57) and (60) are required. Therefore, there are $O(n^2 h)$ decision variables and $O(nh)$ constraints. The problem $1|r_j, s_{ij}|\sum_j w_j T_j$ needs the variables x_{ij}^t, T_j and constraint sets (54), (55), (58), (59) and (60). Thus, there are $O(n^2 h)$ decision variables and $O(nh)$ constraints.

3 Computational Results

An extensive computational experiment is performed to capture the strength and weaknesses of each paradigm. A specific benchmark including different features and characteristics was created for this purpose.

3.1 Benchmark

Six different classes of instances are artificially created. All parameters of the instances are randomly generated from an uniform distribution and their minimal and maximal values are based on specific scale parameters. A similar methodology can be found in Hariri and Potts (1983); Potts and Wassenhove (1983); Razaq et al. (1990); Ho and Chang (1995); Pereira Lopes and de Carvalho (2007); Rocha et al. (2008) and Unlu and Mason (2010). The instance classes and its scale parameters are listed in Table 3.

Table 3 Distribution values of the instances

Input data	*Distribution value*
Processing Time (p_j)	$U(1, \alpha_1 50)$
Setup time (s_{ij})	$U(1, \alpha_2 10)$
Priority (w_j)	$U(1, n)$
Release date (r_j)	$U(0, \frac{\alpha_3 h'}{10})$
Due date (d_j)	$U(\max_j(p_j), \frac{2h'}{\alpha_4})$

The h' was defined as the sum of processing times plus the sum of maximum setup times ($\sum_j p_j + \sum_i \max_j(s_{ij})$). The scale parameters α_1, α_2, α_3 and α_4 define the distribution scenario of "Processing Time", "Setup time", "Release date " and "Due date" respectively. The parameter $\alpha_1 \in \{1,4\}$ modifies the process time extent, $\alpha_2 \in \{1,5\}$ defines the setup time impact, $\alpha_3 \in \{1,4\}$ the availability level and $\alpha_4 \in \{1,4\}$ the congestion level.

In each class (1 to 6) there is a change in one scale parameter. The created classes are namely:

Class 1: all scale parameters have minimum values;
Class 2: α_1 has the maximum value (4) and other scale parameters have minimum values;
Class 3: α_2 has the maximum value (5) and other scale parameters have minimum values;
Class 4: α_3 has the maximum value (5) and other scale parameters have minimum values;
Class 5: α_4 has the maximum value (4) and other scale parameters have minimum values;
Class 6: all scale parameters have maximum values.

Each class presents special characteristics. The "Class 1" is our base scheduling system. The "Class 2" considers a long planning horizon and the system is slightly affected by setup times. This class is closer to single machine scheduling problems without setup times ($p_j >>> s_{ij}$). The "Class 3" considers a moderate planning horizon with setup times having a great impact in the scheduling system. This class is closest to the traveling salesman problem. The "Class 4" presents a moderate planning horizon with longer release dates. The "Class 5" defines a scheduling system with high congestion level, reducing its due date values. The "Class 6" determines a scheduling system with emphasized conditions. The later defines a complex scheduling system, presenting long planning horizons, a moderate impact of setup times, an impact on the job's release dates and a considerable congestion level.

For each class, ten independent instances are considered with size $n \in \{5,7,9,11,13,15,20,30,50,75,100\}$. Thus, 660 instances are randomly and independently generated. All instances are slightly modified to satisfy the triangle inequality of the setup times ($s_{ij} \leq s_{ik} + p_k + s_{k,j}$, where i, j and $k \in J$ and $i \neq j \neq k$).

3.2 Results

The mathematical formulations were modeled and solved using AMPL and CPLEX 12.1 with default settings. The experiments were run on a Linux Maya with a single 2.4 GHz processor and 4GB memory. The runs were concluded after one hour of CPU time.

To analyze the differences between the formulations, it was made a comparison of the optimality gap within 3600 seconds, the linear programming relaxation gap, CPU times and its size. The linear programming relaxation gap is defined as the relative difference between the best integer solution found for each instance and the LP (linear programming) relaxation value. The average results of the experiments are presented in Tables 4 and 5.

The Table 4 depicts the average *GAP* results for the two problems considering both problems for each instance class, while Table 5 shows the average results for each size. It must be highlighted that in several occasions the time-indexed formulations (“Sousa and Wolsey”, “Sousa and Wolsey Improvement” and “Pessoa et al.”) were unable to load the whole problem into the solver. In those cases the *GAP* was defined as 100% and its computational time was defined as 3600 seconds. Individual results for each class and each instance size are presented in the supplementary material.

In the analysis of the LP relaxation, Tables 4 and 5, it is possible to observe that the “Pessoa et al.” formulation presents generally better $\overline{GAP}$ results with higher CPU time values every time when the problem can be loaded into the solver, due to its size (see Table 5). The “Sousa and Wolsey Improvement” formulation also provide tighter lower bounds in general, but with the same disadvantages that “Pessoa et al.”.

The “Sousa and Wolsey Improvement” formulation presents slightly tighter lower bounds than “Sousa and Wolsey” with similar computational time. This difference increases for the problem **F1**. In the analysis of the instance size (Table 5), the “Sousa and Wolsey” presents a larger number of instances with feasible solutions than the “Sousa and Wolsey Improvement”. For the problem **F1**, the time-indexed formulations (“Sousa and Wolsey”, “Sousa and Wolsey Improvement” and “Pessoa et al.”) present better $\overline{GAP}$ results for instance classes **1** and **4** (shortest and moderate planning horizon length) and poorer results for classes **2** and **6** (long planning horizon), while these formulations present the worst results for problem **F2** instance classes **5** and **6** (high congestion level and emphasized conditions).

The “Manne”, “Manne Alternative” and “Wagner” formulations have lower computational time values, but generally produce poorer solutions. However, for the problem **F2**, the “Manne” and “Manne Alternative” formulations produce tighter lower bounds for instance classes **1** to **4**. The “Manne Alternative” formulation presents better $\overline{GAP}$ results in **F1** and longer computational time than “Manne”. As the number of jobs increase the $\overline{GAP}$ difference decrease. There is no noticeable difference between the results for problem **F2**, however the “Manne ” formulation presents lower computational times. The “Wagner” formulation presents poorer $\overline{GAP}$ results in all classes and all problems than the others. For the problem **F1**, the “Manne” and “Manne Alternative” formulations present better $\overline{GAP}$ results for instance classes **4** and **6** (moderate and long planning horizon length) and slightly worse results for instances of class **5** (high congestion level). When considering the problem **F2**, they perform worse for problem instance classes **5** and **6** (high congestion level and emphasized conditions), specially the class **5** (high congestion level).

Time-indexed formulations are known to yield better bounds, but cannot be directly applied to many instances due to their large number of variables and constraints. These formulations are always interesting for column-generation algorithms (van den Akker et al., 1999; Van den Akker et al., 2000; Bigras et al., 2008a; Pessoa et al., 2010) and Lagrangean relaxations algorithms (Sun et al., 1999; Avella et al., 2005; de Paula et al., 2010). However, such as mentioned by Pessoa et al. (2010), the time-indexed bound may still leave a significant duality gap and all exact algorithms based on it sometimes need to explore large enumeration trees.

The analysis of the average results for the MIP formulations, Tables 4 and 5, show that all formulations have difficulty as the number of jobs increases. It is possible to notice that the “Sousa and Wolsey”, “Sousa and Wolsey Improvement ” and “Pessoa et al.” formulations managed to optimality solve some instances (small $\overline{GAP}$ value), but as the number of variables and constraints increase, the problems become rapidly unmanageable by the commercial solver (see 5). In the supplementary material, a detailed description of the results are presented for each instance size in each class. As it can be seen in Tables **S1** to **S6**, the time-indexed formulations (“Sousa and Wolsey”, “Sousa and Wolsey Improvement ” and “Pessoa et al.”) are able to solve instances of up to 20 jobs, depending on the class. These formulations present better $\overline{GAP}$ results for problem **F1** for instance classes **1** and **4** (shortest and moderate planning horizon length), and the worst performance for classes **2** and **6** (long planning horizon). When considering the problem **F2**, they perform worse for problem instance classes **5** and **6** (high congestion level and emphasized conditions), specially the class **6** (emphasized conditions).

Even presenting poor lower bound values, “Manne”, “Manne Alternative” and “Wagner” formulations managed to optimally solve several instances, specially the “Manne” formulation for **F2**. These formulations present generally better $\overline{GAP}$ results than the time-indexed formulation for the problems **F1** and **F2** in all instance classes (**1** to **6**), highlighting the “Manne” formulation $\overline{GAP}$ results. For the problem **F1**, the “Manne” and “Manne Alternative” formulations present better $\overline{GAP}$ results for instance classes **4** and **6** (moderate and long planning horizon length) and slightly worse results for instances of class **5** (high congestion level). When considering the problem **F2**, they perform worse for problem instance classes **5** and **6** (high congestion level and emphasized conditions), specially the class **5** (high congestion level).

Table 4 Average GAP Results for Single Machine Scheduling Problems for Six MIP Formulations for All Classes in All Sizes. For the LP (linear programming) relaxation problem, the $\overline{\overline{GAP}}$ indicates the average value of the average linear relaxation gap for all classes in all sizes, computed as the relative difference between the best integer solution and the LP relaxation value. For the MIP (mixed integer programming) problem, the $\overline{\overline{GAP}}$ is the average value of the average optimality gap for all classes in all sizes. $\overline{\overline{T(s)}}$ indicates the average value of the average CPU time for all classes in all sizes, and **SD** is the standard deviation for each metric. $F1$ and $F2$ denote the objective functions $\sum_j w_jC_j$ and $\sum_j w_jT_j$, respectively.

	Instance Classes	Objective Function	Mixed Integer Program Formulations																							
			Manne				Manne Alternative				Wagner				Sousa and Wolsey				Sousa and Wolsey Improvement				Pessoa et al.			
			$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	*overlineGAP*	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)
LP RELAXATION PROBLEM	Class 1	F1	68.5%	10.4%	0.1	0.2	58.5%	17.7%	6.9	15.3	100.0%	0.0%	62.9	149.5	72.5%	24.3%	1587.8	1713.5	40.1%	47.5%	1956.6	1565.3	36.4%	50.4%	1347.1	1787.7
	Class 2	F1	67.5%	10.5%	3.7	8.1	57.2%	18.1%	10.7	15.2	100.0%	0.0%	83.1	193.2	80.8%	24.6%	2642.2	1471.0	81.9%	39.7%	3132.1	1038.5	43.7%	50.4%	1646.9	1761.6
	Class 3	F1	63.8%	5.7%	3.7	6.7	55.5%	12.4%	3.8	8.0	100.0%	0.0%	133.1	326.2	69.0%	23.8%	1869.0	1414.3	57.3%	42.6%	2225.3	1503.1	38.7%	49.1%	1465.4	1745.8
	Class 4	F1	26.6%	4.6%	0.6	0.9	23.5%	4.8%	14.4	30.0	100.0%	0.0%	129.8	353.2	50.4%	39.3%	1579.5	1658.8	39.0%	48.4%	1924.2	1580.5	36.4%	50.4%	1318.5	1759.9
	Class 5	F1	67.8%	9.7%	6.3	9.4	58.1%	17.0%	31.4	65.1	100.0%	0.0%	69.9	163.1	72.2%	24.6%	1570.6	1728.9	42.4%	46.3%	1886.9	1618.6	38.2%	49.4%	1342.5	1754.3
	Class 6	F1	24.8%	3.1%	0.8	1.9	21.9%	3.4%	60.6	87.0	100.0%	0.0%	113.3	279.9	67.0%	38.3%	2615.1	1349.1	78.5%	37.7%	3182.4	962.6	45.5%	52.1%	1688.4	1774.9
	F1 Average		**53.2%**	**7.4%**	**2.5**	**4.5**	**45.8%**	**12.2%**	**21.3**	**36.8**	**100.0%**	**0.0%**	**98.7**	**244.2**	**68.7%**	**29.1%**	**1977.4**	**1555.9**	**56.6%**	**43.7%**	**2384.6**	**1378.1**	**39.8%**	**50.3%**	**1468.1**	**1764.0**
	Standard Deviation		**21.3%**	**3.3%**	**2.4**	**4.0**	**17.9%**	**6.6%**	**21.5**	**32.0**	**0.0%**	**0.0%**	**30.7**	**87.2**	**10.1%**	**7.5%**	**516.9**	**164.5**	**19.5%**	**4.4%**	**610.5**	**295.8**	**3.9%**	**1.1%**	**163.3**	**15.0**
	Class 1	F2	6.1%	8.6%	0.0	0.1	6.1%	8.6%	12.2	28.1	29.1%	32.7%	54.8	135.5	27.6%	33.5%	2203.7	1602.5	27.5%	33.6%	2352.4	1581.0	23.9%	34.3%	1613.3	1787.2
	Class 2	F2	8.3%	12.2%	0.1	0.1	8.3%	12.2%	16.3	36.7	32.7%	39.3%	41.1	103.4	30.9%	38.3%	2643.1	1333.3	30.9%	38.3%	2904.3	1297.0	30.7%	40.2%	1975.4	1666.3
	Class 3	F2	4.1%	7.9%	0.0	0.1	4.1%	7.9%	9.8	27.1	43.6%	35.6%	48.1	133.1	40.0%	38.0%	2589.9	1472.5	43.6%	35.6%	2729.2	1372.7	32.1%	42.4%	1583.0	1673.8
	Class 4	F2	4.6%	4.4%	0.1	0.1	4.6%	4.4%	17.0	33.1	76.4%	32.0%	67.3	139.7	57.8%	45.4%	2249.7	1534.9	55.1%	46.7%	2331.6	1512.5	37.2%	49.8%	1566.3	1705.9
	Class 5	F2	98.3%	1.2%	4.4	12.6	98.6%	1.3%	58.7	178.3	100.0%	0.0%	85.2	212.1	99.4%	1.4%	2142.9	1680.4	86.8%	15.6%	2436.9	1530.7	66.3%	29.5%	1672.0	1735.0
	Class 6	F2	51.3%	12.9%	5.5	13.5	51.2%	12.6%	24.9	65.7	100.0%	0.0%	89.0	206.4	92.5%	12.5%	2912.9	1239.0	90.4%	21.6%	3158.0	997.4	52.1%	40.3%	2063.3	1647.4
	F2 Average		**28.8%**	**7.8%**	**1.7**	**4.4**	**28.8%**	**7.8%**	**23.2**	**61.5**	**63.6%**	**23.3%**	**64.3**	**155.1**	**58.0%**	**28.2%**	**2457.0**	**1477.1**	**55.7%**	**31.9%**	**2652.1**	**1381.9**	**40.4%**	**39.4%**	**1745.5**	**1702.6**
	Standard Deviation		**38.7%**	**4.5%**	**2.5**	**6.7**	**38.7%**	**4.4%**	**18.2**	**59.0**	**32.7%**	**18.2%**	**19.7**	**43.9**	**31.2%**	**17.3%**	**305.2**	**166.0**	**27.3%**	**11.4%**	**336.0**	**216.3**	**15.8%**	**7.0%**	**216.9**	**51.8**
	LP Relaxation Average		***41.0%***	***7.6%***	***2.1***	***4.5***	***37.3%***	***10.0%***	***22.2***	***49.1***	***81.8%***	***11.6%***	***81.5***	***199.6***	***63.3%***	***28.7%***	***2217.2***	***1516.5***	***56.1%***	***37.8%***	***2518.3***	***1380.0***	***40.1%***	***44.9%***	***1606.8***	***1733.3***
	Standard Deviation		***32.4%***	***3.7%***	***2.4***	***5.3***	***30.1%***	***5.8%***	***19.0***	***47.0***	***29.1%***	***17.3%***	***30.5***	***80.6***	***22.8%***	***12.7%***	***476.0***	***162.9***	***22.6%***	***10.3%***	***490.1***	***247.1***	***11.0%***	***7.4%***	***233.4***	***48.5***
MIP PROBLEM	Class 1	F1	33.3%	35.6%	2070.6	1787.3	40.6%	34.2%	2532.0	1639.6	50.0%	44.2%	2357.4	1734.8	42.1%	47.9%	2281.0	1651.1	38.6%	48.9%	2470.9	1625.7	36.4%	50.5%	1474.6	1714.6
	Class 2	F1	32.0%	35.0%	2049.5	1800.0	40.1%	34.4%	2588.5	1657.6	49.4%	46.6%	2337.5	1736.1	73.5%	37.4%	3396.8	673.8	50.2%	48.6%	2774.9	1266.0	45.5%	52.2%	1881.9	1693.4
	Class 3	F1	28.0%	32.4%	1888.7	1800.4	35.1%	30.9%	2524.3	1644.7	48.9%	43.1%	2414.6	1682.4	53.7%	42.5%	2896.0	1317.6	48.4%	44.2%	2899.1	1331.7	38.3%	49.3%	1664.7	1745.9
	Class 4	F1	8.8%	12.4%	1583.1	1813.2	12.2%	13.9%	2083.0	1764.9	41.4%	45.5%	2218.6	1747.0	38.6%	49.0%	2039.4	1653.7	36.9%	50.1%	2087.0	1652.0	36.4%	50.5%	1435.2	1748.1
	Class 5	F1	32.0%	35.6%	1978.3	1796.8	38.8%	34.6%	2497.0	1641.0	50.1%	45.1%	2350.3	1741.9	46.3%	48.2%	2272.7	1574.4	39.0%	48.7%	2376.4	1579.1	38.2%	49.4%	1428.7	1745.0
	Class 6	F1	8.00%	10.80%	1600.7	1808.3	11.16%	12.35%	2098.9	1770.3	42.96%	46.82%	2194.0	1723.9	62.78%	44.06%	3374.7	585.0	70.21%	43.61%	3261.1	960.9	45.56%	52.12%	1951.5	1662.5
	F1 Average		**23.7%**	**27.0%**	**1861.8**	**1801.0**	**29.7%**	**26.7%**	**2387.3**	**1686.4**	**47.1%**	**45.2%**	**2312.1**	**1727.7**	**52.8%**	**44.9%**	**2710.1**	**1242.6**	**47.2%**	**47.3%**	**2644.9**	**1402.6**	**40.0%**	**50.7%**	**1639.4**	**1718.2**
	Standard Deviation		**12.0%**	**12.0%**	**218.6**	**9.0**	**14.1%**	**10.6%**	**231.5**	**63.3**	**3.9%**	**1.4%**	**86.4**	**23.5**	**13.4%**	**4.5%**	**595.5**	**491.4**	**12.6%**	**2.7%**	**418.3**	**269.0**	**4.3%**	**1.3%**	**232.4**	**35.0**
	Class 1	F2	0.4%	1.3%	188.3	350.9	24.7%	39.0%	1105.7	1547.4	29.2%	45.9%	1208.5	1612.8	43.6%	50.5%	1713.6	1709.3	49.1%	50.1%	1868.5	1755.3	49.1%	50.1%	1935.1	1761.7
	Class 2	F2	3.1%	7.3%	648.1	1280.0	22.3%	38.4%	1077.9	1527.5	28.8%	42.8%	1160.7	1548.8	58.2%	47.7%	2434.2	1413.9	65.5%	45.7%	2541.3	1514.1	60.0%	49.0%	2513.5	1506.4
	Class 3	F2	0.0%	0.0%	52.9	117.8	14.9%	31.3%	904.1	1399.1	25.5%	43.7%	1107.0	1592.1	47.3%	47.6%	1874.5	1644.5	51.6%	48.0%	2042.7	1657.2	54.5%	47.4%	2272.2	1645.0
	Class 4	F2	0.0%	0.0%	172.1	378.5	16.0%	26.7%	1804.4	1701.7	34.4%	47.6%	1461.8	1756.1	42.1%	49.5%	1677.7	1748.5	40.0%	49.0%	1740.6	1708.7	49.1%	50.1%	1896.6	1735.7
	Class 5	F2	42.1%	46.5%	1838.5	1758.3	49.8%	45.8%	2275.7	1723.7	50.7%	49.6%	2177.0	1745.3	62.9%	45.5%	2791.9	1410.9	65.5%	43.3%	2948.0	1253.6	55.4%	44.1%	2780.3	1300.9
	Class 6	F2	13.53%	17.75%	1546.2	1795.6	26.09%	28.78%	2042.6	1709.5	43.89%	47.80%	2052.6	1735.8	84.35%	26.56%	3482.8	388.7	82.73%	32.13%	3371.8	572.2	59.05%	45.00%	2797.6	1292.5
	F2 Average		**9.9%**	**12.1%**	**741.0**	**946.8**	**25.6%**	**35.0%**	**1535.1**	**1601.5**	**35.4%**	**46.2%**	**1527.9**	**1665.2**	**56.4%**	**44.5%**	**2329.1**	**1386.0**	**59.1%**	**44.7%**	**2418.8**	**1410.2**	**54.5%**	**47.6%**	**2365.9**	**1540.4**
	Standard Deviation		**16.6%**	**18.1%**	**769.9**	**755.7**	**12.7%**	**7.3%**	**578.0**	**131.2**	**9.9%**	**2.6%**	**472.2**	**90.9**	**16.0%**	**9.0%**	**716.4**	**509.7**	**15.2%**	**6.6%**	**649.2**	**448.7**	**4.7%**	**2.6%**	**398.6**	**208.8**
	MIP Average		***16.8%***	***19.6%***	***1301.4***	***1373.9***	***27.6%***	***30.9%***	***1961.2***	***1643.9***	***41.3%***	***45.7%***	***1920.0***	***1696.4***	***54.6%***	***44.7%***	***2519.6***	***1314.3***	***53.1%***	***46.0%***	***2531.9***	***1406.4***	***47.3%***	***49.1%***	***2002.7***	***1629.3***
	Standard Deviation		***15.6%***	***16.6%***	***796.1***	***677.2***	***12.9%***	***9.7%***	***611.8***	***107.7***	***9.4%***	***2.1%***	***522.0***	***71.2***	***14.2%***	***6.8%***	***658.8***	***483.2***	***14.7%***	***5.0%***	***533.9***	***352.8***	***8.7%***	***2.5%***	***490.6***	***170.3***

Table 5 Average GAP Results for Single Machine Scheduling Problems for Six MIP Formulations for All Sizes in All Classes. For the LP (linear programming) relaxation problem, the $\overline{\overline{GAP}}$ indicates the average value of the average linear relaxation gap for all sizes in all classes, computed as the relative difference between the best integer solution and the LP relaxation value. For the MIP (mixed integer programming) problem, the $\overline{\overline{GAP}}$ is the average value of the average optimality gap for all sizes in all classes. $\overline{\overline{T(s)}}$ indicates the average value of the average CPU time for all sizes in all classes, and **SD** is the standard deviation for each metric. $F1$ and $F2$ denote the objective functions $\sum_j w_j C_j$ and $\sum_j w_j T_j$, respectively.

	Instance Sizes	Objective Function	Mixed Integer Program Formulations: Manne				Manne Alternative				Wagner				Sousa and Wolsey				Sousa and Wolsey Improvement				Pessoa et al.			
			$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{\overline{GAP}}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)	$\overline{overlineGAP}$	SD($\overline{GAP}$)	$\overline{\overline{T(s)}}$	SD($\overline{T(s)}$)
LP RELAXATION PROBLEM	5	F1	42.1%	13.9%	0.6	1.0	24.7%	6.6%	0.9	1.1	100.0%	0.0%	0.2	0.3	33.0%	9.7%	137.5	302.4	3.5%	3.1%	101.7	168.1	0.0%	0.1%	1.6	3.6
	7	F1	49.0%	14.2%	1.0	1.9	34.4%	8.6%	0.0	0.0	100.0%	0.0%	0.0	0.0	40.5%	13.5%	236.2	350.3	5.9%	3.9%	954.8	1410.1	0.0%	0.0%	2.3	2.8
	9	F1	49.6%	18.1%	0.2	0.5	37.0%	12.6%	0.0	0.0	100.0%	0.0%	0.0	0.0	43.9%	13.7%	873.1	1242.7	31.1%	38.8%	1280.4	1593.1	0.1%	0.1%	13.4	18.1
	11	F1	50.2%	18.3%	0.0	0.1	40.5%	13.8%	1.3	1.9	100.0%	0.0%	0.5	0.8	44.1%	17.4%	757.9	1095.4	36.9%	46.2%	1515.2	1359.2	0.1%	0.1%	35.4	32.9
	13	F1	51.0%	21.0%	1.9	2.1	42.8%	17.2%	2.0	1.7	100.0%	0.0%	4.0	8.9	52.3%	22.2%	1149.8	1282.6	40.1%	47.0%	2135.5	1185.8	0.1%	0.1%	96.8	85.6
	15	F1	52.9%	21.6%	4.9	10.9	45.7%	17.8%	11.4	13.2	100.0%	0.0%	2.1	2.7	67.2%	29.2%	1725.9	1539.0	46.0%	45.7%	2604.2	789.4	0.2%	0.3%	228.5	222.8
	20	F1	52.8%	22.8%	5.0	8.9	47.2%	19.8%	20.1	40.7	100.0%	0.0%	1.8	1.8	75.5%	28.6%	2575.4	839.5	58.7%	46.0%	3238.7	397.1	37.4%	42.4%	1449.4	1369.9
	30	F1	54.7%	25.3%	5.1	11.1	51.4%	23.4%	22.1	49.0	100.0%	0.0%	7.2	5.9	98.7%	3.1%	3495.3	256.5	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3522.0	122.4
	50	F1	55.9%	28.6%	4.6	8.9	54.5%	27.8%	6.5	4.5	100.0%	0.0%	40.6	16.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	75	F1	61.7%	26.2%	2.2	2.5	61.0%	25.9%	79.4	91.7	100.0%	0.0%	223.9	43.9	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	100	F1	64.7%	26.1%	2.1	1.8	64.3%	25.8%	90.8	78.4	100.0%	0.0%	804.9	297.2	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	F1 Average		**53.2%**	**21.5%**	**2.5**	**4.5**	**45.8%**	**18.1%**	**21.3**	**25.7**	**100.0%**	**0.0%**	**98.7**	**34.3**	**68.7%**	**12.5%**	**1977.4**	**628.0**	**56.6%**	**21.0%**	**2384.6**	**627.5**	**39.8%**	**3.9%**	**1468.1**	**168.9**
	Standard Deviation		**6.2%**	**4.9%**	**2.0**	**4.4**	**11.7%**	**7.2%**	**32.6**	**34.0**	**0.0%**	**0.0%**	**243.5**	**88.1**	**27.3%**	**11.1%**	**1429.4**	**583.4**	**37.9%**	**22.9%**	**1264.1**	**652.5**	**49.0%**	**12.8%**	**1724.5**	**404.5**
	5	F2	30.1%	46.1%	0.0	0.0	30.1%	46.2%	0.4	0.5	40.0%	49.0%	0.2	0.2	29.3%	44.9%	26.4	55.8	21.3%	32.9%	143.9	320.0	16.2%	30.1%	0.9	1.0
	7	F2	29.5%	41.1%	0.0	0.0	29.5%	41.1%	0.1	0.1	36.7%	49.7%	0.1	0.0	30.5%	41.9%	297.4	388.0	21.5%	28.1%	444.8	610.9	16.2%	21.8%	14.4	22.4
	9	F2	26.9%	42.0%	0.0	0.0	26.9%	42.0%	0.3	0.6	46.7%	46.8%	0.2	0.4	37.3%	49.1%	1088.0	1326.7	30.5%	41.9%	1670.9	1343.0	10.5%	17.4%	44.8	45.0
	11	F2	28.5%	41.2%	0.0	0.0	28.5%	41.2%	0.3	0.7	56.7%	42.7%	0.1	0.1	40.5%	44.7%	1859.7	1064.7	36.8%	42.3%	2426.4	1069.9	12.2%	21.4%	173.3	163.3
	13	F2	29.6%	40.1%	0.0	0.0	29.6%	40.1%	0.2	0.4	56.7%	42.7%	0.3	0.4	47.3%	39.8%	2610.9	679.4	46.2%	36.8%	3220.2	470.4	9.6%	13.0%	738.8	710.7
	15	F2	28.6%	39.1%	0.0	0.0	28.6%	39.1%	1.0	1.5	53.3%	45.0%	0.8	1.4	46.8%	46.8%	3264.2	429.9	50.0%	43.4%	3266.5	367.6	8.2%	11.8%	1032.5	911.6
	20	F2	36.3%	34.8%	0.3	0.6	36.5%	35.2%	3.5	7.7	70.0%	35.2%	0.8	0.7	66.7%	37.2%	3480.9	291.6	66.7%	37.2%	3600.0	0.0	31.1%	26.9%	2796.2	788.2
	30	F2	26.6%	37.3%	0.4	0.6	26.6%	37.3%	0.4	0.2	70.0%	41.5%	2.5	1.4	70.0%	41.5%	3600.0	0.0	70.0%	41.5%	3600.0	0.0	70.0%	41.5%	3600.0	0.0
	50	F2	23.9%	37.5%	0.3	0.5	23.9%	37.5%	10.7	11.2	80.0%	25.3%	24.8	13.5	80.0%	25.3%	3600.0	0.0	80.0%	25.3%	3600.0	0.0	80.0%	25.3%	3600.0	0.0
	75	F2	28.5%	36.5%	14.6	22.4	28.5%	36.5%	35.8	20.5	100.0%	0.0%	180.8	96.2	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	100	F2	28.2%	39.0%	2.8	4.8	28.2%	39.0%	202.2	198.8	90.0%	16.7%	496.4	146.4	90.0%	16.7%	3600.0	0.0	90.0%	16.7%	3600.0	0.0	90.0%	16.7%	3600.0	0.0
	F2 Average		**28.8%**	**39.5%**	**1.7**	**2.6**	**28.8%**	**39.6%**	**23.2**	**22.0**	**63.6%**	**35.9%**	**64.3**	**23.7**	**58.0%**	**35.3%**	**2457.0**	**385.1**	**55.7%**	**31.5%**	**2652.1**	**380.2**	**40.4%**	**20.5%**	**1745.5**	**240.2**
	Standard Deviation		**3.0%**	**3.1%**	**4.4**	**6.7**	**3.1%**	**3.1%**	**60.3**	**59.0**	**20.4%**	**15.7%**	**153.1**	**49.7**	**24.6%**	**15.2%**	**1405.4**	**463.4**	**27.4%**	**13.4%**	**1318.0**	**468.0**	**36.6%**	**10.8%**	**1666.9**	**367.7**
	LP Relaxation Average		**41.0%**	**30.5%**	**2.1**	**3.6**	**37.3%**	**28.8%**	**22.2**	**23.8**	**81.8%**	**17.9%**	**81.5**	**29.0**	**63.3%**	**23.9%**	**2217.2**	**506.6**	**56.1%**	**26.2%**	**2518.3**	**503.8**	**40.1%**	**12.2%**	**1606.8**	**204.6**
	Standard Deviation		**13.3%**	**10.1%**	**3.3**	**5.6**	**12.1%**	**12.2%**	**47.3**	**47.0**	**23.3%**	**21.3%**	**199.2**	**70.0**	**25.9%**	**17.4%**	**1404.9**	**529.0**	**32.3%**	**19.1%**	**1267.6**	**568.4**	**42.2%**	**14.3%**	**1661.2**	**379.0**
MIP PROBLEM	5	F1	0.0%	0.0%	0.4	0.6	0.0%	0.0%	0.3	0.5	0.0%	0.0%	0.1	0.1	0.0%	0.0%	546.3	769.5	0.0%	0.0%	81.2	158.7	0.0%	0.0%	4.2	6.9
	7	F1	0.0%	0.0%	0.5	1.1	0.0%	0.0%	1.1	0.5	0.0%	0.0%	1.0	0.3	5.2%	9.3%	1223.0	1638.6	0.3%	0.8%	888.1	1193.2	0.0%	0.0%	19.4	29.4
	9	F1	0.0%	0.0%	1.5	2.5	0.0%	0.0%	37.1	30.2	0.0%	0.0%	24.4	13.8	11.4%	14.0%	1741.5	1671.7	3.3%	6.3%	1416.0	1383.1	0.0%	0.0%	61.1	72.9
	11	F1	0.0%	0.0%	15.1	12.1	3.1%	3.3%	1778.9	1369.1	0.0%	0.0%	625.0	394.0	17.0%	26.3%	2261.4	1472.0	11.2%	21.8%	2116.0	1329.3	0.0%	0.0%	143.7	152.5
	13	F1	0.9%	1.4%	458.7	492.5	19.1%	14.7%	2857.4	1151.0	20.7%	15.5%	3182.3	617.3	27.3%	33.9%	2623.3	1205.6	25.1%	38.1%	3124.5	848.6	0.1%	0.3%	389.9	451.3
	15	F1	9.0%	8.2%	2214.9	1651.0	29.8%	18.3%	3585.4	35.8	40.3%	18.4%	3600.0	0.0	51.8%	41.6%	3415.5	393.6	31.2%	37.3%	3468.2	322.9	0.1%	0.2%	777.4	705.3
	20	F1	31.2%	20.2%	3388.8	327.9	40.6%	19.2%	3600.0	0.0	73.8%	8.7%	3600.0	0.0	68.5%	31.8%	3600.0	0.0	48.4%	43.3%	3600.0	0.0	40.2%	47.2%	2238.1	1275.1
	30	F1	44.7%	23.8%	3600.0	0.0	50.0%	23.3%	3600.0	0.0	88.1%	10.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	50	F1	51.3%	27.9%	3600.0	0.0	56.3%	26.8%	3600.0	0.0	95.5%	4.6%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	75	F1	59.9%	25.9%	3600.0	0.0	62.2%	25.3%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	100	F1	63.7%	26.0%	3600.0	0.0	65.2%	24.9%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	F1 Average		**23.7%**	**12.1%**	**1861.8**	**226.2**	**29.7%**	**14.2%**	**2387.3**	**235.2**	**47.1%**	**5.2%**	**2312.1**	**93.2**	**52.8%**	**14.3%**	**2710.1**	**650.1**	**47.2%**	**13.4%**	**2644.9**	**476.0**	**40.0%**	**4.3%**	**1639.4**	**244.9**
	Standard Deviation		**26.7%**	**12.4%**	**1741.0**	**501.4**	**26.5%**	**11.2%**	**1621.6**	**509.2**	**44.7%**	**6.9%**	**1716.9**	**210.0**	**42.3%**	**16.2%**	**1117.9**	**721.0**	**44.2%**	**18.0%**	**1300.7**	**588.3**	**49.0%**	**14.2%**	**1675.3**	**411.9**
	5	F2	0.0%	0.0%	0.0	0.0	0.0%	0.0%	0.3	0.5	0.0%	0.0%	0.1	0.1	3.9%	9.6%	396.3	938.0	3.3%	8.2%	313.1	707.2	0.0%	0.0%	3.8	2.6
	7	F2	0.0%	0.0%	0.0	0.0	0.0%	0.0%	0.9	1.5	0.0%	0.0%	0.6	0.8	7.9%	19.3%	744.2	1413.0	3.5%	8.6%	679.0	1161.8	1.8%	3.0%	343.8	480.2
	9	F2	0.0%	0.0%	0.2	0.4	0.0%	0.0%	5.3	6.6	0.0%	0.0%	6.2	9.6	11.5%	28.3%	1071.6	1381.9	13.8%	27.6%	1202.5	1525.9	4.3%	7.6%	826.0	890.1
	11	F2	0.0%	0.0%	1.5	3.2	0.0%	0.0%	175.8	219.9	0.0%	0.0%	78.1	133.2	22.6%	37.2%	1521.7	1680.5	31.2%	40.0%	1636.5	1691.1	7.5%	14.9%	1408.9	1312.6
	13	F2	0.0%	0.0%	76.7	186.1	4.5%	9.1%	805.5	1132.2	0.4%	1.0%	531.8	902.0	37.3%	38.9%	1907.7	1572.9	44.6%	42.0%	2225.9	1582.0	26.4%	23.7%	2304.5	1464.9
	15	F2	2.6%	6.4%	295.6	716.4	11.2%	20.1%	1426.9	1562.5	12.8%	23.3%	1160.9	1760.6	50.0%	48.6%	2269.5	1344.1	63.3%	42.7%	2639.0	1109.5	59.7%	26.6%	3137.8	601.5
	20	F2	12.6%	25.8%	1031.5	1629.3	28.9%	29.7%	2166.6	1633.9	33.1%	41.1%	1841.3	1473.1	87.2%	15.4%	3309.4	321.0	90.0%	24.5%	3510.9	218.4	100.0%	0.0%	3600.0	0.0
	30	F2	20.1%	37.5%	1201.4	1857.9	27.4%	37.3%	2085.8	1533.5	51.0%	46.5%	2571.8	1167.2	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	50	F2	20.3%	38.1%	1433.8	1705.1	41.3%	32.7%	3018.7	885.2	92.1%	10.0%	3416.4	285.6	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	75	F2	26.4%	37.2%	2155.5	1560.3	77.4%	21.4%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	100	F2	26.4%	39.8%	1954.8	1578.8	91.1%	10.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0	100.0%	0.0%	3600.0	0.0
	F2 Average		**9.9%**	**16.8%**	**741.0**	**839.8**	**25.6%**	**14.6%**	**1535.1**	**634.2**	**35.4%**	**11.1%**	**1527.9**	**521.1**	**56.4%**	**17.9%**	**2329.1**	**786.5**	**59.1%**	**17.6%**	**2418.8**	**726.9**	**54.5%**	**6.9%**	**2365.9**	**432.0**
	Standard Deviation		**11.4%**	**18.5%**	**841.7**	**820.5**	**32.4%**	**14.3%**	**1444.7**	**717.8**	**43.1%**	**17.7%**	**1534.7**	**675.1**	**41.5%**	**18.0%**	**1268.2**	**720.4**	**41.1%**	**18.1%**	**1281.1**	**707.6**	**46.5%**	**10.2%**	**1455.0**	**565.7**
	MIP Average		**16.8%**	**14.5%**	**1301.4**	**533.0**	**27.6%**	**14.4%**	**1961.2**	**434.7**	**41.3%**	**8.1%**	**1920.0**	**307.2**	**54.6%**	**16.1%**	**2519.6**	**718.3**	**53.1%**	**15.5%**	**2531.9**	**601.4**	**47.3%**	**5.6%**	**2002.7**	**338.4**
	Standard Deviation		**21.3%**	**15.6%**	**1452.5**	**734.1**	**29.0%**	**12.5%**	**1560.9**	**640.7**	**43.2%**	**13.5%**	**1639.0**	**534.8**	**40.9%**	**16.8%**	**1182.8**	**706.8**	**42.1%**	**17.8%**	**1265.1**	**647.9**	**47.2%**	**12.1%**	**1575.7**	**492.3**

All the MIP formulations developed in this article present a polynomial number of constraints and variables. The Table 6 shows the number of constraints and binary variables associated with each paradigm. It is worth noting that as $h >>> n$, $h \propto n$, "Pessoa et al.", "Sousa and Wolsey" and "Sousa and Wolsey Improvement" formulations will increase its size faster than other formulations.

Table 6 Model Size for each Formulation Paradigm for Problems $1|r_j,s_{ij}|\sum_j w_jC_j$ and $1|r_j,s_{ij}|\sum_j w_jT_j$. For the formulations, "Variables" indicate the number of associated variables and "Constraints" the number of constraints with each formulation paradigm.

MIP Formulations	Model Order Size for Both Problems	
	Variables	Constraints
Manne	$O(n^2)$	$O(n^2)$
Manne Alternative	$O(n^2)$	$O(n^2)$
Wagner	$O(n^3)$	$O(n^3)$
Sousa and Wolsey	$O(nh)$	$O(n^2h)$
Sousa and Wolsey Improvement	$O(nh)$	$O(n^2h)$
Pessoa et al.	$O(n^2h)$	$O(n^2h)$

4 Concluding remarks

In this article, the computational performance of six different specific MIP formulations were proposed and compared for two single machine scheduling problems with sequence dependent setup times and release dates. In addition, these MIP formulations could be easily adapted to others objective functions and machine environments (parallel machines, flowshop, jobshop and others). The performances of these MIP formulations depend on the problem, the number of jobs, the characteristic of the instances (class) and the length of the planning horizon. "Manne" and "Sousa and Wolsey" formulations seem to be the most widely used formulations in the Scheduling literature. Formulations based on "Manne" and "Wagner" formulations are the oldest and "Pessoa et al." formulation is the newest one.

The "Manne" formulation optimality solves a greater number of instances. Problem $1|r_j,s_{ij}|\sum_j w_jC_j$ (**F1**) is better solved by "Pessoa et al." formulation for small instance sizes (about 15 jobs) and "Manne" formulation for larger instances. The problem $1|r_j,s_{ij}|\sum_j w_jT_j$ (**F2**) is better solved for "Manne" formulation. Furthermore, with small size of instances, the "Pessoa et al." formulation presets good MIP and LP results. However, LP relaxation for time-indexed formulations has better lower bounds, but its problems are harder to solve.

In summary, the results suggest that time-indexed formulations have difficulty to solve the instances with long planning horizon (classes **2** and **6**) for the problem $1|r_j,s_{ij}|\sum_j w_jC_j$, while other formulations have difficulty to solve the instances with high congestion level (class **5**). In the problem $1|r_j,s_{ij}|\sum_j w_jT_j$, all formulations have difficulty to solve the instance class with high congestion level and emphasized conditions (classes **5** and **6**), highlighting considerably poorer results for the instances with high congestion level (class **5**). Furthermore, the results indicate that "Pessoa et al." formulation always perform better, independently of the class for instances with less than 15 jobs. For larger instances, "Manne" formulation manages to solve a greater number of instances. The alternative formulation for "Manne" does not see to work well as there were no significant improvements. On the other hand, the improvements proposed for the "Sousa and Wolsey" formulation improve the lower bounds but increase its computational time.

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