Multicriteria Partitioning Problems

Pedro Miguel Ruivo Catalão Amiguinho

Master in Industrial Engineering and Management

Instituto Superior Técnico, Universidade Técnica de Lisboa-UTL, Lisboa, Portugal

Abstract

In this paper we study the possibility of generalizing a monocriteria model to a multicriteria partition problem and show the reason why exact algorithms are not efficient in solving the problem. The mathematical model studied is a modified version of a mono-objective model from Mehrotra [1] and applied to the biobjetive case using the ε-constraint method. Research conducted concludes that the studied multicriteria model is not suitable to solve the problem due not only to the high computational requirements needed to run the model, as well as the difficulty of ensuring the fundamental constraints of integrality, contiguity and "absence of holes"'. Kruskal's algorithm applied to multicriteria partitioning problem [2] shows satisfactory results, managing to secure the general constraints of contiguity and integrality without requiring a high computational effort, with very low execution times compared with the exact method.

Keywords: multicriteria partitioning problem; ε-constraint; Kruskal's algorithm

1 Introduction

Multicriteria Partitioning Problems have been gaining great importance and are starting to be used to be used in various areas in order to improve the decision making process in these same areas. This problem results from the need to make a partition of a given set (which can be a territory, a group of people, etc.), according to several objectives that, in general, are in conflict. Problems of this type are usually modeled by partitioning a graph into connected sub graphs. Thus, the graph that represents the study set, say the area, is comprised of nodes which represent the elementary territorial units, and edges that represent the boundaries between these elementary units. The resulting sub graphs representing a partition of the partition and are often referred to as clusters.

The research done in this area have some limitations. The most common is related to the computation time. In smaller problems is possible to use exact methods and still reach an optimal solution in a relatively low processing time. However, the problems of larger dimensions, which represent most of the problems found in the literature, take a long time to process the solutions, becoming impeditive to use these methods in large scale problems. In these cases it is necessary to use heuristics and meta-heuristic, which do not guarantee that the obtained solutions are optimal. Another limitation is related to the restrictions necessary for proper formation of clusters, particularly constraints of contiguity, integrality and restrictions relating to the prevention of obtaining solutions with embedded areas, also known as "absence of holes". The implementation of these restrictions in the problem can hinder the resolution of such problems at the computational level. In addition to these two limitations, which are related to restrictions, most works found in the literature use only one objective function, however, in most real problem decision situations are inherently multidimensional and multi-objective.

One of the most popular applications of multicriteria partitioning problems is political districting, this is a common in the United States, and where it is used to avoid manipulation of district boundaries in order to give advantage to a particular party during elections [3-7]. Besides political districting which is the most common problem in the literature, there are also many studies on various topics such as zoning or school districts [8], [9], marine reserves [10], power distribution networks [11] and assignment of commercial zones to vendors [12], [13]. All these problems have in common a need to change the boundaries of the territory in study in order to optimize certain objectives.

Currently in the Portuguese context there is an interesting application of the multicriteria partition problem, it is the case of fusion/ extinction of the parishes of each county. This intervention seeks to improve the efficiency of resource allocation, increasing the coverage area of each municipality.

2 Research Questions

The literature review leads us to make the approach to our problem in light of three key issues:

- Does the monocriteria model can easily be generalized to the case multicriteria without jeopardizing the very definition of clusters?

- If the first issue is resolved or if there is a way to get around it, are there exact efficient algorithms to solve this problem?

- Does the multicriteria model useful in practice and in decision situations in a real context where we can apply it?

In order to answer these questions we are going to: identify, analyze and evaluate different multicriteria optimization methods with respect to computational effort required and the quality of the results obtained, to identify the most common objectives in literature and how to implement them in the model.

3 Problem objectives and restrictions

The partitioning problem is usually addressed using graph theory, with the nodes of the graph corresponding to the territorial units, and the edges representing the boundary between units. Most problems in the literature have common objectives and restrictions that must be met in order to obtain feasible solutions, those will be explained in the following sections of the paper.

3.1 Common Restrictions

The partitioning problem uses some general restrictions that seek to ensure the correct formation of clusters, the most common restrictions are contiguity, integrity and "absence of holes". The contiguity requirement obliges a cluster to be contiguous, i.e. it has to be possible to travel between two points in a cluster without having to pass another cluster

[14], [15]. The “absence of holes” prevents the formation of solutions with embedded areas [16]. And the integrality prevents the use of one node in more than one cluster, each node has one and only one cluster.

Another common constraints are the capacity constraints. These constraints can be used limit both the number of clusters existing in the plan as the number of possible territorial units within each cluster [17].

3.2 Common Objectives

Districting problems are intended to group a set of territorial units in clusters, this grouping is done based on several goals that are chosen by the decision makers. The objectives frequently used in the literature are as follows:

- Compactness, the clusters formed must have a compact geometric shape close to a circle or square, is very common in political districting problems in order to avoid the creation districts with partisan advantage [6], [7];
- Homogeneity, clusters should have similar features, the distribution of the population with respect to gender, age, ethnicity, should be common among clusters, there must be a socio-economic homogeneity [12];
- Similarity, clusters formed should not present major deviations from the initial settings before the plan was implemented. This is particularly important in situations where the reconfiguration of the clusters have a major impact on their administration, or in the lives of its inhabitants [8], [13].

4 Capacitated Clustering Problem

One version of this problem is the Capacitated Clustering Problem (CCP), a case where the problem is formulated with various restrictions. Although those may hinder the resolution at the computational level, they ensure a high level of flexibility in shaping the problem allowing it to be as close to reality as possible.

4.1 Mathematical Model

Given a graph $G = G(V, E)$ with n nodes and weighting coefficients w_{ij} and c_{ij} for the edges $(i, j) \in E$. A cluster is a group of connected nodes and respective edges. And the weight of the cluster is given by the sum of the weights of the edges that belong to the cluster. The problem aims to partition the nodes of the graph into clusters so as to maximize the total weight of the clusters. Be y_{ijk} a binary variable with value one if the edge (i, j) belongs to cluster k, and zero otherwise, the problem can be formulated as follows [1]:

$$Z_1 = Max \sum_{ijk} w_{ij} \times y_{ijk}$$

$Subject\ to$:

$$Z_2 = \sum_{ijk} c_{ij} \times y_{ijk}$$

$$\sum\nolimits_k x_{ik} = 1, \;\; i \in V,$$

$$y_{ijk} \leq x_{ik},$$

$$y_{ijk} \leq x_{jk},$$

$$y_{ijk} \geq x_{ik} + x_{jk} - 1,$$

$$x_{ik} \in \{0,1\},$$

$$y_{ijk} \in \{0,1\}.$$

The restriction $\sum_k x_{ik} = 1$ requires each node to belong to only one cluster, in order to fulfill the integrality constraints. The following three constraints require the nodes belonging to the same edge to be part of the same cluster. The last two constraints indicate that the variables are binary.

In this case, the objective function, Z_1, is maximized, while the function Z_2 is used as a restriction iteratively, each time the model is run ε is incremented to the maximum limit of the function, Z_2, in this way we obtain several of possible combinations of Z_1 and Z_2. This multiobjective optimization method is known ε-constraint [18].

4.2 Results

Using a graph with 13 nodes with random values for the weights w_{ij} and c_{ij} as an example, and requiring the number of clusters to be exactly six, with more than two and less than three nodes per cluster, the time the program took to run the model was 35 minutes, a value quite high considering the small size of the problem and the fact that we are only using two goals. The values obtained for the two functions Z_1 and Z_2 are shown in Figure 1.

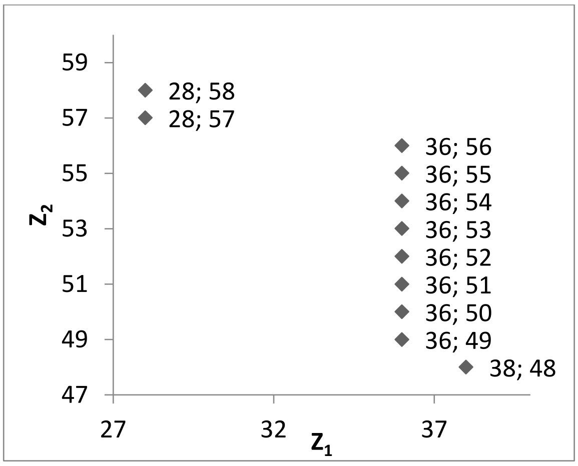

Figure 1 – Solutions for the integer programming method

These solutions are using the ε – constraint method. As seen in the graph, most of the solutions are dominated, the only values worth considering by the decision maker are (28, 58), (36, 56) and (38, 48).

So far the mathematical model allows to solve small districting problems, both mono and multicriteria. However, it is not possible to ensure compliance with contiguity and "absence of holes" restrictions without making a few changes to the model. The model is also not suitable for larger problems.

5 Kruskal's Algorithm

The algorithm consists of selecting the edges in descending order of weight until there have been selected $n-1$ edges, n being the total number of nodes of the graph. The selection of edges is performed so as not to allow the formation of cycles. Thus, if the selection of the highest weighted edge results in the formation of a cycle, it should be selected the following edge. This algorithm is used to obtain the maximum spanning tree [2].

5.1 Kruskal's partitioning

Although Kruskal's main use is to create a spanning tree, it can also be used in the partitioning problem. This is done by cutting the $(k-1)$ lowest weighted edges of the spanning tree obtained from the algorithm, or stopping the algorithm after the selection of $(n-k)$ edges, being k the number of intended clusters. This way we will get k smaller spanning trees each one corresponding to a cluster.

Using this heuristic for partitioning problems allows us to avoid the problems present in the study of the exact method, in particular the contiguity restriction, and the problems associated with the high computational effort required to run the model, which prevented us from working on larger problems. However, this heuristic is only for the monocriteria case and must be adapted to the multicriteria case.

5.2 Multicriteria Kruskal's partitioning

Initially the graph and the respective weights of the edges, (w_1, w_2), are defined. Then is made the deliberation of the weights based on the value λ, which varies between 0 and 1, using the following function:

$$W(x) = (1-\lambda)\mathrm{w}_1(x) + \lambda\,\mathrm{w}_2(x)$$

After the new weights assigned to the edges, Kruskal's Algorithm is used to obtain the solution as explained in the previous section. The function above allows us to obtain different solutions based on the weighting coefficient λ, some solutions will value more w_1, while other will value more w_2, but in the end we will obtain a set of solutions based on those two weights for the decision maker to choose from.

5.3 Capacity constraints

The algorithm as it is does not ensure the compliance with capacity restrictions, the cuts made on the spanning tree are made by the value of the weights, and can result in clusters with several different sizes, being impossible to control how many nodes we allow in each cluster. For that purpose it was made a different algorithm to change the boundaries of the clusters, or in other words, to change nodes between neighbor clusters. This is done in four steps briefly explained bellow.

Step one: Connect isolated nodes to extreme nodes belonging to clusters above the minimum limit, cutting the existing connection to the other cluster. If possible the lowest weighted connected edge is cut and the highest weighted edge is connected. This is only done for the cases where the minimum limit is above one.

Step two: Some nodes of the clusters above the maximum limit are connected to those bellow the maximum limit, always cutting the existing connections before making the new one. This is done to eliminate the clusters above the maximum limit.

Step three: This is exactly the opposite of the step two. The clusters below the minimum limit are connected to nodes of clusters above the minimum limit, also cutting an old connection before creating a new one. This is done to eliminate the clusters below the minimum limit.

Step four: This is used only if the steps above are not enough to get a feasible solution. It usually happens if there are no possible connections between neighbor clusters. For example, if we have three clusters and we want to pass a node from cluster one (above the maximum) to cluster three (below the minimum), but these two clusters are not neighbors, this is, they do not have at least a non-connected edge between them, we have to give one node to cluster two from cluster one, even if cluster two passes the maximum, and then give one node from cluster two to cluster three. It is as if we are pushing a connection from one cluster to another till it reaches the intended cluster.

This algorithm is to be used right after the Kruskal's, the obtained configurations are then converted to the original weights in order to calculate the solutions, in terms of z_1 and z_2, for each value of λ.

5.4 Critical analysis of the algorithm

The use of this algorithm allows the resolution of the partitioning problem ensuring the constraints of contiguity, integrality and capacity, and the resolution of large problems. However, unlike the multicriteria integer programming model, the algorithm does not guarantee the achievement of optimal solutions. The initial solution obtained from the cuts to the Kruskal's spanning tree would be an optimum result if the capacity constrains were not implemented, but in most cases they are implemented and the reorganization of the clusters is necessary, even though this process is made in a way that minimizes the losses with each change, the solution is usually inferior to the optimum.

Other limitation of the algorithm is related to the restriction "absence of holes", the data structures used in the design of the algorithm does not allow the identification of solutions with embedded areas. Therefore it is not possible to prevent these occurrences, as happened with the integer programming model.

6 Applications

The algorithm was used in two graphs based on real cases, the district of Lisbon and municipality of Elvas.

6.1 District of Lisbon

In this case we have 16 nodes, each one representing a municipality, and the objective is to create six clusters between one and four nodes each. The weights are shown in Figure 2.

The solutions obtained from the algorithm are in the graphic from Figure 3.

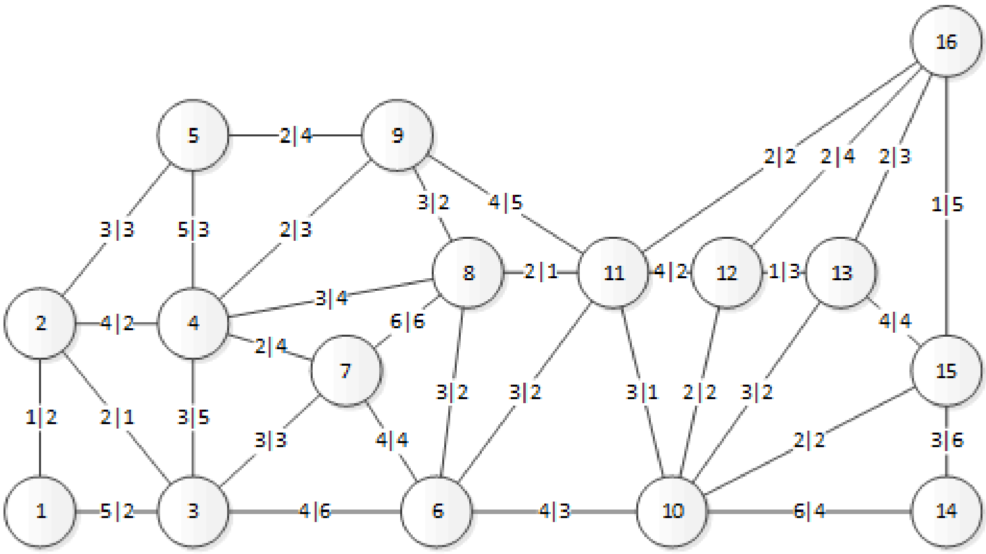

Figure 2 – Graph of the district of Lisbon

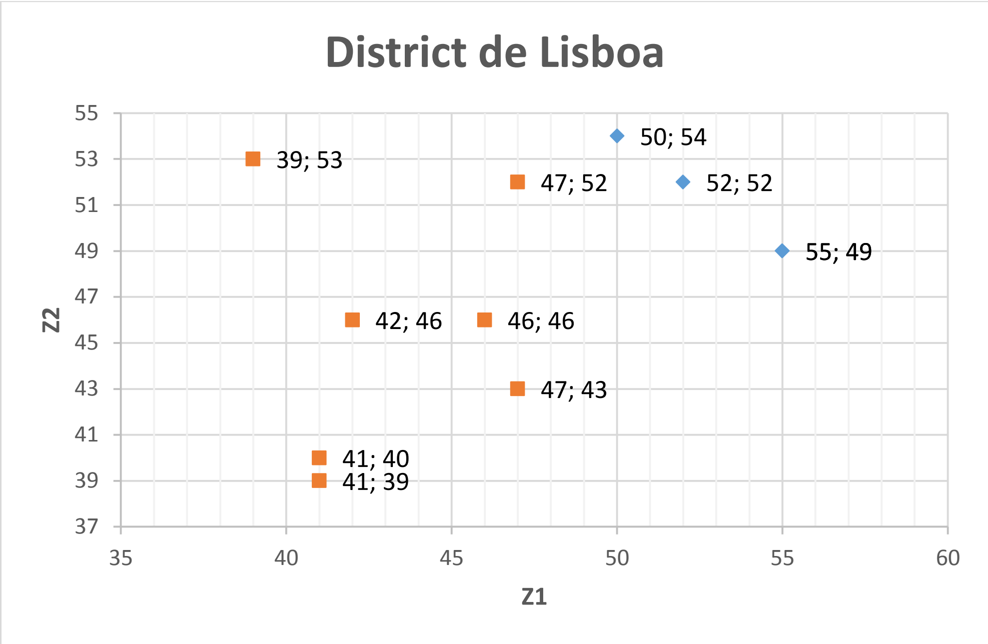

Figure 3 – Solutions for the district of Lisbon

In this example we obtained three non-dominated solutions, represented in the graph as blue paralelograms, and the remaining solutions, represented by squares, are all dominated.

6.2 Municipality of Elvas

In this case we have 11 nodes, each one representing a parish, and the objective is to create four clusters between two and three nodes each. The weights are shown in Figure 4.

The solutions obtained from the algorithm are in the graphic from Figure 5.

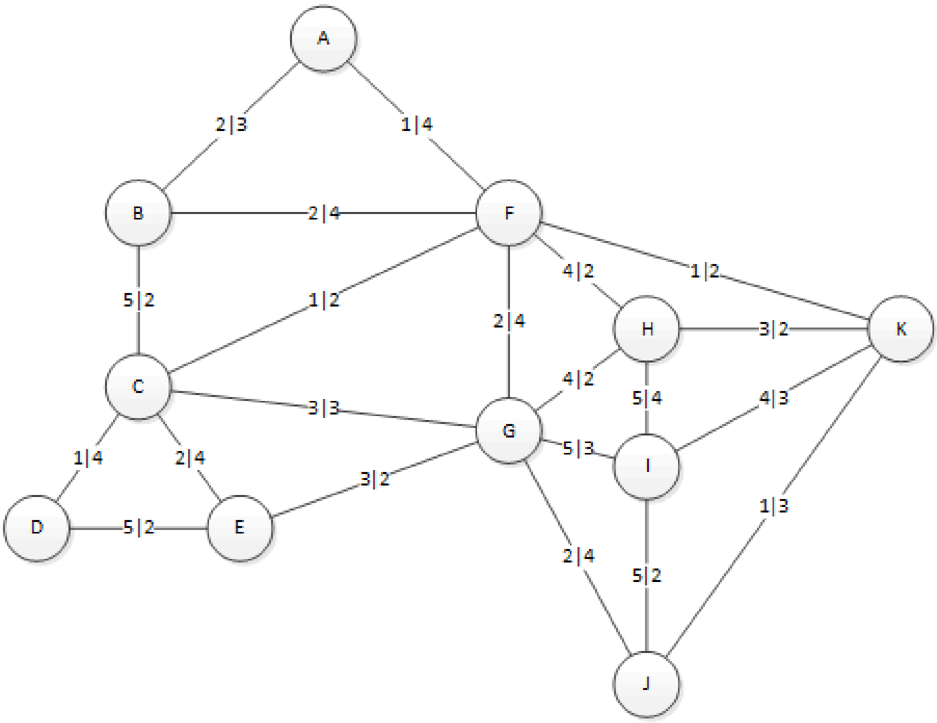

Figure 4 – Graph of the municipality of Elvas

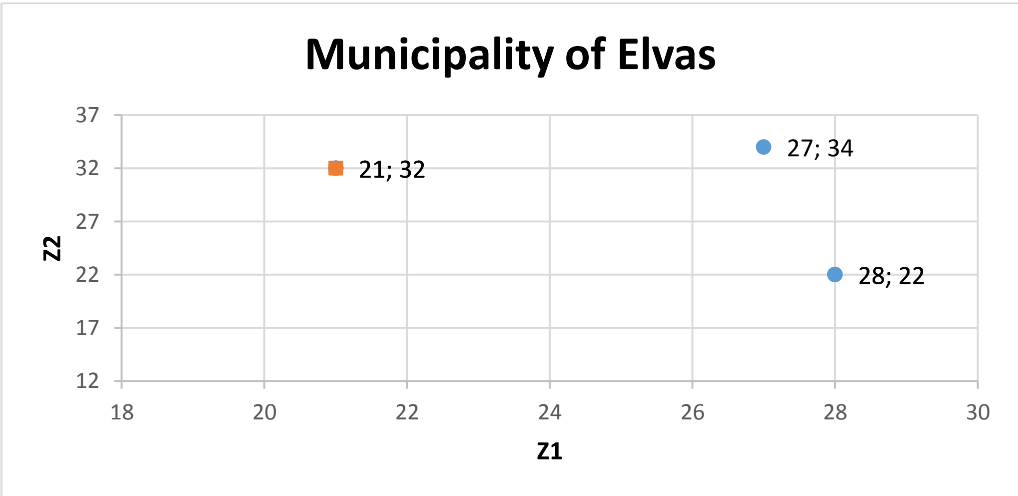

Figure 5 – Solutions for the municipality of Elvas

In this example we obtained two non-dominated solutions (blue circles), and one dominated (orange square).

7 Final Conclusions

In the case of exact methods we conclude that although the solutions are optimal, the model does not guarantee the general constraints of contiguity and “absence of holes, and requires a large computational effort for larger problems, whether in terms of the number of objectives, or the number of elementary units considered.

The Kruskal algorithm applied to the partition problem ensures the general constraints of contiguity and integrality, it is not yet possible to ensure the restriction of "absence of holes", and is easily applied to multicriteria case using a procedure similar to the ε - constraint method. To ensure capacity constraints is necessary to use a complementary algorithm to reorganize the clusters according to these same restrictions. This new algorithm was implemented in JAVA using as starting point the spanning tree obtained from a version of Kruskal's algorithm found in the literature [2].

Although the quality of the solutions is inferior in the algorithm, the lower processing times, and the compliance with the contiguity restriction makes it the most suitable method of the two to solve multicriteria partitioning problems.

8 References

[1] Mehrotra, A., & Trick, M. (1998). Cliques and clustering: A combinatorial approach. *Operations Research Letters*, *22* (1), 1-12.

[2] Sedgewick, R., & Wayne, K. (2011). *Algorithms.* Prentice Hall, New Jersey, United States of America.

[3] Apollonio, N., Becker, R., Lari, I., Ricca, F., & Simeone, B. (2009). Bicolored graph partitioning, or: gerrymandering at its worst. *Discrete Applied Mathematics*, *157* (17), 3601-3614.

[4] Altman, M. (1998). Modeling the effect of mandatory district compactness on partisan gerrymanders. *Political Geography*, *17* (8), 989-1012.

[5] Chou, C., & Li, S. (2006). Taming the gerrymander—statistical physics approach to political districting problem. *Physica A: Statistical Mechanics and its Applications*, *369* (2), 799-808.

[6] Ricca, F., & Simeone, B. (2008). Local search algorithms for political districting. *European Journal of Operational Research*, *189* (3), 1409-1426.

[7] Mehrotra, A., Johnson, E., & Nemhauser, G. (1998). An optimization based heuristic for political districting. *Management Science*, *44* (8), 1100-1114.

[8] Lemberg, D., & Church, R. (2000). The school boundary stability problem over time. *Socio-Economic Planning Sciences*, *34* (3), 159-176.

[9] Teixeira, J., & Antunes, A. (2008). A hierarchical location model for public facility planning. *European Journal of Operational Research*, *185*(1), 92-104.

[10] Meester, G., Mehrotra, A., Ault, J., & Baker, E. (2004). Designing marine reserves for fishery management. *Management Science*, *50* (8), 1031-1043.

[11] Bergey, P., Ragsdale, C., & Hoskote, M. (2003). A decision support system for the electrical power districting problem. *Decision Support Systems*,*36* (1), 1-17.

[12] Salazar-Aguilar, M., Ríos-Mercado, R., & González-Velarde, J. (2011). A bi-objective programming model for designing compact and balanced territories in commercial districting. *Transportation Research Part C: Emerging Technologies*, *19* (5), 885-895.

[13] Ríos-Mercado, R., & López-Pérez, J. (2013). Commercial territory design planning with realignment and disjoint assignment requirements. *Omega*, *41* (3), 525–535.

[14] Ricca, F., Scozzari, A., & Simeone, B. (2008). Weighted Voronoi region algorithms for political districting. *Mathematical and Computer Modelling*, *48* (9), 1468-1477.

[15] Cirincione, C., Darling, T., & O'Rourke, T. (2000). Assessing South Carolina's 1990s congressional districting. *Political Geography*, *19* (2), 189-211.

[16] Datta, D., & Figueira, J. (2011). Graph partitioning by multi-objective real-valued metaheuristics: A comparative study. *Applied Soft Computing*, *11* (5), 3976-3987.

[17] Ji, X., & Mitchell, J. (2007). Branch-and-price-and-cut on the clique partitioning problem with minimum clique size requirement. *Discrete Optimization*, *4* (1), 87-102.

[18] Ehrgott, M. (2005). *Multicriteria optimization*. Springer, 2nd Edition, Auckland, New Zealand.