Repetitive Models in Gas Transportation Networks

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***Abstract*— The problem of distributing gas through a network of pipelines is formulated as an linearized nonstationary differential repetitive model subject to some flow-pressure constraints of material balances and pressure bounds. The linear model is constructed in the neighborhood of the known basic operating regime of gas delivery. The considered problem is to minimize the total supply cost of a gas transmission company with the minimal guaranteed pressure at the nodes. Some aspects of a comprehensive optimization theory based on a 'constructive approach' are discussed.**

I. Introduction

Gas transportation networks are well known to constitute complex and large scale distributed parameter system of great practical interest. Modeling approaches, numerical methods and optimization of operating modes of gas transport networks have, therefore, been of permanent interest for researchers in the last decades, and a large number of papers were published both in civil engineering and the mathematical community, see e.g. [1], [2], [3], [4]. However, optimization and control of complicated gas networks still remains a challenging problem. The general model of a gas transportation network typically includes a large number of nonlinear elements such as pipelines, gasholders, compressor stations and others. In this paper the mathematical model and corresponding optimization problem of gas network units are introduced on the basis of a so-called constructive approach and the 2-D (space and time) system theory setting, which is a starting point for the further investigation of the complex networks and which provide a fairly well-established mathematical framework. For a representative survey of these see, for example, [5]. The main elements of the theory of constructive optimization for the repetitive processes were developed in [6], [7]. Some aspects of control theory for multidimensional systems are investigated in [8], [9] and application of it to gas networks have been considered in [10]. A major part of the method consists in finding the switching points of the optimal controls. There are other approaches dealing with this problem, where on employs search techniques or local descent methods [11], [12]. The methods are, however, not suitable for large scale computations. This is the driving motivation to investigate corresponding problems for linearizations around predefined trajectories along with real-time capable well-scaled algorithms. In this work the proposed linearization for the models of a gas distribution networks leads to some classes of linear differential processes. The models introduced are shown to be suitable for handling problems of optimal control of pressure and flow in gas transport units. The considered problem is to minimize the total supply cost of a gas transmission company with the minimal guaranteed pressure at the nodes. Some aspects of a comprehensive optimization theory based on a 'constructive approach' are discussed.

II. Gas flow model in pipeline networks

The general model of a gas transportation network typically includes a large number of nonlinear elements such as pipelines, gasholders, compressor stations and others. In this paragraph the mathematical model and corresponding optimization problem for single pipeline units of the gas network are introduced on the basis of the 2-D (space and time) system theory setting based on the results given in [5]. The purpose of the modeling presented here is to guarantee a predefined regime for each pipeline unit. This models are used then for the further investigation of the complex networks and which provide a fairly well-established mathematical framework.

A. *Gas flow model in pipeline unit*

The aim of this section is to use the 2-D control theory (in repetitive model setting) for studying control problems in gas pipeline units. The state space parameters are gas pressure p and mass flow Q at the points of the pipe (we write Q for mass flow which is often used in the literature). For calculating the state space parameters for isothermal gas flow in a long pipeline the following system of non-linear differential equations from the theory of gas dynamics can be used (see [2])

$$\begin{aligned} \frac{\partial Q(t,x)}{\partial t} &= -S\frac{\partial p(t,x)}{\partial x} - \frac{\lambda c^2}{2DS}\frac{Q^2(t,x)}{p(x,t)}, \\ \frac{\partial p(t,x)}{\partial t} &= -\frac{c^2}{S}\frac{\partial Q(t,x)}{\partial x}, \end{aligned} \tag{1}$$

where x denotes the space variable, t the time variable, S the cross sectional area, D the pipeline diameter, c the isothermal speed of sound and λ the friction factor.

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It is known that some important dynamic characteristics of the processes can be evaluated from the linearized model of the processes. The most accurate linear model can be realized in some neighborhood of the known basic regime $\overline{Q}(x,t), \overline{p}(x,t)$ of the considered process. In the section below we give such kind of a linearized model.

B. Linearization scheme

Let (Q,p) and $(\bar{Q},\bar{p})$ are the known state variables for a gas pipeline unit. Therefore, they satisfy the system (1):

$$\begin{aligned} \frac{\partial Q(t,x)}{\partial t} &= -S\frac{\partial p(t,x)}{\partial x} - \gamma\frac{Q^2(t,x)}{p(x,t)}, \qquad (2) \\ \frac{\partial p(t,x)}{\partial t} &= \alpha\frac{\partial Q(t,x)}{\partial x},\ \gamma = \frac{\lambda c^2}{2DS}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \bar{Q}(t,x)}{\partial t} &= -S\frac{\partial \bar{p}(t,x)}{\partial x} - \gamma\frac{\bar{Q}^2(t,x)}{\bar{p}(x,t)}, \qquad (3) \\ \frac{\partial \bar{p}(t,x)}{\partial t} &= \alpha\frac{\partial \bar{Q}(t,x)}{\partial x},\ \alpha = -\frac{c^2}{S}. \end{aligned}$$

Let us the current flow and pressure be presented as disturbances of the given regime

$$Q = \bar{Q} + \Delta Q,\ p = \bar{p} + \Delta p. \qquad (4)$$

Then subtracting from (2) the (3)and using the representation

$$\left[\frac{(\bar{Q}+\Delta Q)^2}{\bar{p}+\Delta p}\right] = \frac{(\bar{Q}(1+\frac{\Delta Q}{\bar{Q}}))^2}{\bar{p}(1+\frac{\Delta p}{\bar{p}})} = \frac{\bar{Q}^2}{\bar{p}}\frac{(1+\frac{\Delta Q}{\bar{Q}})^2}{(1+\frac{\Delta p}{\bar{p}})} =$$

$$= \frac{\bar{Q}^2}{\bar{p}}\left(1 + 2\frac{\Delta Q}{\bar{Q}} + (\frac{\Delta Q}{\bar{Q}})^2\right)\left[1 - \frac{\Delta p}{\bar{p}} + (\frac{\Delta p}{\bar{p}})^2 + ...\right]$$

$$= \frac{\bar{Q}^2}{\bar{p}}\left(1 + 2\frac{\Delta Q}{\bar{Q}} - \frac{\Delta p}{\bar{p}} + (more \quad higher \quad order\ )\right)$$

we have the following linearized equations for disturbances $(\Delta Q, \Delta p)$:

$$\begin{aligned} \frac{\partial \Delta Q}{\partial t} &= -S\frac{\partial \Delta p}{\partial x} - 2\gamma\frac{\bar{Q}^2}{\bar{p}\bar{Q}}\Delta Q - \gamma\frac{\bar{Q}^2}{\bar{p}^2}\Delta p, \\ \frac{\partial \Delta p}{\partial t} &= \alpha\frac{\partial \Delta Q}{\partial x}. \end{aligned}$$

Introducing the new variables (we can say about new local coordinates $\Delta Q \to Q,\ \ \Delta p \to p$) we can present the linearized model in the neighbourhood of the known (pre-assigned/basic) regime $(\bar{Q},\bar{p})$ has the following form

$$\frac{\partial Q}{\partial t} = -S\frac{\partial p}{\partial x} - \delta Q + \beta p,\ \frac{\partial p}{\partial t} = -\alpha\frac{\partial Q}{\partial x}, \qquad (5)$$

where

$$\delta = 2\gamma\frac{\bar{Q}}{\bar{p}},\ \beta = \gamma\frac{\bar{Q}^2}{\bar{p}^2}, \gamma = \frac{\lambda c^2}{2DS},\ \alpha = \frac{c^2}{S}.$$

Notice that (5) can be reformulated as a damped wave equation in the pressure:

$$\frac{\partial^2}{\partial t^2}p + \delta\frac{\partial}{\partial t}p = \alpha\frac{\partial^2}{\partial x^2}p - \alpha\beta\frac{\partial}{\partial x}p \qquad (6)$$

Note that a basic/pre-assigned regime $(\bar{Q},\bar{p})$ can be obtained by various different approaches, see e.g. [3]. Wave equations on networks also in the context of optimal control have been considered e.g. in [13].

C. Linearized Pipeline networks model

In this paragraph the obtained linearized pipeline unit model is used for the further investigation of the complex networks and which provide a fairly well-established mathematical framework. A simple part of the complex pipeline network is illustrated in Figure 1. The components of this network are pipes and nodes. Each node can be presented by the following elements: the space discretization point, the multijunction point, the offtake, compressor inlet/outlet or a source outlet. As usually the main parameters of the gas network model are gas pressure p^i and mass flow Q^i at the pipe i. Analogously to the single pipe unit, the hydrodynamics along each pipe of the network is described by the continuity equations and the momentum equations of the form [2]

$$\begin{aligned} \frac{\partial Q^i(t,x)}{\partial t} &= -S\frac{\partial p^i(t,x)}{\partial x} - \frac{\lambda c^{i2}}{2D^iS^i}\frac{Q^{i2}(t,x)}{p^i(x,t)}, \qquad (7) \\ \frac{\partial p^i(t,x)}{\partial t} &= -\frac{c^2}{S^i}\frac{\partial Q^i(t,x)}{\partial x}, \end{aligned}$$

where x denotes the space variable, t the time variable, S^i the cross sectional area, D^i the pipeline diameter, c^i the isothermal speed of sound and λ^i the friction factor.

The linear model can be realized in some neighborhood of the known basic regime of gas delivery through network $\overline{Q^i}(x,t), \overline{p^i}(x,t)$. For each pipe we can apply the linearization approach described in the previous section. Denote by I_v the set of nodes for the considered networks. Analogously (5) the dynamics along each pipe $i \in I_v$ associated with the node v are described by couple of linearized partial differential equations in both the time and spatial dependent variables p^i and Q^i:

$$\begin{aligned} \frac{\partial Q^i(x,t)}{\partial t} &= -S^i\frac{\partial p^i(x,t)}{\partial x} \\ &- \ \delta Q^i(x,t) - \beta^i p^i(x,t) + \mu^i r^i(x,t), \\ \frac{\partial p^i(x,t)}{\partial t} &= -\alpha^i\frac{\partial Q^i(x,t)}{\partial x} + \nu^i q^i(x,t),\ i \in I_v \qquad (8) \end{aligned}$$

where $\mu^i,\ \nu^i$ are some normalizing coefficients. On the other side, the physical meaning of the control functions $r^i(t,x)$ can be treated, for example, as a correcting pressure generated by compressor station and gasholders to increase the velocity $\frac{\partial Q^i(t,x)}{\partial t}$ of the running gas volume for the considered gas unit. Analogously, the control variable $q^i(t,x)$ can be interpreted as a an additional flow (supply/offtake) to change the velocity $\frac{\partial p^i(t,x)}{\partial t}$ of the pressure $p^i(t,x)$. To simulate gas flow in a

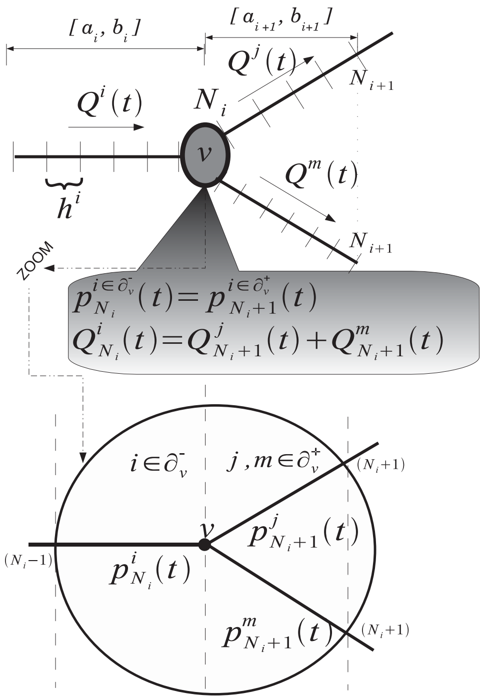

Fig. 1. Network nodes

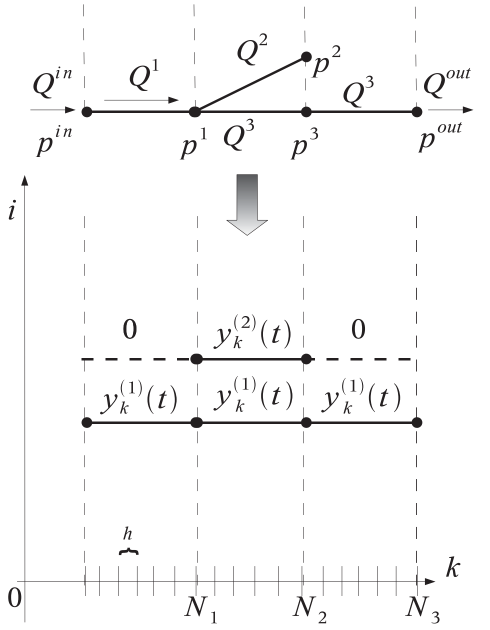

Fig. 2. Gas flow and pressure in discretized network

complex pipeline network, we compute values of pressure (flow) at each discretization point along each pipe, and also the values of pressure (flow) at each node v (offtake or junction), assuming that values of pressure/flow at some selected nodes of the system are known. Modeling the junction (or offtake) points will use two physical principles (see Fig. 1): i) the pipes at the junction point should have a common pressure; ii) the mass flow should be balanced at the junction according to the first Kirchhoff's law.

Next in order to connect these pipes i, we need to add into the model the additional constraints at the nodes where the pipes $i \in I_v$ are linked. The flow dynamic at each node v keep two requirements. Firstly, the Kirchhoff's law must hold for any node. This physical law ensures flow balance in the node. This means that the sum of ingoing gas flows must be equal the sum of outgoing gas flows. Secondly, the pressure at the common node should be equal for all ingoing and outgoing pipes.

The formalization of these requirements at the arbitrary node v can be done as follows. Let ∂_v^+ be the set of outgoing pipes of node v and ∂_v^- be the set of ingoing pipes. For example, at the balance node v on Figure 1 we have

$$i \in \partial_v^-; \quad j, m \in \partial_v^+.$$

Hence, the mentioned pressure requirements give

$$p^i(b_i, t) = p^j(a_{i+1}, t) = p^m(a_{i+1}, t), \tag{9}$$

that can be rewritten in the form

$$p^j(b_i, t)(\forall j \in \partial_v^-) = p^j(a_{i+1}, t), \quad \forall j \in \partial_v^+, \tag{10}$$

where b_i the end of the interval $[a_i, b_i]$ which actually equal to the length L_i of the pipe i, and a_{i+1} is a beginning of the next pipe $(i+1)$. In fact, b_i concises with a_{i+1}. The Kirchhoff's law leads to another constraint of the form

$$\sum_{j\in\partial_v^-} Q^j(b_j, t) = \sum_{m\in\partial_v^+} Q^m(a_m, t) \tag{11}$$

III. Repetitive model in gas pipeline networks

To simplify the net model description we will image the net node pipes collection as the collection of the lines, divided onto segments $[a_i, b_i]$ equal to length of the corresponding pipes (see Fig 2). The total number of the lines associated to the given node v is defined as :

$$M_v = \max\{number \;\; of \; i \in \partial_v^-, \;\; number \;\; of \; i \in \partial_v^+\} \tag{12}$$

Note that the number of the lines involved depends on the node under consideration. This fact produces the nonstationarity of the introduced model. The total lines used in the net model is determined as follows

$$M = \max_v \{M_v | \; v \in I\} \tag{13}$$

where I means the set of nodes for the given network. For each pipeline we introduce the discrete grid formed by the points $x^{i_k} \in [a_i, b_i]$, where

$$x^{i_k} = a_i + kh^i, \; h^i = \frac{b_i - a_i}{N_i}, \; k = 0, ..., N_i.$$

In accordance with the representation of the nodes and associated pipes as a collection of the lines, we divide these lines by segments $[a_1, b_1], [a_2, b_2], ..., [a_{K_v}, b_{K_v}]$. We can then renumbering the discretization points and calculate the values $Q^i(x^{i_k}, t)$ and $p^i(x^{i_k}, t)$ of the unknown functions $Q^i(x, t)$

and $p^i(x,t)$ in the nodes of integer lattice $\{(kh_i)\}$, where $k = 1, 2, \dots, N_1, N_1+1, \dots N_2, N_2+1, \dots, N_{K_v}$, $i = 1, ..., K_v$, K_v denotes the number of segments of the divided lines.

In order to obtain the desired repetitive network model, we approximate the derivatives the linear partial differential equations (8) we by the backward differences

$$\begin{aligned}\frac{\partial Q^i(t,x)}{\partial x} &\simeq \frac{Q^i(t,x) - Q^i(t,x-h)}{h},\\ \frac{\partial p^i(t,x)}{\partial x} &\simeq \frac{p^i(t,x) - p^i(t,x-h)}{h}.\end{aligned} \quad (14)$$

Replace now the spatial derivatives in (8) by the obtained sampled values

$$\begin{gathered}Q^i_k(t) = Q(x^{i_k}, t),\ p^i_k(t) = p(x^{i_k}, t), \\ r^i_k(t) = r(x^{i_k}, t),\ q^i_k(t) = q(x^{i_k}, t)\end{gathered} \quad (15)$$

Then the system (8) can be rewritten as follows

$$\begin{aligned}\dot{Q}^i_k(t) &= -\delta Q^i_k(t) - (\frac{S^i}{h^i} + \beta^i) p^i_k(t) \\ &+ \frac{S^i}{h^i} p^i_{k-1}(t) + \mu^i r^i_k(t),\end{aligned} \quad (16)$$

$$\dot{p^i}_k(t) = \frac{-\alpha^i}{h^i} Q^i_k(t) + \frac{\alpha^i}{h^i} Q^i_{k-1}(t) + \nu^i q^i_k(t) \quad (17)$$

$$N_i \leq k \leq N_{i+1},\ t \in [0,T], i = 1, 2, \dots, M_v,$$

and the conditions (10)—(11) now are:

$$p^s_{N_i}(t) = p^j_{N_i+1}(t), j \in \partial^+_v,\ s \in \partial^-_v \quad (18)$$

$$\begin{aligned}\sum_{s\in\partial^-_v} Q^s_{N_i}(t) &+ \sum_{j\in\partial^+_v} Q^j_{N_i+1}(t) = 0, \quad (19)\\ i &= 1, \dots, M_v\end{aligned}$$

Next in order to get the desired repetitive model denote

$$y^i_k(t) = \begin{bmatrix} Q^i_k(t) \\ p^i_k(t) \end{bmatrix}, u^i_k(t) = \begin{bmatrix} r^i_k(t) \\ q^i_k(t) \end{bmatrix} \quad (20)$$

Then (16) can be rewritten as

$$\frac{dy^i_k(t)}{dt} = \overbrace{\begin{pmatrix} -\delta_i & -(\beta_i + \frac{S_i}{h_i}) \\ \frac{-\alpha_i}{h_i} & 0 \end{pmatrix}}^{A^i} y^i_k(t) \quad (21)$$

$$+ \overbrace{\begin{pmatrix} 0 & \frac{S_i}{h_i} \\ \frac{\alpha_i}{h_i} & 0 \end{pmatrix}}^{D^i} y^i_{k-1}(t) + \overbrace{\begin{pmatrix} \mu_i & 0 \\ 0 & \nu_i \end{pmatrix}}^{B^i} u^i_k(t),$$

where $N_i \leq k \leq N_{i+1},\ t \in [0,T],\ i = 1, \dots, M_v$.

Also the nodal conditions we will rewrite in the suitable form using the following vectors of the suitable dimensions:

$$e = \begin{bmatrix} 1 & 0 \end{bmatrix},\quad l = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Then $ey^i_k(t) = Q^i_k(t),\ ly^i_k(t) = p^i_k(t)$ and hence

$$\begin{aligned}ly^{s\in\partial^-_v}_{N_i}(t) &= ly^{j\in\partial^+_v}_{N_i+1}(t)\ ; \quad (22)\\ \sum_{s\in\partial^-_v} ey^s_{N_i}(t) &+ \sum_{j\in\partial^+_v} ey^j_{N_i+1}(t) = 0,\\ i &= 1, \dots, M_v\end{aligned}$$

Then the dynamical model of a pipeline network (16) is defined over $0 \leq t \leq T$, $k \geq 1$ by the repetitive state space model of the form

$$\dot{y}^i_k(t) = A^i y^i_k(t) + D^i y_{k-1}(t) + B^i u^i_k(t), \quad (23)$$

$$N_i \leq k \leq N_{i+1},\ t \in [0,T],\ i = 1, \dots, M_v.$$

with the state constraints:

$$\begin{aligned}ly^{s\in\partial^-_v}_{N_i}(t) &= ly^{j\in\partial^+_v}_{N_i+1}(t);\\ \sum_{s\in\partial^-_v} ey^s_{N_i}(t) &+ \sum_{j\in\partial^+_v} ey^j_{N_i+1}(t) = 0, \quad (24)\\ i &= 1, \dots, M_v\end{aligned}$$

In order to complete the description of the process for the network pipeline model, it is necessary to specify the boundary and initial conditions. The boundary condition can be treated as a standard pumping regime. The initial conditions describe the values of this standard regime calculated in the starting moment $t = 0$ at the discrete grid points of the pipe.

In order to formulate the optimization problem we need to specify a cost function. In particular, the total gas volume needed to guarantee some technically approved pressure values at the pre-assigned points of the network appears to be an appropriate choice. Another forms of the cost functions are determined by concrete requests.

The (23)—(24) fully describes the dynamics along every pipes of the complex gas network system. Note, that the obtained repetitive model has the following particularities: firstly, it is nonstationary and secondary, there is a phase state constraints of the form (24). These facts are a motivation to investigate the nonstationary repetitive differential with state constraints given in the section below.

IV. Constrained optimization for nonstationary repetitive processes

Here we develop the method to establish optimality conditions in the classic form for a particular case of differential repetitive processes with nonlinear inputs and nonlocal state-phase terminal constraints of general form. The problem statement in the proposed form follows from the mathematical modeling of the distributed gas network given above. The obtained results are traditional for classic optimal control theory. In order to study the model obtained above and give their extension to other cases we will consider the general convex case of objective cost function and state constrains of general form in the given time moments. Also, we consider the case when the model matrixes are functions of the temporary variables that presents the most hard case for nonstationary dynamics. It seems that such kind of mathematical formalization is closely to practical implementation of the model.

In practice, a repetitive process will only ever complete a finite number of passes. Hence we consider repetitive processes modeled by a system of linear differential equations with variable coefficients. Let $T = [0, t^*]$ be a given interval of values of the continuous independent variable $t \in T$ and $K = \{1, 2, ..., N\}$, $N < +\infty$ be a set of values of the discrete variable $k \in K$. Also introduce the control and state vectors as $u_k(t) \in R^r$ and $y_k(t) \in R^n$ respectively. Then the repetitive processes considered in this paper are described by

$$\frac{dy_k(t)}{dt} = A(t)y_k(t) + D(t)y_{k-1}(t) + b_k(u_k(t), t), \quad k \in K,\ t \in T \qquad (25)$$

where the last nonlinear term represents the input signal actually applied to the process. To complete the description, it is necessary to specify the boundary conditions which are here taken to be of the form

$$y_k(0) = \alpha(k), k \in K, \quad y_0(t) = \beta(t), \quad t \in T \qquad (26)$$

Note also that it is possible to augment the above model to include the fact that the pass profile can be a vector valued function of the state dynamics.

Now we define the class of available and admissible input signals for the above model. We say that the function $u : K \times T \to R^r$ is available for (25) if it is measurable with respect to t for fixed $k \in K$, and satisfies the constraint $u_k(t) \in U$, $k \in K$, for almost all $t \in T$, where U is a given compact set from R^r. Also the function $y : K \times T \to R^n$ is a solution of (25) corresponding to the given available control $u_k(t)$ if it is absolutely continuous with respect to $t \in T$ for each fixed $k \in K$ and satisfies (25) for almost all $t \in T$ and each $k \in K$.

We denote the set of available controls by $U(\cdot)$ and use $M_i,\ M_i \subset R^n,\ i = 1, 2, ..., l$ to denote given compact convex sets. The available control $u_k(t)$ is said to be admissible for the process (25) if the corresponding solution $y_k(t) = y_k(t, \alpha, \beta, u)$ of (25) and (26) satisfies

$$y_N(\tau_i) \in M_i,\ i = 1, 2, ..., l \qquad (27)$$

where $0 < \tau_1 < \tau_2 < ... < \tau_l = t^*$ are specified elements of T.

The optimal control problem considered in this paper can now be stated as: Minimize a cost function of the form

$$J(u) = \varphi(y_N(\tau_1), y_N(\tau_2), ..., y_N(\tau_l)) \qquad (28)$$

for processes described by (25) and (26) in the class of admissible controls $u_k(t) \in U(\cdot)$.

We also assume that: the $n \times n$ matrix functions $A(t)$ and $D(t)$ and the $n \times 1$ function $\beta(t)$ are measurable and integrable on T, the function $b : K \times U \times T \to R^n$ is continuous with respect to $(u, t) \in U \times T$ for each fixed $k \in K$ and the function $\varphi : R^{nl} \to R$ is convex. It is easy to see that these conditions guarantee the existence and uniqueness of an absolutely continuous solution of (25) and(26) for any available control $u_k(t)$. To guarantee the existence of optimal control, throughout this paper we assume that the set of admissible controls is non-empty.

At this stage, it is possible to give some motivation for considering a cost function of this form by reference to the general area of iterative learning control. This is a technique for controlling systems operating in a repetitive (or pass-to-pass) mode with the requirement that a reference trajectory $r(t)$ defined over a finite interval $0 \le t \le T$ is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task to high precision, chemical batch processes or, more generally, the class of tracking systems. Motivated by human learning, the basic idea is to use information from previous executions of the task in order to improve performance from pass-to-pass in the sense that the tracking error is sequentially reduced. The objective of such schemes is to use their repetitive process structure (i.e. information propagation from pass-to-pass and along a pass) to progressively improve the accuracy with which the core operation under consideration is performed, by updating the control input progressively from pass-to-pass.

In application, such an iterative learning controller will only ever complete a finite number of passes, say N, and one way to approach control law design is on the basis of minimizing a suitably constructed cost function. The cost function of (28) is an abstraction of this approach. Next we present optimality conditions for the processes described by (25) —(28). The solvability conditions and some properties of the optimization problem were studied in [6], also.

A. *Optimality conditions*

The optimality conditions for (25)—(28) are given by the following theorem.

Theorem. If $u_k^0(t)$, $k \in K$, $t \in T$ is optimal control for the problem (25)–(28) then for almost all $t \in T$ the following conditions

$$\psi_k^T(t) b_{N-k+1}\big(u_{N-k+1}^0(t), t\big) = \min_{v \in U} \psi_k^T(t) b_{N-k+1}(v, t)$$

hold for all $k \in K$. Here the function $\psi : K \times T \to R^n$ is given by

$$\psi_k(t) = \int_0^t \psi_{k-1}^T(\tau) D(\tau) \Phi(\tau, t) d\tau, \qquad (29)$$

$$\psi_1(t) = \lambda(t), \quad k \in K$$

where the function $\lambda(t)$ is the solution of the ordinary linear differential equation

$$\frac{d\lambda(t)}{dt} = -\lambda^T(t) A(t), \qquad (30)$$

with jump conditions

$$\lambda(\tau_j -) - \lambda(\tau_j +) = g_j^0, \quad j = 1, \ldots, l-1$$

and where $g^0 = (g_1^0, ..., g_l^0)^T \in R^{nl}$ is the maximizing vector for the smallest root δ^0 of the equation $\Lambda(\delta) = 0$, where

$$\Lambda(\delta) = \max_{\|g\|_{R^{nl}}=1} \{ g^T c - \max_{z \in K(\delta)} g^T z + \max_{u \in U(\cdot)} g^T S u \}. \quad (31)$$

Here the set $K(\delta)$ is defined as

$$K(\delta) = \{ z \in R^{nl},\ z \in M,\ \varphi(z) \leq \delta \}$$

where $M = M_1 \times M_2 \times \ldots \times M_l \subset R^{nl}$, and δ is a fixed number from R. The mapping $S : U(\cdot) \to R^{nl}$ is the vector valued function $Su = (S_1 u, S_2 u, \ldots, S_l u)^T$ with

$$S_j u = \sum_{i=1}^{N-1} (\mathcal{P}^i Q b_{N-i}(u_{N-i}(\tau_j))) \quad (32)$$

$$+ \int_0^{\tau_j} \Phi(\tau_j, t) b_N(u_N(t), t) dt,\ j = 1, 2, \ldots, l \quad (33)$$

and where the mappings involved are as follows

$$(Qf)(\tau) = \int_0^{\tau} \Phi(\tau, t) f(t) dt, \quad \tau \in (0, \alpha) \quad (34)$$

and

$$(\mathcal{P} f)(\tau) = \int_0^{\tau} \Phi(\tau, t) D(t) f(t) dt, \quad \tau \in (0, \alpha) \quad (35)$$

with its power composition

$$(\mathcal{P}^k f)(\tau) = \mathcal{P}(\mathcal{P}^{k-1} f)(\tau), \quad \tau \in (0, \alpha)$$

The introduced mappings use the $n \times n$ matrix function $\Phi(\tau, t)$ which solves the following differential equation

$$\frac{d\Phi(\tau, t)}{d\tau} = A(\tau)\Phi(\tau, t), \quad \Phi(t, t) = I_n \quad (36)$$

where I_n denotes the $n \times n$ identity matrix.

Proof. In accordance with [6] for the optimal control $u^0 = \{u_k^0(t), k \in K, t \in T\}$ of the problem (25) — (28) the number $\delta^0 = J(u^0)$ is the smallest root of the equation $\Lambda(\delta) = 0$ and

$$u^0 \ = \arg \min_{u \in U(\cdot)} L(g^0, u),\ \text{where} \quad (37)$$

$$L(g^0, u) = \sum_{j=1}^{l} g_j^{0T} \Bigg(\sum_{i=1}^{N-1} \mathcal{P}^i Q b_{N-i}(u_{N-i}, t)(\tau_j) + \int_0^{\tau_j} \Phi(\tau_j, t) b_N(u_N(t), t) dt \Bigg)$$

Introduce the function

$$\lambda(t) = \sum_{i=j+1}^{l} (g_i^0)^T \Phi(\tau_i, t), \quad \tau_j \leq t < \tau_{j+1}. \quad (38)$$

From (36) it follows that $\lambda(t)$ satisfies (30). Then using notations (29) the condition (37) can be represented as

$$\min_{u \in U(\cdot)} L(g^0, u) = \min_{u_k(\cdot) \in U(\cdot)} \Bigg\{ \psi_1^T(t) b_N(u_N(t), t) + \cdots + \psi_N^T(t) b_1(u_1(t), t) \Bigg\} = \sum_{k=1}^{N} \min_{v \in U} \psi_k^T(t) b_{N-k+1}(v, t)$$

which completes the proof.

V. Conclusion

This paper presents some results of mathematical description of the distributed gas networks in framework of multistage modeling. The subject of ongoing work is the development of control algorithms for considered models to apply for gas transportation networks with real data. These results constitute a very promising base for further research towards applications to the real models. It is necessary to add that this note covers only first attempts to investigate the pipeline networks from the point of differential repetitive processes, and hence a rich material to be the subject for further work still remains.

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