STOCHASTIC INTEGER OPTIMIZATION AND APPLICATIONS IN ENERGY SYSTEMS

By

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To my lovely daughter, De Hao Zheng, my wife and my parents

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Everyday, we are faced with a lot of uncertainties and discrete decisions. Stochastic mixed integer programming is well suited to help us handle this situation. However, this type of optimization problems are not easy to solve. The first half this dissertation gives a brief review of stochastic programming and stochastic mixed integer programming, and proposes a solution method, embedded Benders' decomposition. Of all these difficult problems, those arising from energy systems are very urgent and important, since in the modern age, instead of human force, people rely more on other energy sources to keep the whole society running. The second half of this dissertation is about stochastic integer optimization applications in energy systems. Firstly, this dissertation studies the stochastic security constrained unit commitment problem, which includes both day-ahead and real time unit commitment, making it a very typical stochastic mixed integer program. Numerical results show that embedded Benders decomposition method suits well this problem, especially when it has a large number of scenarios. Secondly, this dissertation discusses optimization models and algorithms in the natural gas industry, and proposes natural gas transmission system expansion planning models which include both natural gas transmission network expansion and LNG (Liquified Natural Gas) terminals location planning. These models take into account the uncertainties of demands and supplies in the future, which make the models stochastic integer programs with discrete subproblems. In addition, this dissertation considers risk

control in these models by including probabilistic constraints, such as a limit on CVaR (Conditional Value at Risk). In order to solve the large-scale problems, especially those with large numbers of scenarios, the embedded Benders decomposition algorithm is applied to tackle the discrete subproblems. Numerical results show that this algorithm is efficient for solving large scale stochastic natural gas transportation system expansion planning problems.

CHAPTER 1
INTRODUCTION

Optimization or mathematical programming is a very vibrant and important branch of mathematics, which has a broad area of applications and great implications in our real life. A mathematical programming problem is to minimize an objective function either or not with respect to a set of constraints, which is shown as follows,

$$\begin{aligned} \text{Min} \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0, \end{aligned} \tag{1–1}$$

where x is the decision variable vector in $\mathbb{R}^n$, and $f(x)$ and $g(x)$ are vectors of functions of x in $\mathbb{R}^l$ and $\mathbb{R}^m$ respectively. When $l = 1$, we have a unique objective function, which makes the type of most studied mathematical programming problems. However, there are a lot of problems with $l > 1$, which are called multiobjective optimization. Depending on the properties of the objective function $f(x)$, constraints $g(x)$ and restrictions on the decision variables, many different types of mathematical programming problems are defined, and accordingly different solution techniques are developed.

Following the pioneering research by Dantzig, von Neumann, Kuhn, Tucker, etc, in the 1940s and the 1950s, mathematical programming has been gaining more and more attentions, while influencing the real life more broadly and deeply. Of all research fields of mathematical programming, linear programming is a very basic type, which is rich in a lot application areas, and is the first mathematical programming experience or class for most of the researchers in optimization and operations research. In linear programming problems, both $f(x)$ and $g(x)$ are linear functions of x, shown as follows,

$$\begin{aligned} \text{Min} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b, \\ & x \geq 0, \end{aligned} \tag{1–2}$$

where c and b are column vectors given in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and A is a fixed matrix in $\mathbb{R}^{m\times n}$.

Because of the extensive studies, well developed theories, and solution techniques on linear programming, such as simplex method, interior point method, etc, many other optimization studies, e.g., stochastic programming, mixed integer programming, etc, are based on linear programming.

1.1 Stochastic Programming

In reality, there is a great need for us to incorporate the future uncertainties when we try to make some decisions right now, such as applications in energy, finance, economics, business, transportation, etc. Stochastic programming or optimization has been continuously gaining more and more popularity since its birth in 1950's. Stochastic programming takes into account all possible future outcomes, and assumes that we perform optimally under any situation when the uncertainties unfold, and minimizes the summation of the current cost and the future expected cost. In long run, this actually can help us to achieve a better current decision than it would have been if we only consider some scenario(s) or even the expected outcome.

1.1.1 General Formulation

The most extensively studied stochastic programming problems are the stochastic linear programs, which only involve linear constraints and continuous variables. In the two stage stochastic programs, the randomness is only observed once. Decisions need to be made both before and the after the uncertainties unfold. The general formulation of this type of problems is shown as follows,

$$\begin{aligned} \text{Min} \quad & c^T x + E\left[Q(x, w)\right] \\ \text{s.t.} \quad & Ax \geq b, \end{aligned} \tag{1–3}$$

where w is an random vector, and c and x are respectively a given vector of costs and a decision vector in $\mathbb{R}^n$, and $E\left[Q(x, w)\right]$ is called the value function or recourse function, which is the expected future cost of $Q(x, w)$, which is cost of the decisions made after the uncertainties unfold and is shown as follows,

$$\begin{aligned} Q(x, w) = \text{Min} \quad & [d(w)]^T y \\ \text{s.t.} \quad & F(w)y \geq g(w) - T(w)x, \\ & y \geq 0, \end{aligned} \tag{1–4}$$

where y is the decision vector after the outcome of the random variable w is observed. $F(w)$ is called the recourse matrix, and $T(w)$ is the technology matrix. But in most of the literature, instead of $F(w)$, a fixed recourse matrix F is used, which is considered to be independent of the scenarios. (1–12) and (1–13) are usually referred to as the first and second stage problems respectively. When the random variable is discretely distributed and has a finite number of outcomes, the stochastic program is completely a deterministic linear programming problem as follows,

$$\begin{aligned} \text{Min} \quad & c^T x + \sum_{\xi \in \Xi} Prob(\xi)[d(\xi)]^T y(\xi) \\ \text{s.t.} \quad & Ax \geq b, \\ & F(\xi)y(\xi) + T(\xi)x \geq g(\xi), \quad \forall \xi \in \Xi, \\ & y(\xi) \geq 0, \quad \forall \xi \in \Xi, \end{aligned} \tag{1–5}$$

where $y(\xi)$ is the decision vector for scenario ξ with corresponding probability $Prob(\xi)$, and Ξ is the set of all possible outcomes of random variable w. Due to the special structure of the above problem, decomposition algorithms are very useful when experiencing a big number of scenarios. Benders' decomposition [9] is well suited to handle this situation. A brief introduction of the Benders' decomposition algorithm is presented in the following section.

In the two-stage stochastic programming problems, we assume that the random variable will be realized only once, and decisions are made both before and after that event. However, in reality, we may need to make a series of decisions along a time sequence, within which a sequence of random events happen alternately with the decisions. Multistage stochastic programming is a very good modeling tool for this type of problems. It generalizes the concept of the two-stage stochastic programming, and can be seen as a sequence of two-stage stochastic programs. Its formulation is shown as follows,

$$\begin{aligned} Q_t(x_{t-1}(\zeta),\xi) \;=\; \text{Min} \quad & [d_t(\xi)]^T x_t(\xi) + E\left[Q_{t+1}\left(x_t(\xi)\right)\right] \\ \text{s.t.} \quad & F_t(\xi)x_t(\xi) \geq g_t(\xi) - T_t(\xi)x_{t-1}(\zeta), \\ & x_t(\xi) \geq 0, \end{aligned} \tag{1–6}$$

where the subscript t and $t-1$ denote the stages, ξ and ζ denote the scenarios of stage t and $t-1$ respectively. Multistage stochastic programming problems are much more difficult to solve due to the curse of dimensionality explosion.

1.1.2 Solution Methods

The computational difficulty of stochastic programming lies in the fact that it involves too many decision variables and constraints because for each scenario a whole second stage formulation, (1–13), is required. If the first stage decision is given, the second stage problem can be decomposed to many smaller problems which can be solved separately. Then we can provide some feedback to the first stage to tell whether the given first stage solution is good or not. Benders' cut is a very good media that coordinates this back-and-forth communication. Benders decomposition was proposed by Benders [9] in 1962, which is explained briefly as follows. Suppose we are dealing with the following optimization problem [P],

$$\begin{aligned} \text{[P]:} \quad \text{Min} \quad & c^T x + d^T y \\ \text{s.t.} \quad & x \in \mathbf{X}, \\ & Ex + Fy \geq g, \\ & y \geq 0. \end{aligned} \tag{1–7}$$

Given a solution $\hat{x}$, the above program reduces to a linear programming problem as follows, called [SP],

$$\begin{aligned} \text{[SP]:} \quad \text{Min} \quad & d^T y \\ \text{s.t.} \quad & Fy \geq g - E\hat{x}, \qquad \leftarrow u \\ & y \geq 0. \end{aligned} \tag{1–8}$$

Its corresponding dual program is shown as follows, called [DSP],

$$\begin{aligned} \text{[DSP]:} \quad \text{Min} \quad & (g - E\hat{x})^T u \\ \text{s.t.} \quad & F^T u \leq d, \\ & u \geq 0. \end{aligned} \tag{1–9}$$

[DSP] and [SP] share the same optimal objective value since they are both linear programs. The optimal objective value of [SP] or [DSP] is a piecewise linear function with respect to master problem decision variable, x. In [DSP], the feasible region is not related to the master problem decision, x. [DSP] is a linear program, and then its optimal solution is on the vertex of the feasible region. This means if we can get all the extreme points of the [DSP], we can define the value function of the subproblem. Up to now, we assume the [DSP] is feasible and bounded. If the [DSP] is infeasible, which means the [SP] is unbounded, then the original problem [P] is unbounded. If [DSP] is unbounded, which means the [SP] is infeasible, then the given first stage decision, $\hat{x}$, is not a feasible solution to the original problem [P]. Hence, in order to prevent unboundedness or this kind of first stage solution, we need $(g - Ex)^T v \leq 0$, where v is an extreme ray of the unbounded [DSP]. Then the original problem can be redefined as follows,

$$\begin{aligned} \text{[MP]:} \quad \text{Min} \quad & c^T x + \pi \\ \text{s.t.} \quad & x \in \mathbf{X} \\ & \pi \geq (g - Ex)^T \hat{u}^i, \qquad \forall i \in \mathcal{I}, \end{aligned} \tag{1–10}$$

$$(g - Ex)^T \hat{v}^j \leq 0, \qquad \forall j \in \mathcal{J},$$

where $\mathcal{I}$ and $\mathcal{J}$ are the extreme-point set and extreme-ray set of [DSP] respectively. However, including all extreme points and extreme rays would not make it a very efficient formulation. If we only include subsets of all extreme points and rays, we would be able to get an lower bound of the problem. This problem with subsets of extreme points and rays is called restricted master problem, [RMP], which is shown as follows,

$$\begin{aligned} \text{[RMP]:} \quad \text{Min} \quad & c^T x + \pi \\ \text{s.t.} \quad & x \in \mathbf{X}, \\ & \pi \geq (g - Ex)^T \hat{u}^i, \qquad \forall i \in \mathcal{I}^r, \\ & (g - Ex)^T \hat{v}^j \leq 0, \qquad \forall j \in \mathcal{J}^r, \end{aligned} \tag{1–11}$$

where $\mathcal{I}^r$ and $\mathcal{J}^r$ are the extreme-point subset and extreme-ray subset of [DSP] respectively.

An combination of feasible solutions of both master and sub problems yields an upper bound of the original problem. So we can iteratively solve the [RMP] and [DSP] to update the lower bound and upper bound until they match each other. Given a very small value, ϵ, the Benders' decomposition algorithm is shown as follows,

Step 0. Set UB=∞, LB=$-\infty$, $\mathcal{I}^r = \mathcal{J}^r = \emptyset$;

Step 1. Solve [RMP], and optimal solution and objective value are $\hat{x}$ and w respectively;
LB$\leftarrow$ $max(LB, w)$;

Step 2. Solve [DSP], and optimal solution is $\hat{u}$ or extreme ray $\hat{v}$.
UB$\leftarrow$ $min(UB, c^T\hat{x} + (g - E\hat{x})^T\hat{u})$;
$\mathcal{I}^r \leftarrow \mathcal{I}^r \cup \{\hat{u}\}$ or $\mathcal{J}^r \leftarrow \mathcal{J}^r \cup \{\hat{v}\}$;

Step 3. If UB$-$LB$\leq \epsilon$, stop; O/W go to step 1.

When extended to stochastic programming, information from the solution of every subproblem need to be considered. Generally there are two ways of feeding back the future information, by either the aggregated cuts or disaggregated cuts. Van Slyke and Wets [63] proposed the L-shaped method, which adds a single aggregated cut at each iteration, while Birge and Louveaux [11] proposed a multicut method. Detailed discussion of advantages and disadvantages of both algorithms can be found in [10].

Also we can apply Benders decomposition sequentially to deal with the multistage stochastic linear programs. The most difficult part to solve a multistage stochastic programming problem is the dimensionality explosion. The variables and scenarios grows exponentially as the number of stages goes up. Pereira and Pinto (1991) [50] proposed Stochastic Dual Dynamic Programming (SDDP), which can deal with the curse of dimensionality, to solve a multistage hydro power plant planning problem. Their method iteratively use Bender's cut to approximate the expected-cost-to-go function, and use Monte Carlo forward simulation to avoid the explicit enumeration of all possible scenarios.

SDDP is the stochastic version of Dynamic Dual Programming (DDP), which applies Benders' decomposition in the multistage problem. Suppose we have multistage problem which has following format,

$$
\begin{array}{lllllll}
\text{Min} & c_1^T x_1 & +c_2^T x_2 & +c_3^T x_3 & + \cdots & & \\
\text{s.t.} & A_1 x_1 & & & & & \geq b_1, \\
 & E_1 x_1 & +A_2 x_2 & & & & \geq b_2, \\
 & & E_2 x_2 & +A_3 x_3 & & & \geq b_3, \\
 & \cdots & & & & &
\end{array}
$$

Since the subproblem in stage n is only related to the subproblem in stage $n-1$, Benders' cuts can still be applied to achieve the communication between two consecutive stages. Given a solution of stage $n-1$, i.e., x_{n-1}, we can find a Benders' cut for the stage $n-1$ by solving the stage n problem with x_{n-1} being fixed, where the dual optimal solution is the coefficients of the Benders' cut. The DDP algorithm is composed of two major procedures, i.e., the forward and backward iterations, where forward iterations find a feasible solution of each stage and backward iterations find the Benders' cuts for the previous stage. We can get the upper bound by calculating the cost of a feasible solution, the lower bound by calculating the approximated objective function.

In traditional stochastic programming, all variables denoting every scenario are included in the formulation. For the multistage problem, the variable size could be easily increased to billions, trillions, even more. SDDP is an algorithm that nicely avoids the dimensionality explosion by the Monte Carlo forward simulation. In the backward iteration of SDDP , Benders' cuts for stage $n-1$ are obtained by using the average of the optimal dual solutions corresponding to different scenarios of stage n. For example, $\bar{\pi}_{n-1} = \sum_{j=1}^{m} p_n^j \pi_{n-1}^j$, where π_{n-1}^j is the optimal dual solution of stage n subproblem under scenario j, which is associated with probability p_n^j. At each forward iteration, instead of enumerating all possible scenarios, SDDP only find several sample paths by Monte Carlo simulation, and solve the corresponding problems along the sample paths to obtain a feasible solution.

1.2 Stochastic Mixed Integer Programming

In reality, we also need to make a lot of discrete decisions under uncertainties, which need to include integer variables in the optimization problems, begetting the stochastic mixed integer programs. The formulation of two-stage stochastic mixed integer programs are very similar to the two-stage stochastic linear programs, except that it has integer restrictions on the decision variables either in the first stage (1–12), or the second stage (1–13), or both.

1.2.1 Formulation and Previous Approaches

The general formulation of stochastic mixed integer programs is as follows,

$$\begin{aligned} \text{Min} \quad & c^T x + E\left[Q(x, w)\right] && (1\text{–}12) \\ \text{s.t.} \quad & Ax \geq b, \end{aligned}$$

$$x \in \mathbb{R}_+^{n_1} \times \mathbb{Z}_+^{m_1},$$

where $Q(x, w)$ is the recourse function shown as follows,

$$\begin{aligned} Q(x, w) = \text{Min} \quad & [d(w)]^T y \\ \text{s.t.} \quad & F(w)y \geq g(w) - T(w)x, \\ & y \in \mathbb{R}_+^{n_2} \times \mathbb{Z}_+^{m_2}. \end{aligned} \tag{1–13}$$

Stochastic mixed integer programming (SMIP) has been drawing a lot of attention recently. When integer variables exist only in the first stage, the problem is relatively easier to solve, since generally L-shaped method ([63]) or Benders decomposition ([9]) would work. This is because the value function for the second stage is convex with respect to the first stage variables. However, the second stage value function becomes non-convex and discontinuous, in general, when there are integer variables within the second stage as discussed in [12]. This makes Benders decomposition ([9]) or generalized Benders decomposition ([31]) not readily applicable because of the duality gap of integer programs. Within the last two decades, a lot of research has been done to solve SIMP problems with integer variables in the second stage. Laporte and Louveaux [38] proposed a decomposition-based branch-and-cut method, where both feasibility and optimality cuts are applied, for SMIP with pure binary variables in the first stage. Carøe and Tind [18] proposed a generalized L-shape method by generalized Benders decomposition ([31]), where both Gomory cuts and branch-and-bound algorithm are applied. Sherali and Fraticelli [60] and Sherali and Zhu [61] developed modified Benders decomposition methods by sequentially convexifying the discrete subproblem using reformulation-linearization technique ([59]). Ntaimo and Sen [48], Sen and Higle [57] and Ntaimo [47] proposed decomposition methods for SMIP with random recourse and discrete second stage based on disjunctive programming ([5]). Ahmed et al. [1]

developed a finite branch-and-cut solution algorithm for SMIP with a second stage program of pure integer variables.

1.2.2 Generating Valid Benders Cuts from Discrete Subproblems

In order to tackle this difficulty, we may need to convexify the second stage mixed integer programs to get valid and effective Benders cuts for the first stage or master problem. The approach made by Sherali and Fraticelli [60] is inspiring. They analyze the following two stage mixed integer program,

$$\begin{aligned}
[\mathbf{P}] : \text{Min} \quad & c^T x + d^T y \\
\text{s.t.} \quad & Ax + Dy \geq b, \\
& x \in X, x \in \{0, 1\}^n, y \in Y,
\end{aligned}$$

where y is a vector including integer variables, and X is a nonempty polytope. If we can find the convex hull of the following region,

$$\{Ax + Dy \geq b, y \in Y\}, \tag{1–14}$$

for any given x, then Benders' decomposition can be applied because the linear relaxation of the subproblem (convex hull formulation) will have the same optimal solution as the discrete subproblem, and then the subproblem can simply be treated as a linear programming problem. To achieve this goal, Reformulation-Linearization-Technique or Lift-and-Project cuts are iteratively added to the subproblem in [60]. These are called global cuts, which means that they are valid for the original problem [**P**] but focus on cutting the region of (1–14), which has the following format,

$$\alpha_k^T y + \psi_k^T x \geq \beta_k, \quad k = 1, \dots, K, \tag{1–15}$$

where k denotes the k^{th} cut. There are also other cuts ([6], [58], etc) which possess the same properties (1–15) has. With these cuts added, the relaxed subproblem will be as follows,

$$\begin{aligned} \text{Min} \quad & d^T y \\ \text{s.t.} \quad & Dy \geq b - Ax, \\ & \Gamma y \geq \gamma, \\ & \alpha_k^T y \geq \beta_k - \psi_k^T \hat{x}, \quad k = 1, \dots, K, \end{aligned}$$

where $\Gamma y \geq \gamma$ is the linear relaxation of the set Y. Then we just need to solve the above linear problem to produce Benders cut for the first stage. For convenience, the subproblem is assumed to be feasible given any first stage solution, $\hat{x}$, since we can always add an artificial variable and assign a big penalty on it to make the problem feasible. Suppose the optimal dual solutions are ϕ_1, ϕ_2, and ϕ_3 corresponding to the above three constraints respectively. Then a valid Benders' cut can be obtained as follows,

$$z \geq (b - Ax)^T \phi_1 + \gamma^T \phi_2 + \sum_{k=1}^{K} (\beta_k - \psi_k^T x) \phi_{3_k}.$$

Even when the convex hull of the subproblem is not completely obtained, the Benders cuts are still valid to the first stage problem. This is because that the the relaxed subproblem always provides a lower bound to the subproblem, which means it also provides a valid lower bound for z.

1.3 Outline of this Dissertation

The dissertation is organized in such a way that we first introduce our proposed methods for stochastic programming problems, and then describe some stochastic optimization models, especially in the energy systems area, and finally discuss how to apply our algorithms. In Chapter 2, we introduce the Embedded Benders decomposition method, which also exploits Benders cuts for the second stage subproblems, and these cuts are reusable given any first stage solution. Also our method generates multiple cuts while solving the subproblem of only one scenario, by taking advantage of the special structure of the models.

The second half of this dissertation is about stochastic optimization applications in the area of energy systems, such as stochastic unit commitment problems, optimization models in natural gas industry. Chapter 3 introduces the stochastic security constrained unit commitment model, which tries to solve both day-ahead commitment schedule and real-time commitment schedule, while considering minimum up and down times, spinning reserves and nonspining reserves, unit capacities, and piecewise linear fuel cost functions. We apply the Embedded Benders decomposition on this problem, and the numerical results show that EBD algorithm outperforms the default CPLEX MIP solver for problems with large numbers of scenarios. The computational time almost increase linearly when we increase the size of the problem, which make EBD a very reliable method for solving stochastic security constrained unit commitment problems. Chapter 4 gives a detailed survey of optimization models in the natural gas industry by focusing on the natural gas production, transportation, and market. Chapter 5 proposes expansion planning models which include both natural gas transmission network expansion and LNG (Liquified Natural Gas) terminals location planning. These models take into account the uncertainties of demands and supplies in the future, which make the models stochastic integer programs with discrete subproblems. Also we consider risk control in our models by including probabilistic constraints, such as a limit on CVaR (Conditional Value at Risk). In order to solve the large-scale problems, especially with a large number of scenarios, we also apply the embedded Benders decomposition algorithm. Numerical results show that our algorithm is efficient for large scale stochastic natural gas transportation system expansion planning problems. Chapter 6 concludes this dissertation, while also talking about future research.

CHAPTER 2
EMBEDDED BENDERS' DECOMPOSITION

The method proposed by Sherali and Fraticelli (2002) [60] modifies the Benders' decomposition for discrete subproblem by sequentially convexifying them using Reformulation-Linearization Techniques (RLT) or lift-and-project method. However, it takes a lot of efforts to find a global cut for each subproblem by using RLT or lift-and-project method. At the each iteration, their algorithm needs to add a global cut to get a better convex hull for the discrete subproblem. In this dissertation, a method called embedded Benders' decomposition which apply Bender's cut to help approximate the convex hull of the discrete subproblem is proposed. The Benders' cut added to the subproblem is valid for any first stage decision and then they are reusable along the iterative computations, which save a lot time to generate different new convexification cuts at each iteration. Several variants of Embedded Benders' Decomposition algorithm to solve different types of problems are explained in details in the following sections.

2.1 Two-stage Embedded Benders' Decomposition

A typical deterministic formulation of two stage mixed integer program is as follows,

$$\begin{array}{llllllll} \mathbf{P}_0: & \text{Min} & c_1^T x_1 & +d_1^T z_1 & +c_2^T x_2 & +d_2^T z_2 & & \\ & \text{s.t.} & A_1 x_1 & +B_1 z_1 & & & \geq b_1, & \leftarrow \pi_1 \\ & & E_1 x_1 & +F_1 z_1 & +G_1 x_2 & & \geq h_1, & \leftarrow \mu_1 \\ & & & & A_2 x_2 & +B_2 z_2 & \geq b_2, & \leftarrow \pi_2, \end{array}$$

where x_i is a vector of continuous variables and z_i is a vector of integer variables, for $i = 1, 2$. For convenience, we first discuss how to solve a two-stage deterministic problem, where each stage contains both continuous and integer variables.

When $\bar{x}_1$ and $\bar{z}_1$ are given, we have the following problem,

$$\mathbf{P}_1: \quad \text{Min} \quad c_2^T x_2 + d_2^T z_2$$

$$
\begin{aligned}
\text{s.t.} \quad & G_1 x_1 \geq h_1 - E_1 \bar{x}_1 - F_1 \bar{z}_1, \quad \leftarrow \mu_1 \\
& A_2 x_2 + B_2 z_2 \geq b_2. \quad \leftarrow \pi_2
\end{aligned}
$$

By solving P_1, we can get the optimal solution, $(\bar{z}_2, \bar{x}_2)$. With $\bar{x}_1$, $\bar{z}_1$, $\bar{z}_2$, we can solve the following dual problem, i.e., DP_1,

$$
\begin{aligned}
\textbf{DP}_1\textbf{:} \quad \text{Max} \quad & (h_1 - E_1 \bar{x}_1 - F_1 \bar{z}_1)^T \mu_1 + (b_2 - B_2 \bar{z}_2)^T \pi_2 \\
\text{s.t.} \quad & G_1^T \mu_1 + A_2^T \pi_2 \leq c_2. \quad \leftarrow x_2
\end{aligned}
$$

Suppose the optimal solution to DP_1 is $(\bar{\mu}_1, \bar{\pi}_2)$. Then the new cut, $\beta \geq (h_1 - E_1 x_1 - F_1 z_1)^T \bar{\mu}_1 + (b_2 - B_2 z_2)^T \bar{\pi}_2$, constructed by using $(\bar{\mu}_1, \bar{\pi}_2)$, is a global cut since the feasible region of problem DP_1 does not depend on ($\bar{x}_1$, $\bar{z}_1$, and $\bar{z}_2$). So, by adding these embedded Bender's cuts, we can help approximate the convex hull of the discrete subproblem, and formulate a new subproblem, RLP_1, as follows,

$$
\begin{aligned}
\textbf{RLP}_1\textbf{:} \quad \text{Min} \quad & d_2^T z_2 + \beta \\
\text{s.t.} \quad & \beta \geq (h_1 - E_1 \bar{x}_1 - F_1 \bar{z}_1)^T \mu_1^k + (b_2 - B_2 z_2)^T \pi_2^k, \quad k \in K, \leftarrow \lambda_k \\
& z_2 \leq 1, \leftarrow \rho
\end{aligned}
$$

where K is a subset of the vertex set of problem DP_1, and integer variable z_2 is relaxed to continuous variable. Denote the optimal dual solution of constraint k of RLP_1 by λ_k. A valid Benders' cut for the first stage problem is as follows,

$$
\alpha \geq e^T \rho + \sum_{k \in K} \left[(h_1 - E_1 x_1 - F_1 z_1)^T \mu_1^k + b_2^T \pi_2^k \right] \lambda_k^j \tag{2–1}
$$

The restricted master problem is as follows,

$$
\begin{aligned}
\textbf{RMP}_0\textbf{:} \quad \text{Min} \quad & c_1^T x_1 + d_1^T z_1 + \alpha \\
\text{s.t.} \quad & A_1 x_1 + B_1 z_1 \geq b_1, \\
& \alpha \geq e^T \rho + \sum_{k \in K} \left[(h_1 - E_1 x_1 - F_1 z_1)^T \mu_1^k + b_2^T \pi_2^k \right] \lambda_k^j, \quad j \in J,
\end{aligned}
$$

where J is a subset of the vertex set of dual problem to RLP_1.

The algorithm for solving the two stage problem is shown below,

Step 0. Set UB=∞, LB=$-\infty$, $K = J = \emptyset$.

Step 1. Solve RMP_0, and optimal solution and objective value are ($\bar{x}_1$, $\bar{z}_1$) and w respectively.
LB$\leftarrow max(LB, w)$

Step 2. Solve P_1, and optimal solution is ($\bar{x}_2$, $\bar{z}_2$).
UB$\leftarrow min(UB, c_1^T\bar{x}_1 + d_1^T\bar{z}_1 + c_2^T\bar{x}_2 + d_2^T\bar{z}_2)$

Step 3. If UB$-$LB$\leq \epsilon$, stop. O/W go to step 4.

Step 4. If necessary, solve DP_1, and optimal solution is ($\bar{\mu}_1$, $\bar{\pi}_2$).
$K \leftarrow K \cup (\bar{\mu}_1, \bar{\pi}_2)$.

Step 5. Solve RLP_1, and optimal solution is $\bar{\lambda} = (\bar{\lambda}_k,\ k \in K)$.
$J \leftarrow J \cup \bar{\lambda}$.
Go to step 1.

P_1 is an integer programming problem with a smaller number of integer variables, which need little effort to solve. This algorithm decompose a mixed integer programming problem into multiple smaller MIPs, which make the total computational time less. We can generalize this algorithm to multistage MIPs as DDP is generalization of Benders' decomposition. Eventually, the generalized embedded Benders' decomposition can exploit the SDDP's ideas to avoid the dimensionality explosion.

Below is a simple example of implementing Embedded Benders Decomposition:

$$
\begin{aligned}
\textbf{P}_0\textbf{:}\quad \text{Min}\quad & 2x_1 + 3x_2 + z_1 + 6z_2 + 4x_3 + 3x_4 + z_3 + z_4 \\
\text{s.t.}\quad & x_1 + 2x_2 + 4z_1 + 3z_3 \geq 4, \\
& 2x_1 + x_2 + z_1 + z_2 \geq 2, \\
& \qquad x_2 + 2z_1 + z_2 + 3x_3 + x_4 \geq 3, \\
& \qquad\qquad\qquad 2x_3 + 3x_4 + z_3 + 2z_4 \geq 5, \\
& \qquad\qquad\qquad x_3 + x_4 + 2z_3 + z_4 \geq 2,
\end{aligned}
$$

$$x \geq 0,$$

$$z \in \{0,1\}^4.$$

First we need to solve the first stage subproblem without considering any future information.

$$\begin{aligned}
\textbf{DP}_0\textbf{:}\quad \text{Min} \quad & 2x_1 + 3x_2 + z_1 + 6z_2 \\
\text{s.t.} \quad & x_1 + 2x_2 + 4z_1 + 3z_2 \geq 4, \\
& 2x_1 + x_2 + z_1 + z_2 \geq 2, \\
& x \geq 0, \\
& z \in \{0,1\}^2.
\end{aligned}$$

The optimal objective value is $w = 2$, with optimal solution being $(\bar{x}_1, \bar{x}_2, \bar{z}_1, \bar{z}_2)$ = $(0.5, 0, 1, 0)$. Then we can set $LB = 2$. By fixing the first stage variable as $(\bar{x}_1, \bar{x}_2, \bar{z}_1, \bar{z}_2)$ we solve the second stage mixed integer subproblem to get a feasible solution for the whole problem, which provides an upper bound.

$$\begin{aligned}
\textbf{P}_1\textbf{:}\quad \text{Min} \quad & 2\bar{x}_1 + 3\bar{x}_2 + \bar{z}_1 + 6\bar{z}_2 + 4x_3 + 3x_4 + z_3 + z_4 \\
\text{s.t.} \quad & \bar{x}_2 + 2\bar{z}_1 + \bar{z}_2 + 3x_3 + x_4 \geq 3, \leftarrow \mu_3 \\
& 2x_3 + 3x_4 + z_3 + 2z_4 \geq 5, \leftarrow \pi_4 \\
& x_3 + x_4 + 2z_3 + z_4 \geq 2, \leftarrow \pi_5 \\
& x \geq 0, \\
& z \in \{0,1\}^2.
\end{aligned}$$

The optimal solution of P_1 is $(\bar{x}_3, \bar{x}_4, \bar{z}_3, \bar{z}_4)$=$(0, 1, 0, 1)$. Then the upper bound can be calculated as follows

$$UB = c_1\bar{x}_1 + c_2\bar{x}_2 + d_1\bar{z}_1 + d_2\bar{z}_2 + c_3\bar{x}_3 + c_4\bar{x}_4 + d_3\bar{z}_3 + d_4\bar{z}_4 = 6$$

By fixing $(\bar{z}_3, \bar{z}_4)$, we can solve DP$_1$ as a linear program, of which the optimal dual solution is $(\bar{\mu}_3, \bar{\pi}_4, \bar{\pi}_5)$=$(0, 1, 0)$. With the optimal dual solution, we can construct a inner/convexification cut as in the following linear program,

$$
\begin{aligned}
\textbf{RLP}_1\textbf{:} \quad \text{Min} \quad & z_3 + z_4 + \beta \\
\text{s.t.} \quad & \beta \geq (5 - z_3 - z_4)(\bar{\pi}_4), \leftarrow \lambda \\
& z_3 \leq 1, \leftarrow \rho_3 \\
& z_4 \leq 1 \leftarrow \rho_4
\end{aligned}
$$

Then we solve the above relaxed/convexified linear program, and the optimal primal and dual solution are, $(\hat{\beta}, \hat{z}_3, \hat{z}_4)$= $(3, 0, 1)$ and $(\hat{\lambda}, \hat{\rho}_3, \hat{\rho}_4)$=$(1, 0, -1)$ respectively. The new cut for first stage is constructed as follows,

$$\alpha \geq 5\bar{\pi}_4\hat{\lambda} + \hat{\rho}_3 + \hat{\rho}_3 = 4$$

Then we solve the first stage mixed integer program again with the new cut from the second stage,

$$
\begin{aligned}
\textbf{RMP}_0\textbf{:} \quad \text{Min} \quad & 2x_1 + 3x_2 + z_1 + 6z_2 + \alpha \\
\text{s.t.} \quad & x_1 + 2x_2 + 4z_1 + 3z_2 \geq 4, \\
& 2x_1 + x_2 + z_1 + z_2 \geq 2, \\
& \alpha \geq 4, \\
& x \geq 0, \\
& z \in \{0, 1\}^2.
\end{aligned}
$$

The optimal objective of the above first stage MIP with new outer/feedback cut is $w = 6$, and then we can update LB=6. Now we have UB=LB, and optimal solution is $(x_1^*, x_2^*, z_1^*, z_2^*, x_3^*, x_4^*, z_3^*, z_4^*)$=$(.5, 0, 1, 0, 0, 1, 0, 1)$.

2.2 Stochastic Two-stage Embedded Benders' Decomposition

We can extend the deterministic version of Embedded Benders' Decomposition to solve the stochastic two-stage mixed integer programs. The difference here is that we need to do simulations to get the approximated expected upper bound, which is the sample average of sampled subproblems plus the first stage solution. Or, we need to solve all subproblems for all scenarios to get the exact expected cost given a first stage solution. The algorithm (with the approximated upper bound) is shown as follows,

Step 1. Set UB=∞, LB=$-\infty$, $K = J = \emptyset$.

Step 2. Solve RMP$_0$, and optimal solution and objective value are ($\bar{x}_1$, $\bar{z}_1$) and w respectively.
LB$\leftarrow max(LB, w)$

Step 3. For $i = 1 : n$

Get a sample, ξ, from all scenarios

Solve P$_1(\xi)$, and optimal solution is ($\bar{x}_2(\xi)$, $\bar{z}_2(\xi)$).
$Z_{UB}(\xi) = c_1^T\bar{x}_1 + d_1^T\bar{z}_1 + c_2^T\bar{x}_2(\xi) + d_2^T\bar{z}_2(\xi))$

Solve DP$_1(\xi)$, and optimal solution is ($\bar{\mu}_1(\xi)$, $\bar{\pi}_2(\xi)$).
$K \leftarrow K \cup (\bar{\mu}_1(\xi), \bar{\pi}_2(\xi))$.

Solve RLP$_1(\xi)$, and optimal solution is $\bar{\lambda}(\xi) = (\bar{\lambda}_k(\xi),\ k \in K(\xi))$.
$J(\xi) \leftarrow J(\xi) \cup \bar{\lambda}(\xi)$.

Step 4. Update UB by the sample average of $Z_{UB}(\xi)$'s

Step 5. If UB-LB$< \epsilon$, stop; o/w, go to step 1.

Usually this method is used when there exist a huge number of scenarios, and n is a smaller number as compared to the cardinality of the scenario set. When the total number of scenarios is not so big, we could solve all the subproblems corresponding to the scenarios to get the exact expected future cost.

2.3 Deterministic Multistage Embedded Benders Decomposition

As can be seen in the two stage algorithm, there are two set of cuts :

- Outer/Feedback (OF) Cuts $\mathcal{J}_i$: the cuts which provide information from the future stages.

- Inner/Convexification (IC) Cuts $\mathcal{K}_i$: the cuts which convexify the mixed integer subproblem.

It is easy to note that the first stage does not have IC cuts, and last stage does not have any OF cuts.

For any stage i, we need to solve a mixed integer program to get a feasible solution,

$$
\begin{aligned}
\text{MIP}_i \qquad\qquad \text{Min} \quad & c_i^T x_i + d_i^T z_i + \alpha_i \\
\text{s.t.} \quad & A_i x_i + B_i z_i \geq b_i - E_{i-1}\hat{x}_{i-1} - F_{i-1}\hat{z}_{i-1}, \\
& (\mu_i^j)^T E_i x_i + (\mu_i^j)^T F_i z_i + \alpha_i \geq w_i^j, \quad \forall j \in \mathcal{J}_i, \\
& x_i \geq 0, \\
& z_i \in \{0,1\}^{n_i}.
\end{aligned}
$$

After we get a solution of stage i, we can fix z_i in MIP$_i$ and obtain a linear program, which is denoted by LP$_i$, shown as follows,

$$
\begin{aligned}
\text{LP}_i \qquad \text{Min} \quad & c_i^T x_i + \alpha_i \\
\text{s.t.} \quad & A_i x_i && \geq b_i - E_{i-1}\hat{x}_{i-1} - F_{i-1}\hat{z}_{i-1} - B_i\hat{z}_i, && \leftarrow \pi_i \\
& (\mu_i^j)^T E_i x_i + \alpha_i && \geq w_i^j (\mu_i^j)^T F_i \hat{z}_i, \quad \forall j \in \mathcal{J}_i, && \leftarrow \gamma_i^j \\
& x_i && \geq 0.
\end{aligned}
$$

In order to get an IC cut, which tries to convexify the subproblem MIP$_i$, we need to solve either the linear program LP$_i$ or its dual DLP$_i$ to get the dual optimal solution.

$$
\begin{aligned}
\text{DLP}_i \qquad\qquad \text{Max} \quad & (b_i - E_{i-1}\hat{x}_{i-1} - F_{i-1}\hat{z}_{i-1} - B_i\hat{z}_i)^T \pi_i \\
& + \sum_{j \in \mathcal{J}_i} \left(w_i^j (\mu_i^j)^T F_i \hat{z}_i \right) \gamma_i^j \\
\text{s.t.} \quad & A_i^T \pi_i + \sum_{j \in \mathcal{J}_i} \left((\mu_i^j)^T E_i \right)^T \gamma_i^j \leq c_i, \\
& \sum_{j \in \mathcal{J}_i} \gamma_i^j = 1,
\end{aligned}
$$

$$\pi_{\geq}0,\ \gamma_i^j \geq 0 \in \mathcal{J}_i.$$

Suppose the optimal dual solution is $\hat{\pi}_i^{new}$ and $\hat{\gamma}_i^{j,new}$, $\forall j \in \mathcal{J}_i$. Then a new IC cut is constructed as follows,

$$\begin{aligned}
&\left[(\pi_i^{new})^T B_i + \sum_{j\in\mathcal{J}_i}(\mu_i^j)^T F_i \hat{\gamma}_i^{j,new}\right] z_i + \beta_i \\
\geq\ & (\pi_i^{new})^T b_i + \left(\sum_{j\in\mathcal{J}_i} w_i^j \hat{\gamma}_i^{j,new}\right) \\
& -(\pi_i^{new})^T E_{i-1}\hat{x}_{i-1} - (\pi_i^{new})^T F_{i-1}\hat{z}_{i-1}
\end{aligned} \tag{2–2}$$

After adding the the newly constructed IC cut, we can get the relaxed/convexified subproblem RLP$_i$ as follows,

$$\begin{array}{llllll}
\text{RLP}_i & \text{Min} & d_i^T z_i + \beta_i & & & \\
 & \text{s.t.} & (\theta_i^k)^T z_i + \beta_i & \geq v_i^k & -(\pi_i^k)^T E_{i-1}\hat{x}_{i-1} & \\
 & & & & -(\pi_i^k)^T F_{i-1}\hat{z}_{i-1}, \forall k \in \mathcal{K}_i, & \leftarrow \lambda_i^k \\
 & 0 \leq & z_i & \leq 1, & & \leftarrow \rho_i
\end{array}$$

where

$$\begin{aligned}
\theta_i^k &= (\pi_i^k)^T B_i + \sum_{j\in J_i}(\mu_i^j)^T F_i \hat{\gamma}_i^{j,k}, \\
v_i^k &= (\pi_i^k)^T b_i + \sum_{j\in\mathcal{J}_i} w_i^j \hat{\gamma}_i^{j,k}.
\end{aligned}$$

Since RLP$_i$ is a relaxed linear program of the MIP$_i$, a valid Benders (OF) cut for stage $i-1$ can be constructed by using the optimal dual solution of RLP$_i$. Suppose the newly obtained optimal dual solution is $\hat{\rho}_i^{new}$, $\hat{\lambda}_i^{k,new}$, $\forall k \in \mathcal{K}_i$. The new OF cut for stage $i-1$ is as follows,

$$\alpha_{i-1} \geq e^T \hat{\rho}_i^{new} + \sum_{k\in\mathcal{K}_i} \hat{\lambda}_i^{k,new}\left[v_i^k - (\pi_i^k)^T E_{i-1} x_{i-1} - (\pi_i^k)^T F_{i-1} z_{i-1}\right]$$

If we let

$$\mu_{i-1}^{new} = \sum_{k \in \mathcal{K}_i} \hat{\lambda}_i^{k,new} \hat{\pi}_i^k$$
$$w_{i-1}^{new} = e^T \hat{\rho}_i^{new} + \sum_{k \in \mathcal{K}_i} \hat{\lambda}_i^{k,new} v_i^k$$

Then the new OF cut will be like,

$$(\mu_{i-1}^{new})^T E_{i-1} x_{i-1} + (\mu_{i-1}^{new})^T F_{i-1} z_{i-1} + \alpha_{i-1} \geq w_{i-1}^{new} \tag{2–3}$$

To summarize, the deterministic multistage embedded Benders decomposition algorithm is shown as follows,

Step 0. UB=∞, LB=$-\infty$, $\mathcal{K}_i = \mathcal{J}_i = \emptyset$ for all stages.

Step 1. For stage $i = 1, 2, \dots, T$,

(a) Solve MIP$_i$ with fixed values of $\hat{x}_{i-1}$ and $\hat{z}_{i-1}$, and suppose optimal solution is $(\hat{x}_i, \hat{z}_i)$.
If $i = 1$, LB$\leftarrow max(LB, c_1^T x_1 + d_1^T z_1 + \alpha_1)$.
If $i = T$, UB$\leftarrow min\left(UB, \sum_{i=1}^T (c_i^T x_i + d_i^T z_i)\right)$

(b) If $i > 1$, solve LP$_i$, and suppose the optimal solution is ($\hat{\pi}_i$, α_i) and dual optimal solution is ($\hat{\pi}_i^{new}$, $\hat{\gamma}_i^{j,new}, \forall j \in \mathcal{J}_i$). Construct a new IC cut as in (2–2), and add it into $\mathcal{K}_i$.

(c) If $i > 1$, solve RLP$_i$, and suppose optimal dual solution is ($\hat{\lambda}_i^{k,new}, \forall k \in \mathcal{K}_i$, $\hat{\rho}_i^{new}$).
Construct a new OF cut as in (2–3), and add it into $\mathcal{J}_{i-1}$.

Step 2. If UB$-$LB$\leq \epsilon$, stop; otherwise go to Step 1.

2.4 The Stochastic Multistage Algorithm

In the stochastic case, we assume the uncertainties among stages are independent of each other. This implies that for any stage with any possible outcome, there will be only one future benefit function. This makes the problem easier without losing generality because there is no need to generate the whole scenario tree. Since we have already proposed a method to obtain valid feedback cuts even when the subproblems are mixed integer program, we can still take advantage of this method to handle the dimensionality explosion in SDDP. Also SDDP only allows the random variables on the right hand side, because it only construct an aggregated cut for each stage if a trial vector of previous stage is given. The new approach in this proposal is to construct multiple cuts at a time, which can cope with random variables at anywhere in the subproblem. However, we focus on dealing with problem where random variables appear in the objective function and right hand side.

As is shown in the deterministic algorithm, we calculate the the present stage problem based on the fixed values of previous stage decision. Of course we can calculate the previous stage solution for each node of the decision tree in stochastic case. Doing this will lead to the dimensionality explosion disaster. For example, if we are going to solve a multistage stochastic mixed integer program, each stage of which only has 3 scenarios, there will be $3^{12} = 531441$ leaf nodes, and the number of integer variables in only the leaf nodes will be $3 \times 3^{12} \approx 1.6$million. That is a difficult problem for any state of the art MIP solver. However, this issue actually can be circumvented by using forward simulation, which is shown as follows,

Step 0. Solve MIP$_1$, and suppose the solution is $\hat{x}_1$ and $\hat{z}_1$. Let $x_1^s = \hat{x}_1$, $z_1^s = \hat{z}_1$, $s = 1, 2, \dots, N$.

Step 1. For stage $i = 1, \dots, T$
  For $s = 1, \dots, S$
    Sample a (c_i^s, b_i^s) from $\left\{(c_i^\xi, b_i^\xi), \xi \in \Xi_i\right\}$;
    Solve MIP$_i$ with (c_i^s, b_i^s);
    Save the solution $(\hat{x}_i^s, \hat{z}_i^s)$.

This actually provide an estimate of the upper bound,

$$\hat{UB} = \frac{1}{S}\sum_{s=1}^{S}\sum_{i=1}^{T}\left[(c_i^s)^T\hat{x}_i^s + (d_i^s)^T\hat{z}_i^s\right]$$

.

Also we need to cope with the future benefit functions (OF cuts). The backward recursion shown below is mainly to construct the OF cuts to reflect the information of future stages.

For $i = T, T-1, T-2, \ldots, 2$

For each $\hat{x}_i^s$, $s = 1, 2, \ldots, S$

For each scenario $\xi \in \Xi_i$

Solve LP_i^ξ with $\hat{x}_i^s$

Construct an IC cut for RLP_i^ξ as in (2–2)

Solve RLP_i^ξ

Construct an OF cut for MIP_{i-1} as in (2–3)

Solve MIP_1, which provide a lower bound, $LB = c_1^T x_1 + d_1^T z_1 + \sum_{\xi \in \Xi_2} \alpha_1^\xi$.

The stochastic multistage algorithm actually keeps running the backward recursion and forward simulation until UB and LB are sufficiently close to each other. The multistage stochastic algorithm is shown as follows,

Step 0. Run Forward Simulation and calculate the $\hat{UB}$.

Step 1. Run Backward Recursion and calculate the LB.

Step 2. If $\hat{UB} - LB \leq \epsilon$, stop. Otherwise go to Step 0.

It is interesting to note that $\hat{UB}$ is an estimation of minimum mean value of total cost,

$$\hat{\mu}_{TC} = \hat{UB} = \frac{1}{S} \sum_{s=1}^{S} \sum_{i=1}^{T} \left[(c_i^s)^T \hat{x}_i^s + (d_i^s)^T \hat{z}_i^s \right].$$

So it would be nice we also can calculate the estimation of its standard deviation which is as follows,

$$\hat{\sigma}_{TC} = \sqrt{\frac{1}{S} \sum_{s=1}^{S} \left[(c_i^s)^T \hat{x}_i^s + (d_i^s)^T \hat{z}_i^s - \hat{\mu}_{TC} \right]^2}$$

Hence we may stop when LB falls into the range $\left[\hat{UB} - \hat{\sigma}_{TC}, \hat{UB} + \hat{\sigma}_{TC} \right]$.

CHAPTER 3
STOCHASTIC SECURITY CONSTRAINED UNIT COMMITMENT MODELS

Unit commitment has been a very important problem in the power system, because it is to reduce the production cost of electricity by optimally scheduling the commitments of generation units. This is also a challenging problem since it involves a great amount integer variables. Traditionally, the mostly used method is Lagrangian relaxation. In this chapter, we introduce a new type of unit commitment model which takes into account the uncertainties of demands and the security constraints, e.g., spinning reserves, and non-spinning reserves, to increase the system robustness during contingencies. Also, we apply the EBD method previously proposed in Chapter 2 to these models.

3.1 Introduction

Since the 1980s, the energy sector has been experiencing a dramatic change from regulated market to deregulated market. This introduces a lot of uncertainties to the electricity producers, such as prices, demands, etc. Recently, in order to counter the trend of global climate change, more and more renewable energy sources are introduced into the energy market. This also brings uncertainties, such as wind power, solar power, because of the weather. This makes stochastic programming models very necessary for production companies to achieve profit maximization or cost minimization. There are generally two types of fossil fuel generators, quick-start generators and traditional generators, in use in most of the electricity companies. The traditional generators are usually using coal and takes a long time to get started, i.e., a couple of hours, which has to be scheduled a day ahead.The quick-start generators, instead, are transferring gas or oil energy to electricity, and can get started almost immediately, say, in less than 10 minutes. Then quick-start generators are usually used as remedies to meet the high demands in real time. Because binary variables are used to model whether a generator is on or off, the fuel cost and startup cost minimization problem is a stochastic mixed integer program with discrete second stage. This is very difficult

Table 3-1. SCUC Sets and Indices

N_c	number of coal power units
N_g	number of gas power units
Ξ	the set of all possible scenarios
T	length of planning horizon
i, j	indices of generators
t	time period
ξ, ζ	indices of scenarios

Table 3-2. SCUC Parameters

SU_{it}	start-up cost of unit i in period t
SD_{it}^{g}	shut-down cost of unit i in period t
l_i	minimum down time of unit i
L_i	minimum up time of unit i
$P_i^{\min}$	minimum amount of power generated by unit i
$P_i^{\max}$	maximum amount of power generated by unit i
U_i	ramping up limit of unit i
D_i	ramping down limit of unit i
$S_i^{\max}$	maximum spinning reserve of unit i
RS_t^{ξ}	spinning reserve requirement at time t of scenario ξ
RO_t^{ξ}	operating reserve requirement at time t of scenario ξ
PD_t^{ξ}	real-time system demand at time t of scenario ξ
PL_t^{ξ}	real-time system losses at time t of scenario ξ

to solve directly by any state of the art commercial optimization software when we experience a big number of scenarios. Following this section, we first introduce the model, and then apply the EBD algorithm to solve it, and finally show the numerical results for different settings.

3.2 Problem Formulation

In the stochastic security constrained unit commitment problem, we have both day-ahead and real-time unit commitment scheduling. In the day-ahead scheduling, we need to make commitment plans for all generating units, include both non-fast-start generators and fast-start generators. However, in the real time scheduling, only fast-start units can be rescheduled. Also the power generated by one unit can be adjusted in real time if its status is “on” at that time period. In order to facilitate the description of our model, we list the sets and indices used in this chapter in Table 3-1, parameters in Table

Table 3-3. SCUC Decision Variables

α_{it}	commitment decision of unit i in period t
γ_{it}	startup action of unit i at period t
δ_{it}	shutdown action of unit i at period t
p_{it}^{ξ}	amount of power generated by unit i in period t of scenario ξ
s_{it}^{ξ}	spinning reserve of unit i in period t of scenario ξ
q_{it}^{ξ}	operating reserve of unit i in period t of scenario ξ
γ_{it}^{ξ}	real-time startup action of gas unit i in period t of scenario ξ
δ_{it}^{ξ}	real-time shutdown action of gas unit i in period t of scenario ξ
β_{jt}^{ξ}	commitment decision of gas unit i in period t of scenario ξ
y_{it}^{ξ}	startup reschedule indicator of gas unit i in period t of scenario ξ
z_{it}^{ξ}	shutdown reschedule indicator of gas unit i in period t of scenario ξ

3-2, and decision variables in Table 3-3. α_{it} denotes the commitment status of unit i at time period t, with "0" meaning "off" and "1" vice versa. γ_{it} is the start-up action indicator, of which "1" means there is a start-up action and "0" vice versa, and δ_{it} is the shut-down action indicator. β_{jt}^{ξ} is the rescheduled commitment status variable of fast-start unit j at time t in scenario ξ. So do the start-up action indicator variable, γ_{jt}^{ξ}, and the shut-down action indicator variable, δ_{jt}^{ξ}. y_{jt}^{ξ} is the start-up rescheduling indicator, of which "1" means a start-up action happens in real time but not in the day-ahead schedule, and "0" means real-time schedule is as same as day-ahead one, and "-1" means there is a start-up action in day-ahead schedule but not in the real time. The extensive formulation is shown as the following,

[**ESCUC**]:

$$\min \quad \sum_{t=1}^{T} \sum_{i \in \{N_c \cup N_g\}} (SU_{it}\gamma_{it} + SD_{it}\delta_{it}) \tag{3–1}$$

$$+ \sum_{\xi \in \Xi}^{T} Prob^{\xi} \sum_{t=1}^{T} \left[\sum_{i \in \{N_c \cup N_g\}} F_i\left(p_{it}^{\xi}\right) + \sum_{j \in N_g} \left(SU_{jt} y_{jt}^{\xi} + SD_{jt} z_{jt}^{\xi}\right) \right] \tag{3–2}$$

$$\text{s.t.} \quad \alpha_{it} - \alpha_{i(t-1)} \le \alpha_{i\tau}, \quad \forall i \in N_c, \tau = t, \dots, \min\{t + L_i - 1, T\}, t = 2, \dots, T, \tag{3–3}$$

$$\alpha_{i(t-1)} - \alpha_{it} \le 1 - \alpha_{i\tau}, \quad \forall i \in N_c, \tau = t, \dots, \min\{t + l_i - 1, T\}, t = 2, \dots, T, \tag{3–4}$$

$$\gamma_{it} \ge \alpha_{it} - \alpha_{i(t-1)}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \tag{3–5}$$

$$\delta_{it} \geq -\alpha_{it} + \alpha_{i(t-1)}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \tag{3–6}$$

$$\sum_{i \in \{N_c \cup N_g\}} p_{it}^{\xi} \geq PD_t^{\xi} + PL_t^{\xi}, \quad t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–7}$$

$$\sum_{i \in \{N_c \cup N_g\}} s_{it}^{\xi} \geq RS_t^{\xi}, \quad t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–8}$$

$$\sum_{j \in N_g} q_{jt}^{\xi} \geq RO_t^{\xi}, \quad t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–9}$$

$$y_{jt}^{\xi} \geq \gamma_{jt}^{\xi} - \gamma_{jt}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–10}$$

$$z_{jt}^{\xi} \geq \delta_{jt}^{\xi} - \delta_{jt}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–11}$$

$$\gamma_{jt}^{\xi} \geq \beta_{jt}^{\xi} - \beta_{j(t-1)}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–12}$$

$$\delta_{jt}^{\xi} \geq -\beta_{jt}^{\xi} + \beta_{j(t-1)}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–13}$$

$$p_{it}^{\xi} \geq P_i^{\min} \alpha_{it}, \quad \forall i \in N_c, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–14}$$

$$p_{it}^{\xi} + s_{it}^{\xi} \leq P_i^{\max} \alpha_{it}, \quad \forall i \in N_c, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–15}$$

$$p_{jt}^{\xi} \geq P_j^{\min} \beta_{jt}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–16}$$

$$p_{jt}^{\xi} + s_{jt}^{\xi} \leq P_j^{\max} \beta_{jt}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–17}$$

$$-D_i \leq p_{it}^{\xi} - p_{i(t-1)}^{\xi} \leq U_i, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–18}$$

$$s_{it}^{\xi} \leq S_i^{\max}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \xi \in \Xi, \tag{3–19}$$

$$q_{jt}^{\xi} \leq (1 - \beta_{jt}^{\xi}) P_j^{\max}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–20}$$

$$\beta_{jt}^{\xi}, \gamma_{jt}^{\xi}, \delta_{jt}^{\xi} \in \{0, 1\}, \quad \forall j \in N_g, t = 1, \dots, T, \xi \in \Xi, \tag{3–21}$$

$$\alpha_{it}, \gamma_{it}, \delta_{it} \in \{0, 1\}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \tag{3–22}$$

$$y_{jt}^{\xi}, z_{jt}^{\xi} \in \{-1, 0, 1\}, \quad \forall j \in N_g, t = 1, \dots, T, \forall \xi \in \Xi, \tag{3–23}$$

$$p^{\xi}, s^{\xi}, q^{\xi} \geq 0, \quad \forall \xi \in \Xi, \tag{3–24}$$

where we can just treat both γ_{it} and δ_{it} as positive continuous variables since there are positive costs related to them in the objective function. For convenience, let p^{ξ} be a vector composed of all p_{it}^{ξ}, $i = 1, \dots, N_c$, $t = 1, \dots, T$. So do s^{ξ}, q^{ξ}, α, β^{ξ}, γ, γ^{ξ}, δ, $\delta^{g,\xi}$, y^{ξ} and z^{ξ} through the rest of this chapter.

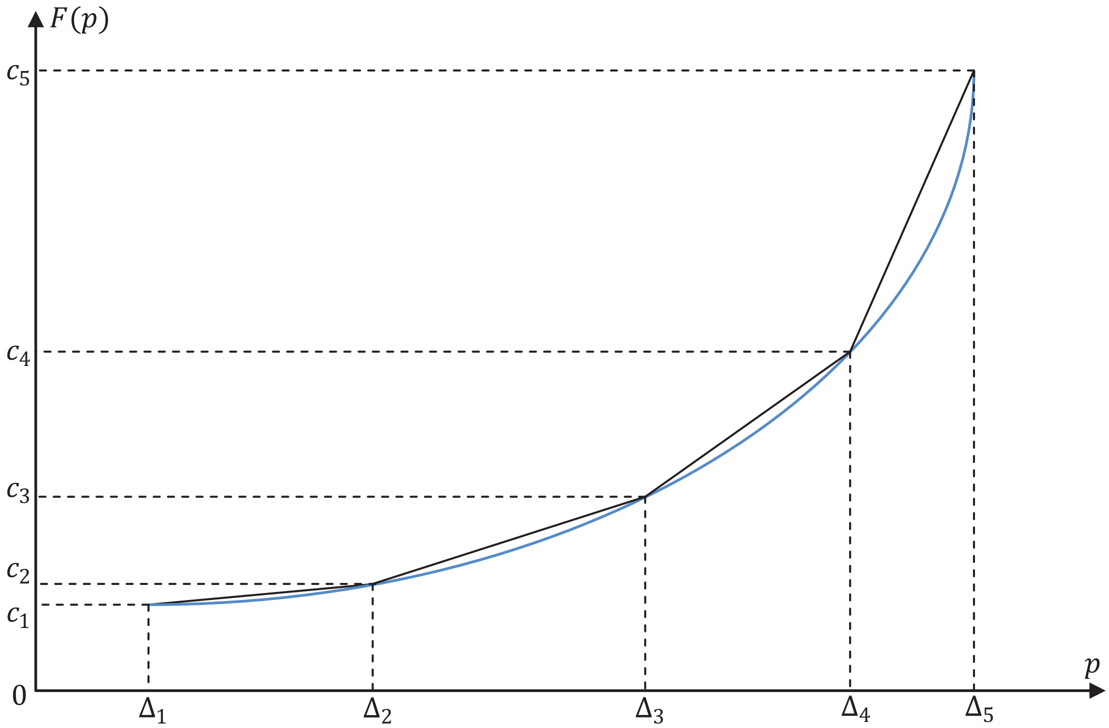

Figure 3-1. Linear Approximation of the Fuel Cost Function

Because the cost function itself is convex, usually a quadratic function with a positive second order derivative, its piecewise linear approximation function is still convex. Hence we can use the following function and constraints to approximate the original function $F(p)$ in the objective function, as is shown in Figure 3-1.

$$F(p) = \sum_{k=1}^{K} C_k \lambda_k$$

And we need to add the followings into the constraints,

$$p = \sum_{k=1}^{K} \Delta_k \lambda_k$$

$$\sum_{k=1}^{K} \lambda_k = 1$$

$$\lambda_k \geq 0, \quad k = 1, \dots, K$$

3.3 Problem Decomposition

It is nontrivial to have a second stage with discrete variables especially when there are a huge number of scenarios. In this chapter, we are going to adopt the same method proposed by [67]. The extensive formulation, [ESCUC], is too difficult to solve because of the huge number of scenarios. So we would like to decompose the problem into two stages: the master problem and subproblems.

The restricted master problem is shown as follows,

[**RMP**]:

$$\min \quad \sum_{t=1}^{T} \sum_{i \in \{N_c \cup N_g\}} (SU_{it}\gamma_{it} + SD_{it}\delta_{it}) + \sum_{\xi \in \Xi} Prob^{\xi} \chi^{\xi} \tag{3–25}$$

$$\text{s.t.} \quad \alpha_{it} - \alpha_{i(t-1)} \leq \alpha_{i\tau}, \quad \forall i \in N_c, \tau = t, \ldots, \min\{t + L_i - 1, T\}, t = 2, \ldots, T, \tag{3–26}$$

$$\alpha_{i(t-1)} - \alpha_{it} \leq 1 - \alpha_{i\tau}, \quad \forall i \in N_c, \tau = t, \ldots, \min\{t + l_i - 1, T\}, t = 2, \ldots, T, \tag{3–27}$$

$$\gamma_{it} \geq \alpha_{it} - \alpha_{i(t-1)}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \ldots, T, \tag{3–28}$$

$$\delta_{it} \geq -\alpha_{it} + \alpha_{i(t-1)}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \ldots, T, \tag{3–29}$$

$$\alpha_{it} \in \{0, 1\}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \ldots, T, \tag{3–30}$$

$$\gamma_{it}, \delta_{it} \geq 0, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \ldots, T, \tag{3–31}$$

$$\sum_{i \in N_c} \hat{\mathsf{x}}_{it}^{\xi,n} \alpha_{it} + \sum_{ji \in N_g} \left(\hat{\mathsf{d}}_{jt}^{\xi,n} \gamma_{jt} + \hat{\mathsf{e}}_{jt}^{\xi,n} \delta_{jt} \right) + \chi^{\xi} \geq \mathsf{b}^{\xi,n}, \quad \forall n \in \mathcal{J}^{\xi}, \xi \in \Xi, \tag{3–32}$$

where γ_{it} and δ_{it} are relaxed to nonnegative continuous variables, because they are related to positive costs and are determined by binary variables α_{it} and $\alpha_{i(t-1)}$. χ^{ξ} is a upper bound variable for the recourse function of scenario ξ, and $\hat{\mathsf{x}}_{it}^{\xi,n}$, $\hat{\mathsf{d}}_{jt}^{\xi,n}$, $\hat{\mathsf{e}}_{jt}^{\xi,n}$ and $\mathsf{b}^{\xi,n}$ are the coefficients of cut n, which will be explained in details later. When the first stage decision variables are fixed, the second stage would be decomposed to $|\Xi|$ separate subproblems since only one scenario will happen in reality. The only difference between two subproblems are the demands as shown in the following formulation. For each scenario $\xi \in \Xi$, the subproblem, with fixed values of first stage decision variables, is as follows,

[**SP**$^\xi$]:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} \left[\sum_{i\in\{N_c\cup N_G\}} \sum_{k\in K_i} C_{i,k}\lambda_{it,k}^{\xi} + \sum_{j\in\{N_g\}} \left(SU_{jt}y_{jt}^{\xi} + SD_{jt}z_{jt}^{\xi} \right) \right] \\
\text{s.t.} \quad & p_{it}^{\xi} = \sum_{k\in K_i} \Delta_{i,k}\lambda_{it,k}^{\xi}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \\
& \sum_{k\in K_i} \lambda_{it,k}^{\xi} = 1, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \\
& \sum_{i\in\{N_c\cup N_g\}} p_{it}^{\xi} \geq PD_t^{\xi} + PL_t^{\xi}, \quad t = 1, \dots, T, \\
& \sum_{i\in\{N_c\cup N_g\}} s_{it}^{\xi} \geq RS_t^{\xi}, \quad t = 1, \dots, T, \\
& \sum_{j\in N_g} q_{jt}^{\xi} \geq RO_t^{\xi}, \quad t = 1, \dots, T, \\
& y_{jt}^{\xi} \geq \gamma_{jt}^{\xi} - \hat{\gamma}_{jt}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& z_{jt}^{\xi} \geq \delta_{jt}^{\xi} - \hat{\delta}_{jt}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& \gamma_{jt}^{g,\xi} \geq \beta_{jt}^{\xi} - \beta_{j(t-1)}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& \delta_{jt}^{g,\xi} \geq -\beta_{jt}^{\xi} + \beta_{j(t-1)}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& P_i^{\min}\hat{\alpha}_{it} \leq p_{it}^{\xi} + s_{it}^{\xi} \leq P_i^{\max}\hat{\alpha}_{it}, \quad \forall i \in N_c, t = 1, \dots, T, \\
& P_j^{\min}\beta_{jt}^{\xi} \leq p_{jt}^{\xi} + s_{jt}^{\xi} \leq P_j^{\max}\beta_{jt}^{\xi}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& -D_i \leq p_{it}^{\xi} - p_{i(t-1)}^{\xi} \leq U_i, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \\
& s_{it}^{\xi} \leq S_i^{\max}, \quad \forall i \in \{N_c \cup N_g\}, t = 1, \dots, T, \\
& q_{jt}^{\xi} \leq (1 - \beta_{jt}^{\xi})P_j^{\max}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& \beta_{jt}^{\xi}, \gamma_{jt}^{\xi}, \delta_{jt}^{\xi} \in \{0, 1\}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& y_{jt}^{\xi}, z_{jt}^{\xi} \in \{-1, 0, 1\}, \quad \forall j \in N_g, t = 1, \dots, T, \\
& p^{\xi}, s^{\xi}, q^{\xi} \geq 0,
\end{aligned}$$

For any given $\hat{\alpha}$, $\hat{\gamma}$ and $\hat{\delta}$, SP^{ξ} might be infeasible, which means we may need to solve its dual in order to construct a cut for [RMP]. However, we can relax the subproblem to make it always feasible by introducing a dummy generator with higher costs. We call the new problem the relaxed subproblem, which is shown as follows,

[**RSP**$^{\xi}$]:

$$\min \quad \sum_{t=1}^{T}\left[\sum_{i\in\{N_c\cup N_G\cup d\}}\sum_{k\in K_i} C_{i,k}\lambda_{it,k}^{\xi} + \sum_{j\in\{N_g\cup d\}}\left(SU_j y_{jt}^{\xi} + SD_j z_{jt}^{\xi}\right)\right] \tag{3–33}$$

$$\text{s.t.} \quad p_{it}^{\xi} = \sum_{k\in K_i}\Delta_{i,k}\lambda_{it,k}^{\xi}, \quad \forall i\in\{N_c\cup N_g\cup d\}, t=1,\dots,T, \tag{3–34}$$

$$\sum_{k\in K_i}\lambda_{it,k}^{\xi} = 1, \quad \forall i\in\{N_c\cup N_g\cup d\}, t=1,\dots,T, \tag{3–35}$$

$$\sum_{i\in\{N_c\cup N_g\cup d\}} p_{it}^{\xi} \ge PD_t^{\xi} + PL_t^{\xi}, \quad t=1,\dots,T, \tag{3–36}$$

$$\sum_{i\in\{N_c\cup N_g\cup d\}} s_{it}^{\xi} \ge RS_t^{\xi}, \quad t=1,\dots,T, \tag{3–37}$$

$$\sum_{j\in\{N_g\cup d\}} q_{jt}^{\xi} \ge RO_t^{\xi}, \quad t=1,\dots,T, \tag{3–38}$$

$$y_{jt}^{\xi} \ge \gamma_{jt}^{\xi} - \hat{\gamma}_{jt}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–39}$$

$$z_{jt}^{\xi} \ge \delta_{jt}^{\xi} - \hat{\delta}_{jt}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–40}$$

$$\gamma_{jt}^{g,\xi} \ge \beta_{jt}^{\xi} - \beta_{j(t-1)}^{\xi}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–41}$$

$$\delta_{jt}^{g,\xi} \ge -\beta_{jt}^{\xi} + \beta_{j(t-1)}^{\xi}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–42}$$

$$p_{it}^{\xi} \ge P_i^{\min}\hat{\alpha}_{it}, \quad \forall i\in N_c, t=1,\dots,T, \tag{3–43}$$

$$p_{it}^{\xi} + s_{it}^{\xi} \le P_i^{\max}\hat{\alpha}_{it}, \quad \forall i\in N_c, t=1,\dots,T, \tag{3–44}$$

$$p_{jt}^{\xi} \ge P_j^{\min}\beta_{jt}^{\xi}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–45}$$

$$p_{jt}^{\xi} + s_{jt}^{\xi} \le P_j^{\max}\beta_{jt}^{\xi}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–46}$$

$$-D_i \le p_{it}^{\xi} - p_{i(t-1)}^{\xi} \le U_i, \quad \forall i\in\{N_c\cup N_g\cup d\}, t=1,\dots,T, \tag{3–47}$$

$$s_{it}^{\xi} \le S_i^{\max}, \quad \forall i\in\{N_c\cup N_g\cup d\}, t=1,\dots,T, \tag{3–48}$$

$$q_{jt}^{\xi} \leq (1-\beta_{jt}^{\xi})P_j^{\max}, \quad \forall j \in \{N_g \cup d\}, t = 1, \dots, T, \tag{3–49}$$

$$\beta_{jt}^{\xi}, \gamma_{jt}^{\xi}, \delta_{jt}^{\xi} \in \{0, 1\}, \quad \forall j \in \{N_g \cup d\}, t = 1, \dots, T, \tag{3–50}$$

$$y_{jt}^{\xi}, z_{jt}^{\xi} \in \{-1, 0, 1\}, \quad \forall j \in \{N_g \cup d\}, t = 1, \dots, T, \tag{3–51}$$

$$p^{\xi}, s^{\xi}, q^{\xi} \geq 0, \tag{3–52}$$

where we assume $\hat{\gamma}_{dt} = \hat{\delta}_{dt} = 0$, $t = 1, \dots, T$. Because the relaxed subproblem is still a mixed integer program, we need to further decompose it in order to find a legitimate dual optimal solution to construct a cutting plane for the [RMP]. Our strategy is to keep convexifying the subproblem while returning cuts to the [RMP] along the iterations. When the binary vector β is fixed, the subproblem associated with scenario ξ reduce to a linear program, [LP$^{\xi}$], shown as follows,

[LP$^{\xi}$]:

$$\min \quad \sum_{t=1}^{T} \sum_{i \in \{N_c \cup N_g \cup d\}} \sum_{k \in K_i} C_{i,k} \lambda_{it,k}^{\xi} \tag{3–53}$$

$$\text{s.t.} \quad p_{it}^{\xi} = \sum_{k \in K_i} \Delta_{i,k} \lambda_{it,k}^{\xi}, \qquad \forall i \in \{N_c \cup N_g \cup d\}, t = 1, \dots, T, \tag{3–54}$$

$$\sum_{k \in K_i} \lambda_{it,k}^{\xi} = 1, \qquad \forall i \in \{N_c \cup N_g \cup d\}, t = 1, \dots, T, \quad \leftarrow l_{it}^{\xi} \tag{3–55}$$

$$\sum_{i \in \{N_c \cup N_g \cup d\}} p_{it}^{\xi} \geq PD_t^{\xi} + PL_t^{\xi}, \quad t = 1, \dots, T, \quad \leftarrow h_t^{\mathrm{I},\xi} \tag{3–56}$$

$$\sum_{i \in \{N_c \cup N_g \cup d\}} s_{it}^{\xi} \geq RS_t^{\xi}, \quad t = 1, \dots, T, \quad \leftarrow h_t^{\mathrm{II},\xi} \tag{3–57}$$

$$\sum_{j \in \{N_g \cup d\}} q_{jt}^{\xi} \geq RO_t^{\xi}, \quad t = 1, \dots, T, \quad \leftarrow h_t^{\mathrm{III},\xi} \tag{3–58}$$

$$P_i^{\min} \hat{\alpha}_{it} \leq p_{it}^{\xi} + s_{it}^{\xi} \leq P_i^{\max} \hat{\alpha}_{it}, \quad \forall i \in N_c, t = 1, \dots, T, \quad \leftarrow u_{it}^{\xi\mp} \tag{3–59}$$

$$P_j^{\min} \hat{\beta}_{jt}^{\xi} \leq p_{jt}^{\xi} + s_{it}^{\xi} \leq P_j^{\max} \hat{\beta}_{jt}^{\xi}, \quad \forall j \in \{N_g \cup d\}, t = 1, \dots, T, \quad \leftarrow v_{jt}^{\xi\mp} \tag{3–60}$$

$$-D_i \leq p_{it}^{\xi} - p_{i(t-1)}^{\xi} \leq U_i, \qquad \forall i \in \{N_c \cup N_g \cup d\}, t = 1, \dots, T, \quad \leftarrow w_{it}^{\xi\mp} \tag{3–61}$$

$$s_{it}^{\xi} \leq S_i^{\max}, \qquad \forall i \in \{N_c \cup N_g \cup d\}, t = 1, \dots, T, \quad \leftarrow r_{it}^{\mathrm{I},\xi} \tag{3–62}$$

$$q_{jt}^{\xi} \leq (1-\hat{\beta}_{jt}^{\xi})P_j^{\max}, \qquad \forall j \in \{N_g \cup d\}, t=1,\dots,T, \qquad \leftarrow r_{jt}^{\mathrm{II},\xi} \tag{3–63}$$

$$p^{\xi}, s^{\xi}, q^{\xi}, \lambda^{\xi} \geq 0. \tag{3–64}$$

Solving the above [LP$^{\xi}$] can generate a global cut, which is shown in (3–65), to convexify the following [RLP$^{\xi}$] which capture the integer part of [RSP].

$$\begin{aligned}
\pi \geq & \sum_{t=1}^{T}\sum_{i\in N_c}\hat{\alpha}_{it}\left(P_i^{\min}\hat{u}_{it}^{\xi-}+P_i^{\max}\hat{u}_{it}^{\xi+}\right)\\
&+\sum_{t=1}^{T}\sum_{j\in\{N_g\cup d\}}\beta_{jt}^{\xi}\left[P_j^{\min}\hat{v}_{jt}^{\xi-}+P_j^{\max}\left(\hat{v}_{jt}^{\xi+}-r_{jt}^{\mathrm{II},\xi}\right)\right]\\
&+\sum_{t=1}^{T}\left[\hat{h}_t^{\mathrm{I},\xi}\left(PD_t^{\xi}+PL_t^{\xi}\right)+\hat{h}_t^{\mathrm{II},\xi}RS_t^{\xi}+\hat{h}_t^{\mathrm{III},\xi}RO_t^{\xi}\right]\\
&+\sum_{t=1}^{T}\left[\sum_{i\in\{N_c\cup N_g\cup d\}}\left(\hat{l}_{it}^{\xi}-D_i\hat{w}_{it}^{\xi-}+U_i\hat{w}_{it}^{\xi+}+S_i^{\max}\hat{r}_{it}^{\mathrm{I},\xi}\right)+\sum_{j\in\{N_g\cup d\}}P_j^{\max}\hat{r}_{it}^{\mathrm{II},\xi}\right]
\end{aligned} \tag{3–65}$$

where $\hat{l}^{\xi}$, $\hat{h}^{\mathrm{I},\xi}$, $\hat{h}^{\mathrm{II},\xi}$, $\hat{h}^{\mathrm{III},\xi}$, $\hat{u}^{\xi\mp}$, $\hat{v}^{\xi\mp}$, $\hat{w}^{\xi\mp}$, $\hat{r}^{\mathrm{I},\xi}$, $\hat{r}_{jt}^{\mathrm{II},\xi}$ are the optimal dual solution corresponding to constraints (3–55) – (3–63) respectively. By "global", it means that the cut is valid for [RLP$^{\xi}$] given any first stage solution, i.e., the solution from the master problem, $\hat{\alpha}$, $\hat{\gamma}$ and $\hat{\delta}$. For convenience, we rewrite (3–65) in vector format as follows,

$$b^{\xi}\beta^{\xi}+\pi^{\xi} \geq f^{\xi}+a^{\xi}\hat{\alpha}$$

Then we can include these global cuts to construct a relaxed version of the subproblems as follows,

[RLP$^{\xi}$]:

$$\min \quad \sum_{t=1}^{T}\left[\sum_{j\in\{N_g\cup d\}}\left(SU_{jt}y_{jt}^{\xi}+SD_{jt}z_{jt}^{\xi}\right)\right]+\pi^{\xi}-\sum_{t=1}^{T}\sum_{j\in\{N_g\cup d\}}(SU_{jt}+SD_{jt}) \tag{3–66}$$

$$\text{s.t.} \quad y_{jt}^{\xi}-\gamma_{jt}^{\xi} \geq 1-\hat{\gamma}_{jt}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \qquad \leftarrow \theta_{jt}^{\xi} \tag{3–67}$$

$$z_{jt}^{\xi}-\delta_{jt}^{\xi} \geq 1-\hat{\delta}_{jt}, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \qquad \leftarrow \sigma_{jt}^{\xi} \tag{3–68}$$

$$\gamma_{jt}^{\xi}-\beta_{jt}^{\xi}+\beta_{j(t-1)}^{\xi} \geq 0, \quad \forall j\in\{N_g\cup d\}, t=1,\dots,T, \tag{3–69}$$

$$\delta^{\xi}_{jt} + \beta^{\xi}_{jt} - \beta^{\xi}_{j(t-1)} \geq 0, \quad \forall j \in \{N_g \cup d\}, t = 1, \ldots, T, \tag{3–70}$$

$$b^{\xi}_{k}\beta^{\xi} + \pi^{\xi} \geq f^{\xi}_{k} + a^{\xi}_{k}\hat{\alpha}, \quad \forall k \in \mathcal{K}^{\xi}, \qquad \leftarrow \eta^{\xi}_{k} \tag{3–71}$$

$$\beta^{\xi}_{jt} \leq 1, \quad \forall j \in \{N_g \cup d\}, t = 1, \ldots, T, \qquad \leftarrow \rho^{\xi}_{jt} \tag{3–72}$$

$$y^{\xi}_{jt} \leq 2, \quad \forall j \in \{N_g \cup d\}, t = 1, \ldots, T, \qquad \leftarrow \phi^{\xi}_{jt} \tag{3–73}$$

$$z^{\xi}_{jt} \leq 2, \quad \forall j \in \{N_g \cup d\}, t = 1, \ldots, T, \qquad \leftarrow \psi^{\xi}_{jt} \tag{3–74}$$

$$\beta^{\xi}, y^{\xi}, z^{\xi}, \gamma^{\xi}, \delta^{\xi} \geq 0, \tag{3–75}$$

where $\mathcal{K}^{\xi}$ is the set containing all global cuts for scenario ξ.

Proposition 3.1. *The cut, (3–65), obtained by solving [LP$^{\xi}$] is a global cut for [RLP$^{\xi}$] given any first stage solution, $\hat{\alpha}$, $\hat{\gamma}$ and $\hat{\delta}$.*

Since the above [RLP$^{\xi}$] is a linear program, whose feasible region is an approximation of the convex hull of [RSP$^{\xi}$]s feasible region, we can derive a valid Benders cut for [RMP], shown in (3–76), by solving its dual problem optimally.

$$\begin{aligned} &-\sum_{t=1}^{T}\sum_{i\in N_c}\left(\sum_{k\in K^{\xi}}\hat{\eta}^{\xi}_{k}a^{\xi}_{k,it}\right)\alpha_{it} + \sum_{t=1}^{T}\sum_{j\in N_g}\left(\hat{\theta}^{\xi}_{jt}\gamma_{jt} + \hat{\sigma}^{\xi}_{jt}\delta_{jt}\right) + \chi^{\xi} \\ &\geq \sum_{k\in\mathcal{K}^{\xi}} f^{\xi}_{k}\hat{\eta}^{\xi}_{k} + \sum_{j\in\{N_g\cup d\}}\left(\hat{\theta}^{\xi}_{jt} + \hat{\sigma}^{\xi}_{jt}\right) + \sum_{t=1}^{T}\sum_{i\in\{N_g\cup d\}}\left(\hat{\rho}^{\xi}_{jt} + 2\hat{\phi}^{\xi}_{jt} + 2\hat{\psi}^{\xi}_{jt}\right), \end{aligned} \tag{3–76}$$

where $\hat{\eta}^{\xi}_{k}$ is the optimal dual solution corresponding to the k^{th} global cuts in [RLP$^{\xi}$], and $\hat{\theta}^{\xi}$, $\hat{\sigma}^{\xi}$, $\hat{\rho}^{\xi}$, $\hat{\phi}^{\xi}$ and $\hat{\psi}^{\xi}$, are the optimal dual solutions corresponding to constraints (3–67), (3–68), (3–72), (3–73) and (3–74) respectively.

Proposition 3.2. *The cut from [RLP$^{\xi}$] is a valid Benders cut for [RMP].*

It is interesting to note that all [LP$^{\xi}$]s share the same dual space (dual feasible region) since the costs in the objective functions and left-hand-side coefficients are the same. Hence the dual solution obtained by solving a specific [LP$^{\xi}$] could be used to construct the global cuts for other scenarios. In constraint (3–71), the set of cuts, $\mathcal{K}^{\xi}$, is designated to only one single scenario, ξ. However, we can generalize this set to all scenarios, which is supported by the following corollary.

Corollary 1. *The global cut of scenario ξ, (3–65), with right hand side being changed to $f^{\zeta} - a^{\xi}\hat{\alpha}$, is also valid for [RLP$^{\zeta}$] given any first stage solution, for all $\zeta \in \Xi$.*

For all scenarios, this constraint, (3–65), is almost the same except f_k^{ζ}, which is calculated as follows,

$$f_k^{\zeta} \;=\; \sum_{t=1}^{T}\left[\hat{h}_{k,t}^{\mathrm{I},\xi}\left(PD_t^{\zeta} + PL_t^{\zeta}\right) + \hat{h}_{k,t}^{\mathrm{II},\xi}RS_t^{\zeta} + \hat{h}_{k,t}^{\mathrm{III},\xi}RO_t^{\zeta}\right] + f_k^{c,\xi}$$

where the (k) in the superscripts denotes k^{th} optimal dual solution in $\mathcal{K}^{\xi}$, and

$$f_k^{c,\xi} = \sum_{t=1}^{T}\left[\sum_{i\in\{N_c\cup N_g\cup d\}}\left(\hat{l}_{k,it}^{\xi} - D_i\hat{w}_{k,it}^{\xi-} + U_i\hat{w}_{k,it}^{\xi+} + S_i^{\max}\hat{r}_{k,it}^{\mathrm{I},\xi}\right) + \sum_{j\in\{N_g\cup d\}} P_j^{\max}\hat{r}_{k,it}^{\mathrm{II},\xi}\right]$$

which is independent of scenario ζ. So f_k^{ζ} can be considered as an affine function of PD_t^{ζ}, PL_t^{ζ}, RS_t^{ζ} and RO_t^{ζ}, in which $f_k^{c,\xi}$ is the constant. Then all [RLP$^{\xi}$]s can share the same global cut set except for the right hand sides. Instead of multiple global cut sets for all the scenarios, we only need to maintain a single global cut set as follows,

$$b_k\beta^{\zeta} + \pi^{\zeta} \geq f_k^{\zeta} + a_k\hat{\alpha}, \quad \forall k \in \mathcal{K}, \tag{3–77}$$

where the only differences between the scenarios are the names of variables and right hand sides.

After we replace (3–71) in [RLP$^{\xi}$] by (3–77), the left-hand-side coefficients of [RLP$^{\xi}$] are not dependent of the scenarios any more because b_k is the same for all scenarios. This also means that all [RLP$^{\xi}$]s have the same dual feasible region because they are all linear programs. So an optimal dual solution to one scenario is also a feasible dual solution to another scenario. Hence the dual optimal solutions, $\hat{\theta}^{\xi}$, $\hat{\sigma}^{\xi}$, $\hat{\psi}^{\xi}$, $\hat{\phi}^{\xi}$, $\hat{\eta}^{\xi}$, and $\hat{\rho}^{\xi}$ to [RLP$^{\xi}$] can help construct valid Benders cuts from all other scenarios, but with different f_k^{ζ}'s, which is stated in the following theorem.

Proposition 3.3. *For all $\zeta \in \Xi$,*

$$-\textstyle\sum_{t=1}^{T}\sum_{i\in N_c}\left(\sum_{k\in K^{\xi}}\hat{\eta}_k^{\xi}a_{k,it}\right)\alpha_{it} + \sum_{t=1}^{T}\sum_{j\in N_g}\left(\hat{\theta}_{jt}^{\xi}\gamma_{jt} + \hat{\sigma}_{jt}^{\xi}\delta_{jt}\right) + \chi^{\zeta} \tag{3–78}$$

$$\geq \sum_{k\in\mathcal{K}} f_k^{\zeta}\hat{\eta}_k^{\xi} + \sum_{j\in\{N_g\cup d\}}\left(\hat{\theta}_{jt}^{\xi} + \hat{\sigma}_{jt}^{\xi}\right) + \sum_{t=1}^{T}\sum_{i\in\{N_g\cup d\}}\left(\hat{\rho}_{jt}^{\xi} + 2\hat{\phi}_{jt}^{\xi} + 2\hat{\psi}_{jt}^{\xi}\right)$$

is a valid Benders cut for [RMP] given any first stage solution, where $\hat{\theta}^{\xi}$, $\hat{\sigma}^{\xi}$, $\hat{\psi}^{\xi}$, $\hat{\phi}^{\xi}$, $\hat{\eta}^{\xi}$, and $\hat{\rho}^{\xi}$ are the dual optimal solutions to [RLP$^{\xi}$].

With these disaggregated cuts being added into the [RMP], we need to include $|\Xi|$ recourse variables, χ^{ξ}s. In the case of a big number of scenarios, this could increase the computational burden of solving the restricted master problem. However, all the cuts generated by the same dual solution of [RLP$^{\xi}$] can be aggregated to one single cut by adding them together while multiplying each of them by the probability of its corresponding scenario. The aggregated cut is shown as follows,

$$\begin{aligned} & -\sum_{t=1}^{T}\sum_{i\in N_c}\left(\sum_{k\in K^{\xi}}\hat{\eta}_k^{\xi}a_{k,it}\right)\alpha_{it} + \sum_{t=1}^{T}\sum_{j\in N_g}\left(\hat{\theta}_{jt}^{\xi}\gamma_{jt} + \hat{\sigma}_{jt}^{\xi}\delta_{jt}\right) + \chi \\ \geq & \sum_{t=1}^{T}\left[\bar{h}_t^{\mathrm{I},\xi}\left(\overline{PD_t} + \overline{PD}_t\right) + \bar{h}_t^{\mathrm{II},\xi}\overline{RS}_t + \bar{h}_t^{\mathrm{III},\xi}\overline{RO}_t\right] + \sum_{k\in\mathcal{K}}\hat{\eta}_k^{\xi}f_k^{c} \\ & + \sum_{j\in\{N_g\cup d\}}\left(\hat{\theta}_{jt}^{\xi} + \hat{\sigma}_{jt}^{\xi}\right) + \sum_{t=1}^{T}\sum_{i\in\{N_g\cup d\}}\left(\hat{\rho}_{jt}^{\xi} + 2\hat{\phi}_{jt}^{\xi} + 2\hat{\psi}_{jt}^{\xi}\right) \end{aligned} \tag{3–79}$$

where $\sum_{\xi\in\Xi}\chi^{\xi}$ is replaced by χ, and $\overline{RS}$, $\overline{RO}$, $\overline{PD}$ and $\overline{PL}$ are the expectations of the random spinning reserve, operating reserve and demand. $\bar{h}_t^{\mathrm{I},\xi}$, $\bar{h}_t^{\mathrm{II},\xi}$ and $\bar{h}_t^{\mathrm{III},\xi}$ are aggregated optimal dual solutions as follows,

$$\begin{aligned} \bar{h}_t^{\mathrm{I},\xi} &= \sum_{k\in\mathcal{K}}\hat{\eta}_k^{\xi}\hat{h}_{k,t}^{\mathrm{I}}\ , \\ \bar{h}_t^{\mathrm{II},\xi} &= \sum_{k\in\mathcal{K}}\hat{\eta}_k^{\xi}\hat{h}_{k,t}^{\mathrm{II}}\ , \\ \bar{h}_t^{\mathrm{III},\xi} &= \sum_{k\in\mathcal{K}}\hat{\eta}_k^{\xi}\hat{h}_{k,t}^{\mathrm{III}}\ . \end{aligned}$$

If we choose to add the aggregated cuts to the relaxed master problem, RMP, the term, $\sum_{\xi\in\Xi}\chi^{\xi}$, in its objective function can be then simply replaced by χ. According to the number of scenarios, we could choose different strategies to add valid Benders cuts. As discussed in [10], the disaggregated scheme is chosen in the case of a small number of scenarios, and vice versa. We prefer the aggregated scheme because the aggregated cuts contain more information from all scenarios and are very easy to generate due to the sharing of same dual space.

3.4 Solution Algorithm

If there is no Benders' cut being added in the [RMP], without including recourse variable χ in the objective function, its optimal solution is $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) = 0$. It is because that all the variables are nonnegative and the costs of startup and shutdown are positive. Hence 0 is the best objective value that [RMP] can achieve. Then we can use this as the initial solution for the Embedded Benders Decomposition. At each iteration, after we solve the [RMP], its optimal objective value is used as a lower bound of [ESCUC]. An upper bound can be obtained as follows,

$$\mathrm{Z}_{UB} = \mathrm{Z}_{RMP} - \hat{\chi} + \sum_{\xi \in \Xi} Prob^{\xi} \mathrm{Z}^{\xi}_{RSP} \tag{3–80}$$

where Z^{ξ}_{RSP}, Z_{RMP} and $\hat{\chi}$ are the optimal objective values of [RSP$^{\xi}$] and [RMP], and the solution of χ respectively. This actually represents the cost of a feasible solution to the relaxed [ESCUC] with a dummy costly generator being added.

The lower bound based on the solution of [RMP] could improve very slowly in practice when the UB and LB are very close to each other. One of the methods to avoid slow convergence or even stalling is to apply the integer L-shaped cut since it is an optimality cut which ensures to improve the lower bound if there exists a solution with a higher objective value. An integer L-shaped "optimality" cut is as follows,

$$z \geq (Q(\hat{x}) - L)\left(\sum_{j \in T} x_j - \sum_{j \in F} x_j - |T| + 1\right) + L$$

where $Q(\hat{x})$ is the recourse function of $\hat{x}$, the first stage solution, and L is a lower bound for the second stage problem, and $T = \{j|\hat{x}_j = 1\}$ and $F = \{j|\hat{x}_j = 0\}$, if x is the first stage decision variable and $\hat{x}$ is the current solution. This follows from the fact that the right hand side will be equal to $Q(\hat{x})$ if $x = \hat{x}$, and less than L otherwise since

$\left(\sum_{j\in T} x_j - \sum_{j\in F} x_j - |T| + 1\right) \leq 0$ if $x \neq \hat{x}$. We refer interested readers to [38] for the detailed proof. Without having to define the two sets, T and F, after rearranging terms the cut can be expressed by the following equivalent inequality,

$$(Q(\hat{x}) - L)\sum_j (1 - 2\hat{x}_j)x_j + z \geq Q(\hat{x}) - (Q(\hat{x}) - L)\sum_j \hat{x}_j.$$

The first stage problem, [RMP], is a pure integer program with only 0-1 variables, and then we can apply the integer L-shaped “optimality” cut in [RMP], which is shown as follows,

$$[Q(\hat{\alpha}) - L]\left[\sum_{t=1}^{T}\sum_{i\in\{N_c\cup N_g\}}(1 - 2\hat{\alpha}_{it})\alpha_{it}\right] + \chi \geq Q(\hat{\alpha}) - [Q(\hat{\alpha}) - L]\left[\sum_{t=1}^{T}\sum_{i\in\{N_c\cup N_g\}}\hat{\alpha}_{it}\right] \tag{3–81}$$

where the recourse $Q(\cdot)$ is a function of only α, the commitment status of both coal and gas power generators, since the best optimal objective value of the second stage is uniquely defined once they are determined. If all generators remain “on” at each time periods, there will be no startup and shutdown cost, and then the optimal objective cost can be used as an lower bound of the second stage, L. The embedded Benders decomposition algorithm is shown as follows,

Step 0. Set $\mathrm{UB} = \infty$, $\mathrm{LB} = 0$, $\mathcal{K} = \emptyset$, $(\hat{\alpha}, \hat{\gamma}, \hat{\delta}) = 0$, $Z_{UB} = 0$, $Z_{RMP} = 0$, and $\hat{\chi} = 0$;

Step 1. Solve [RSP$^{\xi}$], $\forall \xi \in \Xi$, and suppose that optimal solution and objective value are ($\hat{\beta}^{\xi}$, $\hat{p}^{\xi}$, $\hat{q}^{\xi}$, $\hat{\lambda}^{\xi}$, $\hat{y}^{\xi}$, $\hat{z}^{\xi}$, $\hat{\gamma}^{g,\xi}$, $\hat{\delta}^{g,\xi}$), and Z^{ξ}_{RSP}, $\forall \xi \in \Xi$;
Update Z_{UB};
$\mathrm{UB} \leftarrow \min(\mathrm{UB}, Z_{UB})$.

Step 2. Solve [LP$^{\xi}$], and dual optimal solutions are ($\hat{w}$, $\hat{h}$, $\hat{u}$ and $\hat{v}$);
Add this new dual solution to the set, $\mathcal{K}$;
Repeat this for all $\xi \in \Xi$;

Step 3. Solve [RLP$^{\xi}$], and suppose the optimal dual solution is ($\hat{\psi}$, $\hat{\phi}$, $\hat{\eta}$, $\hat{l}$, $\hat{r}$ and $\hat{\rho}$);
Add a new aggregated cut, as in (3–79), into [RMP];
Repeat this for all $\xi \in \Xi$;

Step 4. Add an integer L-shaped “optimality” cut, as in (3–81), into [RMP];

Step 5. Solve [RMP], and suppose that optimal solution and objective value are $(\hat{\alpha}, \hat{\gamma}, \hat{\delta}, \hat{\chi})$ and Z_{RMP} respectively;
LB$\leftarrow \max(LB, \mathrm{Z}_{RMP})$;

Step 6. If $\mathrm{UB} - \mathrm{LB} \leq \epsilon$, stop; Otherwise, go to Step 1.

where ϵ is a small value for the gap tolerance. As is shown above, we repeatedly solve [LP$^{\xi}$] and [RLP$^{\xi}$] for all scenarios in step 2 and 3 respectively. However, in order to improve computational efficiency we do not need to repeat for all scenarios since all [LP$^{\xi}$]s and [RLP$^{\xi}$]s are corresponding to the same first stage decision. One way is to sample from all the scenarios and only solve a limited amount of [LP$^{\xi}$]s and [RLP$^{\xi}$]s.

In this algorithm, we maintain two sets of cuts: one for convexifying the mixed integer subproblems, and one for constructing the future benefit functions. The first set of cuts are called inner convexification (IC) cuts, and the second set of cuts are referred to as outer feedback (OF) cuts. Because both types of cuts are actually Benders cuts, and IC cuts are embedded in the subproblems to provide valid OF cuts, we call this algorithm Embedded Benders Decomposition algorithm. When the algorithm actually terminate, we may need to check the solution in order to determine if the original [ESCUC] is feasible or not. Any variable related to the dummy costly generator should be equal to zero. Otherwise, [ESCUC] is infeasible because even the all the available generators are turned on, some of requirement constraints (3–7), (3–8) or (3–9) cannot be satisfied, which means the demands are actually greater than the total generation capacity of all units.

3.5 Numerical Examples

In this section, we present numerical results of our algorithm on serval problems with different sizes and settings. We code our embedded Benders decomposition algorithm in Microsoft Visual C++ while calling CPLEX 10 (Concert Technology) to solve the decomposed problems. All programs are run in Microsoft Windows XP Professional 2002 SP2 on a Dell Desktop with Intel Pentium 4 CPU 3.40 GHz and 2 GB of RAM.

Table 3-4. Generators Data

Unit	Startup Cost (MBtu)	Generation Cost (MBtu/MW)	Power (MW) Max	Power (MW) Min	10 Minutes Spinning (MW)
G1	100	8	120	10	10
G2	80	10	100	20	10
G3	150	12	50	20	10
G4	180	15	60	10	10

Table 3-5. Load forecast of a simple example

ξ	$Prob(\xi)$	Loads (MW) $t=1$	Loads (MW) $t=2$
1	0.2	220	260
2	0.5	250	280
3	0.3	270	300

A security constrained unit commitment problem with four generators, of which G3 and G4 are fast-start generators, is discussed below. The generator data are shown in Table 3-4. For convenience, we solve a problem with two time period and three scenarios, with data shown in Table 3-5.

By applying Embedded Benders Composition, after 5 iterations with 10 cuts added in the first stage, the algorithm reaches the optimality and returns the same optimal solution as the complete model solved by CPLEX, which takes 23 interations and adds 8 cuts . The results are shown in Table 3-6. Computational times (in milliseconds) of more examples are shown in Table 3-7, in which we list the total computational times, and computing times for [RMP], [SP], [LP] and [RSP]. As can be seen in Table 3-7, computing times almost increase linearly with respect to the number of scenarios, which

Table 3-6. Solution of the 4-unit SCUC with 3 scenarios

ξ	Cost (MBtu)	t	G1	G2	G3	G4
1	4730	1	120	100	0	0
		2	120	100	40	0
2	5540	1	120	100	30	0
		2	120	100	50	10
3	6080	1	120	100	50	0
		2	120	100	50	30

means our EBD method is well suited to problems with a large number of scenarios. Also, the EBD algorithm spends a big portion of time to solve [SP] and [LP]. Hence it is possible to further reduce computing time if we do not calculate new IC and OF cuts for each scenario in Step 1 because looping through all scenarios takes a lot of time, especially when we have a huge number of scenarios. More advanced implementation could help to achieve this and improve the overall performance.

Table 3-7. Computational Results of SCUC

Instance Group	$\lvert\Xi\rvert$	ESCUC	RMP	RSP	LP	RLP	TTL	ItCnt	TTL/ESCUC	RMP/ESCUC
	1	47	220	187	62	61	530	5	11.28	4.68
$\lvert N_c\rvert = 2$	3	62	234	517	62	31	844	5	13.61	3.77
$\lvert N_g\rvert = 3$	8	79	232	1356	77	47	1712	5	21.67	2.94
$\lvert K\rvert = 4$	20	141	203	3674	79	63	4019	5	28.50	1.44
	100	781	202	18005	80	32	18319	5	23.46	0.26
	1000	24922	187	148439	62	47	148735	4	5.97	0.01
	1	47	203	187	126	77	593	5	12.62	4.32
$\lvert N_c\rvert = 2$	3	62	205	515	93	63	876	5	14.13	3.31
$\lvert N_g\rvert = 3$	8	79	218	1464	126	63	1871	5	23.68	2.76
$\lvert K\rvert = 6$	20	156	218	3708	124	64	4114	5	26.37	1.40
	100	1000	236	18534	112	63	18945	5	18.95	0.24
	1000	50485	187	184502	126	48	184863	4	3.66	0.00
	1	62	203	220	78	80	581	5	9.37	3.27
$\lvert N_c\rvert = 2$	3	78	63	249	31	16	359	2	4.60	0.81
$\lvert N_g\rvert = 3$	8	125	172	1375	47	48	1642	5	13.14	1.38
$\lvert K\rvert = 8$	20	406	156	3154	48	31	3389	4	8.35	0.38
	100	7611	156	16802	45	48	17051	4	2.24	0.02
	1000	152526	125	133855	77	21	134078	3	0.88	0.00
	1	63	233	265	77	79	654	5	10.38	3.70
$\lvert N_c\rvert = 2$	3	78	47	238	62	31	378	2	4.85	0.60
$\lvert N_g\rvert = 3$	8	141	172	1628	109	47	1956	4	13.87	1.22
$\lvert K\rvert = 11$	20	422	172	3281	77	47	3577	4	8.48	0.41
	100	8890	140	16436	79	46	16701	4	1.88	0.02
	1000	227877	126	133635	48	31	133840	3	0.59	0.00

CHAPTER 4
OPTIMIZATION MODELS IN NATURAL GAS INDUSTRY

With the surge of the global energy demand, natural gas plays an increasingly important role in the global energy market. To meet the demand, optimization techniques have been widely used in the natural gas industry, and yielded a lot of promising results. In this chapter, we give a detailed discussion of optimization models in the natural gas industry with the focus on the natural gas production, transportation, and market.

4.1 Introduction

Concerned about global warming and shortage of crude oil, people become more interested in natural gas which is a relatively clean energy source and abundant in many places. Natural gas mainly consists of methane, and when burnt, it releases a fair amount of energy and less green house gases (*e.g.*, CO_2) than oil and coal. As we can see from Fig. 4-1, the world gas consumption/production is linearly growing since 1980 from approximately 52,890 billion cubic feet to approximately 104,424 billion cubic feet in 2006, according to the International Energy Annual 2006 from US Department of Energy, Energy Information Administration (EIA). Moreover, the natural gas consumption is expected to continue to grow linearly to approximately 153 trillion cubic feet in 2030, which is an average growth rate of about 1.6 percent per year according to the International Energy Outlook 2009 from EIA.

In 2008, the residential use of natural gas accounted for 21%, the commercial use for 13%, the industrial use for 34%, the transportation for 3% and the electric power production for 29% the Annual Energy Review 2009 from EIA. The industrial sector is expected to remain the largest end-use sector for natural gas through 2030 with an expected share of 40% according to the International Energy Outlook 2009 from EIA. The electric power generation from natural gas was the second largest consumer of natural gas after the industrial sector in 2006. The electricity generation accounted in 2006 for 32% of the world's total natural gas consumption. Due to the worldwide

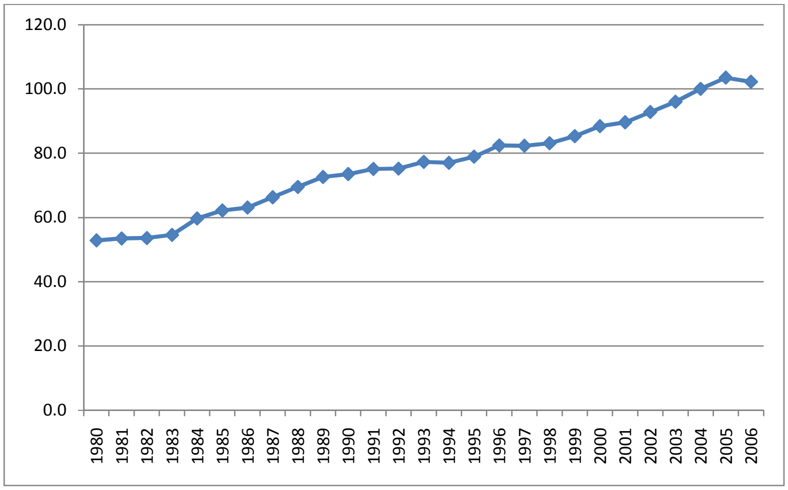

Figure 4-1. World Gas Consumption in Billion Cubic Feet

discussions/attempts to reduce green house gas emissions, the electricity generation via natural gas is expected to become even more important and its share of the world's total natural gas consumption is expected to increase to 35% in 2030 according to the International Energy Outlook 2009 from EIA. Hence, natural gas remains an important source of energy for both the industrial and the electricity sectors.

This chapter discusses different optimization models in the natural gas industry. We focus on three key applications: the natural gas production, the natural gas transportation, and the natural gas market. This chapter is organized in such a way that we start with the introduction of the problem itself, and then discuss a mathematical formulation of the problem and finally review solution techniques to solve these models. However, when well known algorithms, such as Branch & Cut, are used to solve the mathematical programs, we do not go into details but refer to the literature instead.

Section 4.2 discusses the optimization applications in gas recovery and production. We focus on the production scheduling problem and the maximal recovery problem. Section 4.3 focuses on gas transportation, where the network design problems and the optimal fuel cost problem are discussed. The natural gas market is discussed in Section 4.4. We conclude with Section 4.5.

4.2 Optimization in Gas Production (Recovery)

There is still a huge amount of gas natural gas reserves in the world: in 2009, the reserves were estimated at 6,254 trillion cubic feet; 69 trillion cubic feet above the estimate for 2008. This follows the general upwards trend of the world natural gas reserves over years. With a share of approximately 40.7%, the Middle East has the largest natural gas reserves of the world, followed by Eurasia with 32.2% and Africa with 7.8%. On the country level, Russia has approximately 26.9% of the worlds natural gas reserves and holds together with Iran (15.9%) and Qatar (14.3%) approximately 57% of the world's natural gas reserves while the top 20 countries hold together 90.7%. Interestingly, for most regions, the reserves-to-production rates are substantial, with an worldwide estimate of 63 years according to BP 2008 report. Hence, natural gas production and recovery will continue to be an important task in the future.

Optimization models and techniques are applied extensively in natural gas recovery processes, such as production scheduling, placement of well head, gas recovery systems or facilities designs. For a survey on gas and oil recovery and production, we refer the reader to Horne [34]. These optimization problems are computationally difficult to solve. One reason is that a huge number of parameters are subject to uncertainties. Another reason are the nonlinear/nonsmooth/nonconvex functions and constraints, due to the properties of gas production operations as explained in [8]. In the following, we discuss some specific optimization problems occurring in the gas production.

4.2.1 Production Scheduling Considering Well Placement

Usually, a gas reservoir is accessed by drilling multiple wells on its surface. Also gas withdrawal from any of the wells will lead to pressure reductions at all wells drilled on the same reservoir. Then the pressure reductions will come back to decrease the withdrawal rate at every well for the next period. The optimal production scheduling problem is to find the optimal withdrawal rate at every drilled well at each time period while determining the well location at the same time.

4.2.1.1 Mixed Integer Linear Programming Formulation

Murray and Edgar [44] formulate this problem as a mixed integer linear programming (MILP) problem. They try to determine the optimal well configuration (withdrawal rates) while satisfying the demand schedule without exceeding it. Drilling or not at a particular location, i, can be denoted by a binary variable, say, y_i. Hence, the drilling decision can only be made at particular locations i which have to be identified beforehand. Also, use q_i^k to denote the withdrawal rate from well i at time period k. The interaction between withdrawal rates and pressures at all the wells can be delineated by the following gas flow equation,

$$\nabla k_g \nabla \Phi + q = \phi c_t \frac{\partial \Phi}{\partial t}, \tag{4–1}$$

where $\Phi = 2\int_0^p \frac{\rho}{z(\rho)\mu(\rho)} d\rho$. Including this constraint in a mathematical programming formulation leads to huge computational difficulties. However, as stated in [44], this nonlinear constraint has a very good linearization substitute, called influence equations [2, 64]. In these equations, the pressure drop at well i is a linear function of withdrawal flow rates from all drilled wells. This is defined by influence function matrices, Φ^k, $k = 1, \dots, m$, where Φ_{ij} denotes the pressure drop at well i for a unit flow at well j during time period k. The maximal profit problem can be formulated as follows,

$$\max \quad \sum_{k=1}^{m} \sum_{i=1}^{n} b_i^k q_i^k \tag{4–2}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \Phi_{ij}^k q_j^k = p_i^k, \qquad i = 1, \dots, n, k = 1, \dots, m, \tag{4–3}$$

$$\sum_{j=1}^{n} \Phi_{ij}^k q_j^k \le \bar{p}_i^k, \qquad i = 1, \dots, n, k = 1, \dots, m, \tag{4–4}$$

$$\sum_{k=1}^{l} \sum_{j=1}^{n} \Phi_{ij}^k q_j^k \le \hat{p}_i^l, \qquad i = 1, \dots, n, l = 1, \dots, m, \tag{4–5}$$

$$\sum_{j=1}^{n} q_j^k \le d^k, \qquad k = 1, \dots, m, \tag{4–6}$$

$$q_i^k \le M_i y_i, \qquad i = 1, \dots, n, \tag{4–7}$$

$$q_i^k \geq 0, \qquad i = 1, \ldots, n, k = 1, \ldots, m, \tag{4–8}$$

$$y_i \in \{0, 1\}, \qquad i = 1, \ldots, n, \tag{4–9}$$

where, from well i during time period k, b_i^k is the benefit of one unit gas flow, p_i^k is the pressure reduction, and $\bar{p}_i^k$ is the maximal pressure reduction at period k. $\hat{p}_i^l$ is the maximal total pressure drop allowed from the initial time point to time period l and d^k is the demand at time k. M_i is a big number to bound the withdrawal flow rate if $y_i = 1$. Its objective function is the total benefit from the withdrawal of gas. Constraints (4–3) compute the pressure drop at each well location during every time period. Constraints (4–4) specify the upper bound by which the pressure can drop during a specific single period for each well location. Also there is an upper bound by which the pressure can drop during the period between the initial time point and the current time period, which is stated in constraints (4–5). Constraints (4–6) ensure that the total gas withdrawal from all wells does not exceed the demand at each time period. Constraints (4–7) show that only drilled wells can have a positive withdrawal flow rate. This results in a mixed integer programming (MIP) problem, which can be solved by well Branch & Bound or Branch & Cut techniques. We refer the reader to [35, 37, 39, 46, 65] for comprehensive discussions of these techniques.

Let us discuss now the drawbacks of the proposed model (4–2) - (4–9). The model does not include any other cost such as well drilling cost, it does not take into account the relationship between the profit coefficient b_i^k and the demand d^k, and it assumes that the operator can choose any flow rate without considering the concurrent wellhead pressure. Also, after the deregulation of the natural gas market, the constraint (4–6) is not necessary and can be incorporated into the objective function instead. Furthermore, the different periods are intercorrelated to each other. For instance, the price of gas at time period t will affect the demand at the next time period $t + 1$ and vice versa. By incorporating all these factors, a nonlinear mixed integer programming problem can be formulated.

4.2.1.2 Nonlinear Programming Formulation

A multiple-stage nonlinear optimization problem is also proposed by Murray and Edgar in [44]. They formulate a nonlinear problem for each time period taking into account the interactions between two consecutive stages. The objective function for each time period k incorporates more factors such as the well placement cost, compressor operating cost, compressor setup cost, and the price of gas, which is shown as follows,

$$f^k = \sum_{j=1}^{n} \left(A q_j^k - C_w \frac{q_j^k}{q_j^k + \epsilon} - U_j^k q_j^k - C_j^k q_j^k + D_j^k \right), \tag{4–10}$$

where A is the price per unit gas flow, and C_w is the setup cost of any well placement. Instead of using the binary variables y_i to denote whether a well is drill or not, this nonlinear programming formulation uses the term $\frac{q_j^k}{q_j^k+\epsilon}$ to approximate y_i, where ϵ is a small constant compared to the magnitude of gas withdrawal flow rates q_j^k, $j = 1, \dots, n,$, $k = 1, \dots, m$. To be able to use this approximation, the magnitude of the flow rates are assumed to be known. U_j^k is the operating cost of the compressors for a unit flow of q_j^k. $-C_j^k q_j^k$ and D_j^k approximate the setup cost of a compressor at this location before time period k. Setting $D_j^k = C_j^k q_j^{k-1}$ makes the summation of these two terms equal to 0, which ensures that the compressor setup cost only occur once. For the nonlinear formulation, the deliverability equations are considered besides the constraints in the MIP formulation. The deliverability constraints specify the relationship between the withdrawal rate and well head pressure, which is also approximated by linear functions and shown as follows,

$$q_j^k \le e_j^1 + e_j^2 \rho_j^k, \qquad j = 1, \dots, n, k = 1, \dots, m, \tag{4–11}$$

where e_j^1 and e_j^2 are the linear coefficients and ρ_j^k is the bottom-hole pressure at well site j after time period k.

Also a multi-stage based algorithm is proposed in [44], in which all stages (time periods) are solved in an sequential order from 1 to m. We describe this algorithm as follows:

Step 1: Set up the problem: obtain parameters, e_j^1 and e_j^2, by some regression techniques; assume that no compressor is needed initially and set $U_j^1 = C_j^k = D_j^k = 0$; start from the first period problem.

Step 2: Solve the period k problem with an appropriate nonlinear programming algorithm, such as the gradient projection method [55].

Step 3: Examine the dual variables of the deliverability constraints. If none is positive, an optimal solution has been found for time period k, then go to Step 6. Otherwise, go to Step 4.

Step 4: If all positive dual variables are associated with deliverability constraints of the lowest feasible delivery pressure, an optimal solution is found for time period k, then go to Step 6. Otherwise, go to Step 5.

Step 5: Select the deliverability constraint with the largest associated dual variable, and then relax this constraint to the next lowest delivery pressure. Go to Step 2.

Step 6: By using the current period optimal solution, update the parameters in the next period problem. If $k = m$, terminate the whole program. Otherwise, set $k = k + 1$, and go to Step 2.

The drawback of the proposed model is that it does not consider all time periods together but considers them separately. Obviously, with this approach, an optimal solution to the practical problem cannot be obtained, as the interactions among all time periods are not taken into account.

4.2.2 Total Gas Recovery Maximization

In order to withdraw as much natural gas from a reservoir as possible, one option is to use waterflooding. This leads to the following immediate question. What is an optimal water injection rate with respect to different objectives, such as the maximal ultimate recovery, or the total revenues? A lot of models have been proposed for this problem. Mantini and Beyer [41] proposed optimal control models to this system and defined several objective functions due to different aspects of the problem.

Now, suppose there are two wells drilled on the surface of the gas reservoir, one for gas recovery and one for water injection. Therefore, let $r(t)$ denote the withdrawal rate of gas which is bounded by the maximum rate of gas extraction $r_m(t)$. Through the water injection, well water is injected into the reservoir at the nonnegative rate $s(t)$. This model assumes a constant g which is the ratio of gas entrapped behind the injected water to the volume of water at any time. The model to maximize the ultimate gas recovery can then be stated as

$$\begin{aligned}
\max \quad & \int_0^\infty r(t)dt && (4\text{–}12)\\
\text{s.t.} \quad & PV = NRT, && (4\text{–}13)\\
& \frac{dV}{dt} = -s(t) - gs(t), && (4\text{–}14)\\
& \frac{dN}{dt} = -r(t) - \frac{gs(t)P(t)}{RT}, && (4\text{–}15)\\
& r_m(t) \geq r(t) \geq 0,\\
& s(t) \geq 0,
\end{aligned}$$

where $P(t)$, $V(t)$ are the pressure and volume of the gas reservoir, $N(t)$ is the amount of gas which is not entrapped at time t. R is the universal constant of gas, and T is the temperature. Constraint (4–13) is the ideal gas law, constraint (4–14) shows the entrapped gas equals to constant g times the volume of the water while constraint (4–15) states that gas is entrapped at the current pressure in the reservoir and remains at the same pressure and has no effect on the reservoir. By introducing another variable $Q = P/RT$ and plugging constraint (4–13) into constraint (4–15), a more concise model can be obtained as follows,

$$\begin{aligned}
\max \quad & \int_0^\infty r(t)dt\\
\text{s.t.} \quad & \frac{dV}{dt} = -(1+g)s(t),\\
& \frac{dQ}{dt} = \frac{-r(t) + P(t)s(t)}{V(t)},
\end{aligned}$$

$$r_m(t) \geq r(t) \geq 0,$$

$$s(t) \geq 0.$$

Mantini and Beyer [41] also discuss several other objective functions. For example, the objective function to maximize the present worth value of the net revenues for internal rate of return, ρ, not equal to 0, is

$$\beta \int_0^\infty e^{-\rho t}[r(t) - \alpha s(t)]dt,$$

where α is the ratio of the water price (per cubic meter) to the gas price (per mole), and β is the gas price per mole. Due to the presents of the differential equations, these problems are generally computationally difficult to solve. However, Mantini and Beyer established a very interesting theorem, characterizing the properties of (some) optimal solutions of the control variable $r(t)$ and $s(t)$. Let us re-state this theorem here.

Theorem 4.1. *[41] The objective function $\int_0^\infty r(t)dt$ is maximized by any functions $\hat{r}$ and $\hat{s}$ such that,*

$$\int_0^{t_1} \hat{r}(t) = V_0(P_0 - P_c), \quad (4\text{–}16)$$

$$\int_0^{t_2} \hat{r}(t) = \frac{P_c(V_0 - V_c)}{1+g}, \quad (4\text{–}17)$$

$$\hat{r}(t) = 0, \forall t > t_2, \quad (4\text{–}18)$$

$$\hat{s}(t) = \begin{cases} 0, & 0 \leq t < t_1, \\ \frac{\hat{r}(t)}{P_c}, & t_1 \leq t \leq t_2, \\ 0, & t > t_2. \end{cases} \quad (4\text{–}19)$$

for t_1 and t_2 are any numbers with $0 < t_1 < t_2$, where P_0 and V_0 are the initial pressure and volume respectively and gas recovery stops when $P \leq P_c$ or $V \leq V_c$.

This theorem leads to the interesting statement that it is optimal to start the waterflooding when the first time P is lower than P_c; that is, the entrapped gas is at the lowest possible pressure. Although, in practice, this may not be valid for some specific gas wells due to discrepancies between modeling and reality.

4.3 Natural Gas Pipeline Network Optimization

Originally natural gas was treated as a byproduct of crude oil or coal mining and was spared. The flares in the mining field were usually natural gas [62]. Not until the introduction of pipelines did the natural gas become one of the major sources of energy. The earliest gas pipelines were constructed in the 1890's and they were not as efficient as those that we are using nowadays. The modern gas pipelines did not come into being until the second quarter of twentieth century. Because of the properties of natural gas, pipelines were the only way to transport it from the production sites to the demanding places, before the concept of Liquefied Natural Gas (LNG). The transportation of natural gas via pipelines remains still very economical, but it is highly impractical across oceans. Although LNG market is burgeoning in high speed now, pipeline network remains the main transportation system for natural gas.

Gas pipelines play a major role in energy supply and security. The Nord Stream Gas Pipeline (NSGP) project, transporting Russian gas to Germany, is one of the recent large scale pipeline projects. The NSGP is planned as a twin-pipeline with a total capacity of 55 billion cubic meters per annum. The estimated investment cost are 4 billion euros, financed by a joint venture of the three companies JSC Gazprom, BASF AG and E.ON AG. Not least, the decision to build the marine pipeline was driven politically, passing by Poland, Lithuania, Estonia, Belarus and Ukraine, in order to increase the natural gas supply security for Germany, mainly.

After the post war gas pipeline boom, a lot of research has been done in optimization applications to pipeline networks; for instance, how to setup the pipeline network, how to determine the optimal diameter of the pipelines, how to allocate compressor stations in the pipeline network, and what is the minimal fuel consumption of the network. Typically, the mathematical programming formulations of the pipeline optimization problems contain a lot of nonlinear/nonconvex/nonsmooth constraints and functions. The most common constraints are the so-called Weymouth panhandle equations, which relate the pressure and flow rate through a segment of pipeline (i, j). They read as follows

$$\text{sign}(f_{ij})f_{ij}^2 = p_i^2 - p_j^2, \qquad (i, j) \in A_p, \tag{4–20}$$

where f_{ij} is the flow rate of pipeline (i, j), p_i and p_j are the pressures at node i and j respectively. Hence, the direction of the gas flow depends on the pressure difference of the two nodes i and j. Therefore, the nonsmooth function $\text{sign}(f_{ij})$ is needed.

Recently, more research is related to the network optimization of gas transmission; given the network structure other than the design of the network topology. One of the few papers dealing with the design of network topology is the one by Rothfarb et al. [56], where the authors propose a tree generating algorithm to design the network topology.

4.3.1 Compressor Station Allocation Problem

Once a network topology is chosen, one problem is to determine the optimal configuration of the pipelines and the location of the compressor stations in this network. Because of the high setup cost and high maintenance cost, it is desirable to have the best network design with the lowest cost. This problem concerns a lot of variables: the number of compressor stations which is an integer variable, the pipeline length between two compressor stations, the diameters of the pipelines, and the suction and discharge gas pressures at compressor stations. This problem is computationally very challenging since it includes not only nonlinear functions in both objective and constraints but, in addition, also integer variables.

A simple and typical network for this type of problem is shown in Fig. 4-2. Node s is the supply node where the gas is produced. Nodes a and b are the demand nodes where the gas is consumed. The trapezoids 1 through 6 denote the compressor stations. There are three branches: s to 3 is the first branch, 3 to a is the second branch, and 3 to b is the third branch.

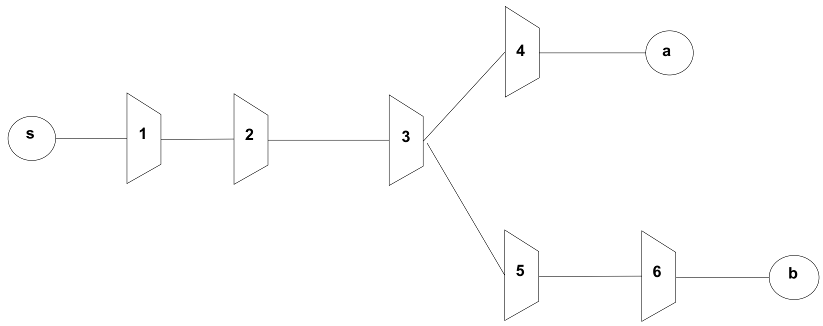

Figure 4-2. A gas pipeline network configuration problem with three branches.

Suppose there are at most n compressor stations to be set up, and n_1, n_2, and n_3 denote the number of compressor stations on branch 1, 2, and 3 respectively. For each pipeline segment i, there are five associated parameters: the flow rate f_i, the discharge pressure (from the upstream compressor) p_i^d, the suction pressure (from the downstream compressor) p_i^s, the diameter d_i, and the length l_i.

The formulation for the three branches problem by Edgar et al. [26, 27] reads as,

$$\text{Min} \quad \sum_{i=1}^{n}(O_y + C_c)\alpha\frac{T_s}{\eta_s}\frac{\gamma}{\gamma-1}\left[1-\left(\frac{p_i^d}{p_i^s}\right)^{\frac{z(\gamma-1)}{\gamma}}\right] + \sum_{i=1}^{n+1} C_l l_i d_i \tag{4–21}$$

$$\text{s.t.} \quad p_i^d \geq p_i^s, \qquad i = 1, \dots, n, \tag{4–22}$$

$$p_i^d \leq K_i p_i^s, \qquad i = 1, \dots, n, \tag{4–23}$$

$$\underline{p}_i^d \leq p_i^d \leq \overline{p}_i^d, \qquad i = 1, \dots, n, \tag{4–24}$$

$$\underline{p}_i^s \leq p_i^s \leq \overline{p}_i^s, \qquad i = 1, \dots, n, \tag{4–25}$$

$$\underline{l}_i \leq l_i \leq \overline{l}_i, \qquad i = 1, \dots, n, \tag{4–26}$$

$$\underline{d}_i \leq p_i^d \leq \overline{p}_i^d, \qquad i = 1, \dots, n, \tag{4–27}$$

$$f_i = Ad_i^{\frac{8}{3}}\left[\frac{(p_i^d)^2-(p_i^s)^2}{l_i}\right]^{\frac{1}{2}} \qquad i = 1, \dots, n, \tag{4–28}$$

$$\sum_{i=1}^{n_1} l_i + \sum_{i=n_1+1}^{n_1+n_2} l_i = L_1, \tag{4–29}$$

$$\sum_{i=1}^{n_1} l_i + \sum_{i=n_1+1}^{n_1+n_3} l_i = L_2, \tag{4–30}$$

where γ is the ratio of specific heats, T_s is the suction temperature, z is the gas compressibility factor, η_s is the efficiency factor, O_y and C_c are cost functions with respect to horsepower. The objective function (4–21) contains two parts, of which the first is the compressor station costs and the second is the maintenance costs of the pipeline segments. Constraints (4–22)-(4–27) are the upper and lower bounds on pressures, pipeline lengths and diameters. L_1 and L_2 are the distances between the supply node and two demand nodes.

Model (4–21) - (4–30) can be solved by applying Branch and Bound techniques using reduced gradient nonlinear optimization method to solve the subproblem at each node in the Branch and Bound tree [26, 27]. The drawback of this model is that it highly depends on the topology of the network.

4.3.2 Least Gas Purchase Problem and Optimal Dimensioning

In the modern natural gas industry, the gas production companies are rarely affiliated with the gas transmission and distribution companies. Thus, for gas distribution companies, one problem is to determine the best flow rate and gas pressures in each pipeline by which the least cost on purchasing gas from producers is achieved. This problem can be formulated as a optimization problem with linear objective function and nonlinear/noconvex constraints.

Consider now Fig. 4-3. s_1 and s_2 are the supplies for source nodes 1 and 2, the set of which is denoted by N_s. Nodes 6 to 9 are demand nodes with demands $-s_i$, $i = 6, 7, 8, 9$. In this model, there are two kinds of arcs: those with compressor stations such as $(1, 4)$ and $(2, 4)$, which is denoted by A_c; and those without, which are also called pipeline arcs and denoted by A_p. Flows on arcs with compressors are directed such that $f_{ij} \geq 0$, $\forall (i, j) \in A_c$, and flows on pipeline arcs are undirected and the direction depends on the pressures of both ends of this arc.

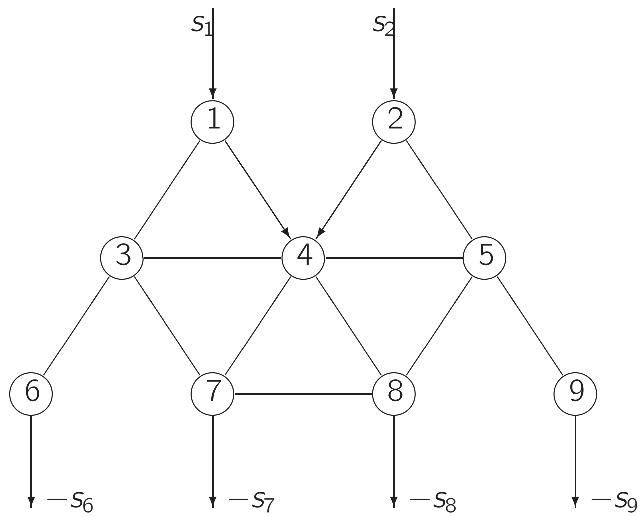

Figure 4-3. Least cost problem network.

A mathematical programming formulation can be stated as

$$\min \quad \sum_{i \in N_s} c_i s_i \tag{4–31}$$

$$\text{s.t.} \quad \sum_{j \in A_i^+} f_{ij} - \sum_{j \in A_i^-} f_{ji} = s_i, \qquad \forall i \in N, \tag{4–32}$$

$$\operatorname{sign}(f_{ij}) f_{ij}^2 = C_{ij}(p_i^2 - p_j^2), \qquad \forall (i,j) \in A_p, \tag{4–33}$$

$$f_{ij}^2 \geq C_{ij}(p_i^2 - p_j^2), \qquad \forall (i,j) \in A_c, \tag{4–34}$$

$$\underline{s}_i \leq s_i \leq \bar{s}_i, \qquad \forall i \in N, \tag{4–35}$$

$$\underline{p}_i \leq p_i \leq \bar{p}_i, \qquad \forall i \in N, \tag{4–36}$$

$$f_{ij} \geq 0, \qquad \forall (i,j) \in A_c, \tag{4–37}$$

where p_i is the gas pressure at node i, c_i is the purchase cost per unit gas from supplier i, and C_{ij} an coefficient for arc (i,j), which is determined by the length, diameter and so on. A_i^+ denotes the set of arcs which are emanating from node i, while A_i^- denote the one of incoming arcs to node i.

The nonlinear constraints of the model above can be simplified by letting π_i substitute p_i^2. Then, constraints (4–33), (4–34), and (4–36) can be replaced by

$$\text{sign}(f_{ij})f_{ij}^2 = C_{ij}(\pi_i - \pi_j), \forall (i,j) \in A_p,$$

$$f_{ij}^2 = C_{ij}(\pi_i - \pi_j), \forall (i,j) \in A_p,$$

$$\underline{\pi}_i \le \pi_i \le \bar{\pi}_i, \forall i \in N$$

With this substitution, the 'only' nonlinear functions left are $\text{sign}(f_{ij})$ and f_{ij}^2.

De Wolf and Smeers [25] propose a piecewise linear programming algorithm to solve this problem, in which they construct a piecewise linear approximation to the nonlinear constraints and solve the relaxed problem by simplex algorithm extensions [22]. The performance of the algorithm depends highly on the choice of the initial point. It is crucial to have a good starting solution, which can be obtained by solving the following problem:

$$\text{Min} \quad \sum_{(i,j)\in A} \frac{|f_{ij}|f_{ij}^2}{3C_{ij}^2} \tag{4–38}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j\in A_i^+} f_{ij} - \sum_{j\in A_i^-} f_{ji} = s_i, && \forall i \in N, \\ & \underline{s}_i \le s_i \le \bar{s}_i, && \forall i \in N. \end{aligned}$$

The objective function (4–38) in this problem is the amount of mechanical energy consumed in the gas pipeline per unit time. Its KKT necessary conditions (see [7]) is equivalent to the constraints (4–32), (4–33), and (4–35). The KKT necessary point is a good approximation starting point which does not take into account pressures' bounds and the existence of compressors. The algorithm proposed by [25] is as follows:

(o) **Initialization:** Let (f^0,p^0, s^0) be a vector of flows, pressures, and net supplies that satisfy constraints (4–32), (4–33), (4–34), (4–35), and (4–37). Replace the nonlinear function $\text{sign}(f_{ij})f_{ij}^2$ by a piecewise linear approximation including f_{ij}^0 as a breakpoint. Use f_{ij}^0 as starting point for the piecewise linear programming approach. Also set $k = 1$.

(i) **Iteration** k**:** Solve the approximation problem by the piecewise linear programming approach. Let (f^k,p^k,s^k) be the solution.

(ii) **Stopping rule:** Compute $\bar{f}_{ij}^k$ by the following equation:

$$\bar{f}_{ij}^k = \text{sign}(p_i^k - p_j^k) C_{ij} |(p_i^k)^2 - (p_j^k)^2|^{\frac{1}{2}}$$

If the error $e_{ij}^k = \bar{f}_{ij}^k - f_{ij}^k$ is greater than a given tolerance, for example, 10^{-5}, then add $\bar{f}_{ij}^k$ as a new discretization point and return to step (i). Otherwise stop and the incumbent solution is optimal.

It can be noticed that the optimal objective function value of problem (4–31) is a function of the diameters of the pipelines, say, $Q(D)$, because the parameter C_{ij} of pipeline (i, j) is a function of the diameter, where $C_{ij} = K_{ij} D_{ij}$ and K_{ij} is a coefficient. If the network structure and the length of each pipeline are fixed, the investment problem is to find the best pipeline diameters which achieve the lowest investment cost including both the gas purchase cost $Q(D)$ and the pipeline construction cost $C(D)$. They are given as

$$C(D) = \sum_{(i,j)\in A} = (k_G D_{ij}^2 + k_G^I D_{ij} + k_G^{II}) l_{ij},$$

where l_{ij} is the length of pipeline (i, j). Then the investment problem becomes

$$\begin{aligned} \text{Min} \quad & C(D) + Q(D) && \text{(4–39)} \\ \text{s.t.} \quad & D_{ij} \geq 0, \qquad \forall (i, j) \in A, \end{aligned}$$

which is a bilevel programming problem. The second part of the cost function, $Q(D)$, is nonconvex/nodifferential and has an implicit domain. De Wolf and Smeers [23] propose how to get one generalized subgradient, as in the next proposition.

Proposition 4.1. *Denote by f^*, s^*, π^* an optimal solution of the operations problem (4–31). Let w_{ij}^* be an optimal value of the dual variable associated to constraint (4–33). Then*

$$(\dots, w_{ij}^* 5 K_{ij}^2 D_{ij}^4, \dots) \in \partial Q(D), \qquad \text{(4–40)}$$

where $\partial Q(D)$ is the generalized subdifferential.

The investment problem (4–39) can be solved by a bundle method which performs well for nondifferential optimization problems. By using a bundle method, we do not

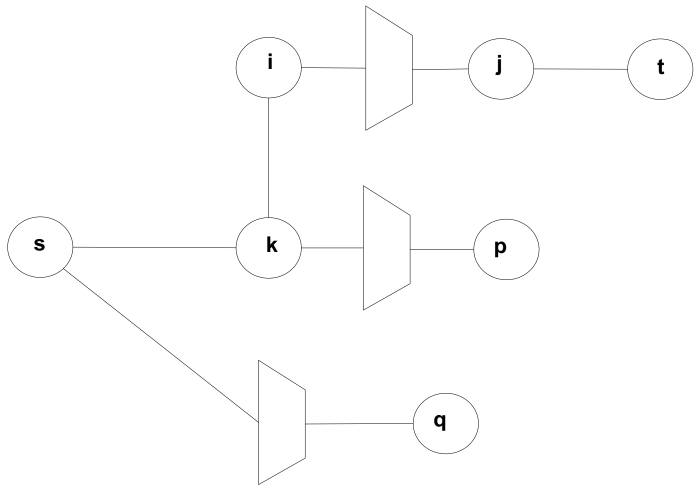

Figure 4-4. A gas pipeline network.

need to know the explicit domain of the objective function. Hence it is a good fit for the investment problem because the objective function domain is implicit. At each step, it only needs the value of the objective function and one of the generalized subgradient, which can be computed by (4–40). The dual variables, w_{ij}, can be obtained while solving the operations problems by using simplex algorithm extensions. Readers may find more comprehensive discussions of the bundle method in [33].

4.3.3 Minimum Fuel Consumption Problem

To let the consumer receive an acceptable withdrawal rate of gas, the pipeline needs to maintain a certain pressure. This is achieved by adding compressor stations in the network. One well known problem is the minimal fuel cost problem due to the fuel consumption of compressor stations, which are usually considered as special arcs in the network of this type of models. The minimal fuel cost problem has been widely discussed in the literature; see for instance [20, 32, 51–53, 66].

An typical gas pipeline network is shown in Fig. 4-4. Node s is the source node, and t, p, and q are the demand nodes. Arc (j, t) is an ordinary pipeline arc, arcs (i, j), (k, p), (s, q) are compressor station arcs. In each compressor station (i, j), there are C_{ij} compressors, and the pressures at i and j are denoted by p_i and p_j respectively. Let A^I denote the set of compressor station arcs, A^{II} denote the set of ordinary pipe arcs, V

denote the node set. Then, the minimal fuel cost problem can be stated as

$$\text{Min} \quad \sum_{(i,j)\in A^I} g_{ij}(x_{ij}, p_i, p_j) = \sum_{(i,j)\in A^I} \frac{\frac{x_{ij} Z_i R T_i}{\omega}[(\frac{p_j}{p_i})^{\omega} - 1]}{\mu_{ij}} \tag{4–41}$$

$$\text{s.t.} \quad \sum_{j\in A_i^+} x_{ij} - \sum_{j\in A_i^-} x_{ji} = b_j, \quad \forall i \in V \tag{4–42}$$

$$p_i^2 - p_j^2 = R_{ij} x_{ij}^2, \quad \forall (i,j) \in A^{II} \tag{4–43}$$

$$0 \le x_{ij} \le u_{ij}, \quad \forall i,j \in A \tag{4–44}$$

$$p_i^L \le p_i \le p_i^U, \quad \forall i \in V \tag{4–45}$$

$$(\frac{x_{ij}}{n_{ij}}, p_i, p_j) \in \mathcal{D}_{ij}, \quad \forall (i,j) \in A^I \tag{4–46}$$

$$n_{ij} \in 0, 1, 2, \dots, N_{ij}, \quad \forall (i,j) \in A^I \tag{4–47}$$

where p_i^L and p_i^U are the lower and upper bounds on the pressure of node i. At each compressor station (i,j), u_{ij} is the capacity, $N_{i,j}$ is the total number of compressor, and x_{ij}, n_{ij} are the gas flow rate and number of compressor in use respectively. Also there are several other related parameters for (i,j): z_i is the gas compressibility factor, T_i is the gas temperature, μ_{ij} is the compressor adiabatic efficiency, and R_{ij} is a gas constant. The most complicated constraint is (4–46) in which $\mathcal{D}_{ij}$ is the feasible domain of compressor station (i,j) as for variable triplet $(\frac{x_{ij}}{n_{ij}}, p_i, p_j)$. The feasible domain is stated below by the set of equations,

$$\frac{h_{ij}}{s_{ij}^2} = A_H + B_H(\frac{q_{ij}}{s_{ij}}) + C_H(\frac{q_{ij}}{s_{ij}})^2 + D_H(\frac{q_{ij}}{s_{ij}})^3 \tag{4–48}$$

$$\mu_{ij} = \frac{C_E(\frac{q_{ij}}{s_{ij}})^2 + B_E(\frac{q_{ij}}{s_{ij}}) + A_E}{100} \tag{4–49}$$

$$S_{min} \le s_{ij} \le S_{max} \tag{4–50}$$

$$Surge \le \frac{q_{ij}}{s_{ij}} \le Stonewall \tag{4–51}$$

$$h_{ij} = \frac{Z_i R T_i}{\omega}[(\frac{p_j}{p_i})^{\omega} - 1] \tag{4–52}$$

$$q_{ij} = Z_i R T_i \frac{x_{ij}}{p_i n_{ij}} \tag{4–53}$$

In the above equations, q_{ij} denote the flow through the compressor unit, s_{ij} denote the speed of the compressor(s), and A_H, B_H, C_H, D_H, C_E, B_E, A_E are the compressor unit's constants.

This problem is very difficult to solve, and its solution algorithms are highly dependent on the topology of underlying network. Most of the algorithms for this problem are based on dynamic programming [51–53] and gradient search approaches [66]. Also meta heuristic approaches have been conducted, such as ant colony optimization [20] or genetic algorithms [32].

4.4 Natural Gas Market Models

Government regulation over the gas industry dates back to the early days of natural gas usage. At the first glance, this seams to be reasonable, as government and the public are the main users of natural gas and investments in the natural gas industry are tremendous. Not until the 1980s began the deregulation of this industry to improve both equity and efficiency of the natural gas market. Between the original producers and end users, there exists a variety of participants, each of which acts to optimize its own benefits. Under different government policies, a lot of natural gas market models are proposed. In this section we discuss optimization models of both a regulated and a deregulated gas market.

4.4.1 Reallocation Problem in a Regulated Natural Gas Market

O'Neil et al. [49] propose a model on how to allocate gas to users with different priorities under the government regulations when encountered a gas shortage emergency. In this model there are multiple gas transmission systems among which any two systems are not necessarily connected physically. All users are divide into 9 categories with priorities 1 through 9. The transportation network is composed of two types of arcs and nodes: the physical arcs and nodes which really exist in practice - denoted by A_{phy} and N_{phy}, respectively - and the pseudo counterparts which are for convenience of modeling - denoted by A_{pseudo} and N_{pseudo}, respectively. Let K_w be the set of users who withdraw gas from gas system w. This model also includes the panhandle constraints (4–20) for each of the pipeline arcs. However, instead of using the actual nonlinear constraints, this model incorporates two linearized approximation constraints in each iteration, which read as

$$-\epsilon_{ij} \leq -f_{ij} + \alpha_i p_i - \alpha_j p_j \leq \epsilon_{ij}, \quad \forall (i,j),$$

$$\epsilon^1_{ij} \leq p_i - p_j \leq \epsilon^2_{ij}, \quad \forall (i,j),$$

where ϵ, ϵ^1_{ij}, and ϵ^2_{ij} are parameters determined at each iteration through

$$\begin{aligned}
\epsilon_{ij} &= \alpha_1 |f^{new}_{ij}|, \\
\epsilon^1_{ij} &= (1-\alpha_1)(p^{new}_i - p^{new}_j), \\
\epsilon^2_{ij} &= (1+\alpha_1)(p^{new}_i - p^{new}_j), \\
\alpha_1 &= \max\{\alpha(\gamma_2)^m, \frac{1}{2}\delta_2\},
\end{aligned}$$

with the positive constants α, γ_2, δ_2.

The allocation algorithm proposed by O'Neil et al. [49] is as follows:

Step 0: Allocate the minimum amounts that all users must receive. If no feasible solution exists, then stop; no allocation exists under the specified parameters.

Step 1: Allocate gas according to the priorities within each transporter's system, starting with priority 1 and proceeding in ascending order of priority.

Step 2: Determine if priorities 1 through 5 are satisfied. If so, go to step 4. Otherwise, fix the lower (6 through 9) priority users, in pipelines with a shortage in any higher priority, at their lower bounds.

Step 3: Allocate gas according to the priorities within the entire system.

Step 4: Incorporate the linearized nonlinear constraints and find the optimal solution minimizing the amount transferred between systems, as in the optimization problem (4–54) - (4–64).

The linear programming formulation used in the allocation problem [49] can be stated as follows,

$$
\begin{aligned}
\text{Min} \quad & \sum_{(i,j)\in I} |f_{ij}| + \sum_{(i,j)\in S} f_{ij} && && \text{(4–54)}\\
\text{s.t.} \quad & \sum_{j\in A_i^+} f_{ij} - \sum_{j\in A_i^-} f_{ji} = s_i - \sum_{k\in K}\sum_{l=0}^{9} d_{ikl}, && \forall i \in N, && \text{(4–55)}\\
& \sum_{i\in N}\sum_{k\in K} d_{ikl} + u_l = \bar{d}_l, && l = 0, \ldots, 9, && \text{(4–56)}\\
& \sum_{l=1}^{5}\sum_{k\in K_w}\sum_{i\in N} d_{ikl} + r_w = g_w, && \forall w \in W, && \text{(4–57)}\\
& -\epsilon_{ij} \le -f_{ij} + \alpha_i p_i - \alpha_j p_j \le \epsilon_{ij}, && \forall (i,j) \in A_{ps}, && \text{(4–58)}\\
& \epsilon^1_{ij} \le p_i - p_j \le \epsilon^2_{ij}, && \forall (i,j) \in A_{vc}, && \text{(4–59)}\\
& 0 \le s_i \le \bar{s}_i, && \forall i \in N, && \text{(4–60)}\\
& \underline{d}_{ikl} \le d_{ikl} \le \bar{d}_{ikl}, && \forall i \in N, k \in K, l = 0, \ldots, 9 && \text{(4–61)}\\
& \underline{p}_i \le p_i \le \bar{p}_i, && \forall i \in N, && \text{(4–62)}\\
& u_l \ge 0, && l = 0, 1, \ldots, 9, && \text{(4–63)}\\
& r_w \ge 0, && \forall w \in W, && \text{(4–64)}
\end{aligned}
$$

where s is the supply, d is the demand, u is the slack variable for the demand of each priority, and r is the slack variable for the demand of priority 1 through 5. In constraints (4–58), $-f_{ij} + \alpha_i p_i - \alpha_j p_j$ is the linearized version of the panhandle equation, where α_i and α_j are the coefficients of the first order Taylor series expansion. A_{ps} and A_{vc} denote the pipeline arc set and the compressor arc set, respectively. The objective function is the amount of gas transferred between two systems, I is the set of physical arcs that connect two systems, and S is the set of pseudo arcs that realize swapping by allowing flow into redistribution node. This is one of earliest mathematical models describing the natural gas market under regulation.

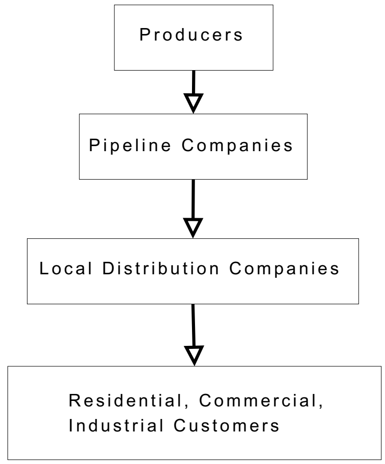

Figure 4-5. Participants Relationship in Regulated Gas Market.

4.4.2 Deregulated Natural Gas Market Models

In North America, before the 1980's, the natural gas market had been greatly regulated by the government since the 1930's. In the regulated market, there were primarily four participants: the gas producers, the gas pipeline companies, local gas distribution companies, and customers. The relationship of these participant is shown in Fig. 4-5, where producers sold gas to pipeline companies, and pipeline companies sold the gas to local gas distribution companies, and then local distribution companies sold the gas to various customers, such as industrial, commercial, and residential customers. In this regulated market, gas prices in each of the above transactions are tightly regulated by Federal and State governments as pipeline companies and local distribution companies had monopolies in the gas market. Since the mid 1980's, a series of deregulation policies have been announced. These polices encourage pipeline companies to switch from their traditional role as owners of natural gas by allowing producers and buyers to bypass the pipeline companies in that the buyers can transport their own gas through the pipeline system by paying some fees.

The deregulation of the gas market not only changed the roles of the former participants but also helped to create more participants, such as the gas marketing companies. Many models have been proposed for the deregulated gas market, especially for North America and Europe. Optimal purchasing strategies considering storage, contract, spot prices, peak day demands local distribution companies under North America gas market conditions have been studied by Avery et al. [4]. A model based on generalized network to provide optimal strategies for the marketing companies and local distribution companies, and a system, GRIDNET, to store all the dealed information were proposed by Brooks and Neill [17] and Brooks [15]. The Natural Gas Transmission and Distribution Module (NGTDM) is an important model of the North American gas market, which is a submodule of the U.S. Department of Energy's National Energy Modeling System (NEMS). The Gas System Analysis Model (GSAM) is another North American gas market model, which tries to maximize the social welfare function to get the equilibrium, see for instance Gabriel et al. [30]. One of the most recent North American gas market models is the Mixed Complementarity-Based Equilibrium Model of Natural Gas Markets; see Gabriel et al. [29].

Gabriel et al. [29] consider six types of participants: the pipeline operators, the production operators, the marketers/shippers, the storage reservoir operators, the peak gas operators, and the customers. Each participant is trying to minimize cost or maximize profit for itself. For the sake of simplicity, this model assumes only linear relationship within each problem faced by a participant. Hence, every participant faces a linear programming problem. Because natural gas is a highly seasonal product, the model specifies three seasons in each year, which are denoted by $s = 1, 2, 3$. Every year has index $y \in Y$.

- $s = 1$: low demand season, Apr.-Oct.;
- $s = 2$: high demand season, Nov., Dec., Feb., Mar.;
- $s = 3$: peak demand season, Jan.

In this formulation, pipeline gas is available for all three seasons, and gas is injected to storage reservoir in season 1 and extracted in season 2 and 3, and peak gas is only used in the peak season.

The operator of pipeline a is trying to maximize its own profit by solving the following problem,

$$\text{Max} \quad \sum_{y \in Y} \sum_{s=1}^{3} days_s \tau_{asy} f_{asy} \tag{4–65}$$

$$\text{s.t.} \quad f_{asy} \le \bar{f}_a, \qquad \forall s, y, \tag{4–66}$$

$$f_{asy} \ge 0, \qquad \forall s, y, \tag{4–67}$$

where $days_s$ is the number of days in season s, τ_{asy} and f_{asy} are the prices and flow rates respectively of pipeline a in season s of year y. Constraints (4–66) are the upper bound constraints of the flows. τ_{asy} are the equilibrium show prices determined by the optimization problems of the other participants. Other than τ_{asy}, there are some other conditions relating this pipeline operator problem to the other pipelines and other kinds of participants. These conditions are usually called Market-Clearing conditions. The corresponding Market-Clearing conditions for the gas pipeline operator problem reads

$$days_1 f_{a1y} = \sum_{r \in R(n_1(a))} days_1 g_{ary} + \sum_{m \in M(n_1(a))} days_1 h_{am1y} \qquad \tau_{a1y} \text{ free} \quad \forall y \in Y, \tag{4–68}$$

$$days_s f_{asy} = \sum_{m \in M(n_s(a))} days_s h_{amsy} \qquad \tau_{asy} \text{ free} \quad s = 2, 3, \forall y \in Y. \tag{4–69}$$

These two Market-Clearing conditions state that all the supplies equal all the demands. g_{ary} is the flow rate of gas to storage operator r from the producers of season 1 through arc a, and h_{amsy} is the gas flow rate from producers of season s to marketer m through arc a.

The production operator's problem, for production company $c \in C$ at node $n \in N$, is to maximize its profit by solving the following problem,

$$\text{Max} \quad \sum_{y \in Y} \sum_{s=1}^{3} days_s \left(\pi_{nsy} q_{csy} - c_c^{pr} q_{csy} \right) \tag{4–70}$$

$$\text{s.t.} \quad q_{csy} \le \bar{q}_c, \qquad \forall s, y, \tag{4–71}$$

$$\sum_{y\in Y}\sum_{s=1}^{3} days_s q_{csy} \leq prod_c \tag{4–72}$$

$$q_{csy} \geq 0, \qquad \forall s, y, \tag{4–73}$$

where π_{nsy} and c_c^{pr} are the price of gas sold by the production company and cost to produce one unit of gas, respectively, for company c, and q_{csy} is the production rate of the company in season s of year y. Constraints (4–71) specify the upper bounds of the production rate in each period, and constraints (4–72) give the total production capacity for the whole planning horizon. Except this optimization problem, the coupling conditions for the production company c at node n are as follows,

$$\sum_{c\in C(n)} days_1 q_{c1y} = \sum_{a\in A_n^+}\Big(\sum_{r\in R(n_1(a))} days_1 g_{ary} + \sum_{m\in M(n_1(a))} days_1 h_{am1y}\Big), \quad \pi_{n1y} \text{ free} \quad \forall y, \tag{4–74}$$

$$\sum_{c\in C(n)} days_s q_{csy} = \sum_{a\in A_n^+}\sum_{m\in M(n_s(a))} days_s h_{asy}, \quad \pi_{asy} \text{ free} \quad s = 2, 3, \forall y \in Y. \tag{4–75}$$

The storage reservoir operator's problem, the marketer's problem, and the peak gas operator's problem are all described in the same way, first the linear programming problem and then the market-clearing conditions. Since all operator's problems are linear programming problems, the KKT conditions are necessary and sufficient. Combining all the KKT conditions and market-clearing conditions of every operator's problem, we then get a Linear Complementarity Problem (LCP), which is a special case of nonlinear complementarity problem (NCP) or variational inequality problem (VI). Gabriel et al. [29] proved that there exists a solution of the system and the prices are unique in this case. For more details about LCP, NCP, and VI, we refer the reader, for instance, to [21, 28, 40, 45].

Also a lot of models for the European gas markets have been proposed. A stochastic Stackelberg-Nash-Cournot equilibrium model for natural gas producers are proposed by De Wolf and Smeers [24]. Breton and Zaccour [14] propose a duopoly

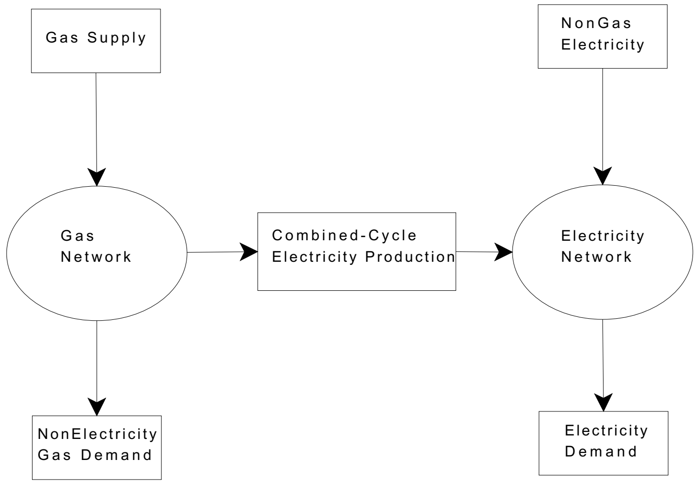

Figure 4-6. Relationship between Gas Network and Electricity Network.

producer model. A recent European gas market model similar to the model in [29] is GASTALE proposed by Boots et al. [13].

4.4.3 Combining Natural Gas System and Electricity System

Natural gas is widely used in electricity production. Because combined-cycle plants are highly efficient and have less damage to the environment, more and more power plants of this type are build around the world. Hence the electricity and the gas system are now highly correlated. Here we discuss some related optimization applications regarding this relationship.

4.4.3.1 Electricity System Reliability Study

Due to the increasing number of combined-cycle power plants being built, electricity production relies more and more on the amount of gas the power plants can get. However, the electricity plants are not the only users of natural gas; see Sec. 4.1. In order to perform a reliability analysis of the electricity system, it is important to study the maximal amount of gas which the gas network can supply to the electricity plants. The relation between gas network and electricity network is shown in Fig. 4-6. Munoz et al. [43] studied the problem of the maximal gas supply the electricity system can receive, taking into account the other gas users, the pipeline capacity and the production capacity. The formulation is very similar to the gas pipeline operations problem (4–31).

Instead of minimizing the gas purchase cost as in (4–31), this problem maximizes the total electricity which can be produced by using gas from the gas system. It can be formulated as

$$\text{Max} \quad \sum_{i \in N_e} A_i e_i + B_i e_i^2 + C_i e_i^3 \tag{4–76}$$

$$\text{s.t.} \quad \sum_{j \in A_i^+} f_{ij} - \sum_{j \in A_i^-} f_{ji} = s_i - d_i - e_i, \qquad \forall i \in N, \tag{4–77}$$

$$\text{sign}(f_{ij}) f_{ij}^2 = C_{ij}(p_i^2 - p_j^2), \qquad \forall (i,j) \in A_p, \tag{4–78}$$

$$f_{ij}^2 \geq C_{ij}(p_i^2 - p_j^2), \qquad \forall (i,j) \in A_c, \tag{4–79}$$

$$\underline{s}_i \leq s_i \leq \bar{s}_i, \qquad \forall i \in N, \tag{4–80}$$

$$\underline{p}_i \leq p_i \leq \bar{p}_i, \qquad \forall i \in N, \tag{4–81}$$

$$\underline{d}_i \leq d_i \leq \bar{d}_i, \qquad \forall i \in N, \tag{4–82}$$

$$\underline{e}_i \leq e_i \leq \bar{e}_i, \qquad \forall i \in N, \tag{4–83}$$

$$f_{ij} \geq 0, \qquad \forall (i,j) \in A_c, \tag{4–84}$$

where the objective function is a polynomial function of withdrawal of gas from the gas network. e_i is the gas withdrawal to produce electricity. d_i is the demand not related to electricity production. A_i^+ denotes the set of arcs which are emanating from node i, while A_i^- denotes the set of incoming arcs to node i.

Munoz et al. [43] solve the above problem in two phases. First, by dropping all nonlinear constraints, a mixed integer linear programming problem is obtained and then solved, where the integer variables denote the directions of flows in the pipeline segments. Second, by knowing the directions of flows from the phase I problem, a nonlinear problem is solved. However, two theoretical questions still remain in the correctness of optimality obtained by the method. First, it remains unanswered whether the solution from phase I will ensure the phase II problem to be feasible. Second, it is not true that the second phase problem is a convex problem for which a simple counterexample can easily be constructed, such as the single pipeline segment problem.

4.4.3.2 Optimization in Natural Gas Contracts

Many electricity production plants use a lot of sources among which natural gas is a very reliable alternative to meet the high electricity demand. The optimization of fuel contracts for a hydro-based power system is a very good example. In hydro power systems, precipitation varies from season to season. For the low precipitation seasons, the plants need to buy gas to generate electricity.

Let us now discuss a model which deals with the optimal dispatch strategy while considering the particular specifications of gas supply contracts as in Chabar et al. [19]. This model assumes a take-or-pay contract, which is widely adopted, especially in Europe. If a take-or-pay contract is signed, specifying a monthly amount and a total annual amount, then at least $X\%$ of the monthly amount has to be bought every month and at least $Y\%$ of the contracted annual amount for the year has to be bought. Hence, there might be some gas excess based on contracts of this type. Two reservoirs are added into this model to accommodate the situations where gas excess exists. All excesses of gas not consumed monthly are stored in the gas reservoir A, the difference between the annual take-or-pay amount and the sum of all monthly take-or-pay amounts of the year is stored in reservoir B. Also, one of the gas contract provisions state that the gas purchased at any time point cannot “stay in the reservoir”, or actually hold by the gas provider by more than N time periods, which means that if any amount of gas stays in the reservoir more than N time periods, it will have to be discarded. GD_t is used to denote the amount of gas discarded at time t. Figure 4-7 shows how the model, based on reservoirs, deals with the contract provisions.

Also the maintenance schedule is modeled by reservoirs. A fictitious remaining-hours reservoir is assigned to every power unit for each maintenance cycle. For a 3 power units 3 cycles problem, there will be 9 reservoirs. The length of each kind of cycle is

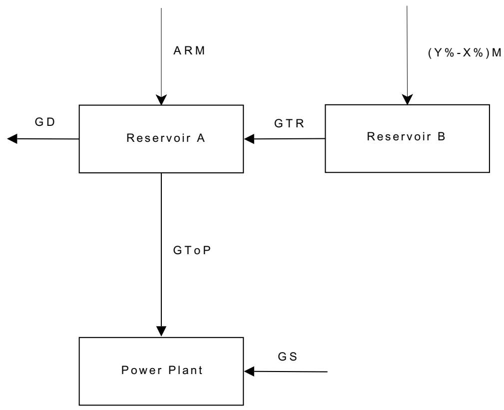

Figure 4-7. Gas Contracts Modeled by Reservoirs.

Table 4-1. Maintenance cycle length

Cycle	Frequency	Average Duration	Cost(MMR$)
Combustor	8000 hours	7 days	3.5
Hot path circuit	24000 hours	14 days	10
Major maintenance	48000 hours	21 days	20

shown in Table 4-1. For each power unit, the reservoirs are filled with the amount of remaining hours of operation until next maintenance. The capacity of each reservoir is the length of the cycle. As the unit operates, all reservoirs for that unit are reduced by the quantity of the elapsed hours. After maintenance, the fictitious maintenance reservoir is filled to its capacity.

Considering also the maintenance scheduling of the thermal plant, a dynamic programming formulation of the problem, for a given stage and price, is proposed by Chabar et al. [19]:

$$FBF_t^k(VA_t, VB_t, \{VH_t^{i,j}, \forall i,j\}, \pi_t^k) \tag{4–85}$$

$$= \text{ Max } RI_t + \sum_{s=1}^{S} p_{t+1}(k,s) FBF_{t+1}^s(VA_{t+1}, VB_{t+1}, \{VH_{t+1}^{ij}, \forall i,j\}, \pi_{t+1}^k) \tag{4–86}$$

$$\text{s.t.} \quad VA_{t+1} = VA_t + ARM_t - GToP_t + GTR_t - GD_t, \tag{4–87}$$

$$VB_{t+1} = VB_t - GTR_t \tag{4–88}$$

$$VH_{t+1}^{i,j} = VH_t^{i,j}(1 - x_t^{i,j}) + \overline{VH}^j x_t^{i,j} - \gamma EG_t^i, \quad \forall i,j, \tag{4–89}$$

$$\sum_{i=1}^{n} \psi_t^i EG_t^i = H_c(CToP_t + GToP_t + \nabla G_t), \tag{4–90}$$

where VA_t and VB_t are the volume of gas in reservoirs A and B, respectively, and $VH_t^{i,j}$ is the "volume" of remaining hours of operation that the unit i has until the next maintenance of cycle j. $GToP_t$ is the amount of gas actually used to generate electricity, and GS_t is the amount of gas purchased or sold to the gas spot market. ARM_t is the amount of gas purchased from the gas distributor, and should be bounded below by $X\%M$. GTR_t is the amount of gas transfer from A to B, and GD_t is the amount of gas discarded when it is in the reservoir more than the maximum storage time, N. n and m are the total number of power units and total number of maintenance cycles, respectively. RI_t is the immediate revenue in stage t. π_t^s is the spot price in stage t of scenario s. $p_{t+1}(k,s)$ is the transition probability of the spot price of scenario k in stage t to the spot price of scenario s in stage $t+1$. $x_t^{i,j}$ is the binary decision variable associated with the schedule of maintenance of cycle j for unit i at stage t. $\overline{VH}^j$ is the maximum capacity of the reservoir of remaining hours of operation until the next maintenance of cycle j. EG_t^j is the energy generated by unit i at stage t. γ is an inverse coefficient of the power unit, and ψ_t^i is conversion factor from MMBTU to MWh of unit i at stage t, and H_c is the heat rate of the gas.

Constraints (4–87)-(4–89) are the fictitious reservoir balance constraints and (4–90) is the transformation from gas to electricity. Except constraints (4–87)-(4–90), there are also a lot of other constraints, such as gas consumption priority constraints, maximum and minimum gas consumption constraints, maintenance constraints, constraints related to the mechanism implemented for the modeling of the contracts and so on. For this problem, each stage is a mixed integer linear programming problem. And the whole problem is solved by using stochastic dual dynamic programming, first proposed by Pereira and Pinto [50].

Also the natural gas market can be modeled as a natural gas value chain. The primary component is natural gas in this chains. Various market models are proposed and utilized in reality at different stages along this value chain, e.g., production, transportation and processing, storage, import terminals and markets, wholesale and retail markets. Please refer to [42] for more details about market models within the natural gas value chain.

4.5 Conclusion

This chapter discuss various optimization models occurring in the natural gas industry; focusing on three aspects: production, transportation, and market. As we can see, the natural gas industry is a complex system and in great need of optimization techniques to improve performance. Especially the nonlinear and nonconvex nature of the problems makes it computationally challenging to find good solutions. We observe that linearization techniques are a common method to tackle these nonconvex functions, often reducing the problem to a (series) of linear or mixed integer liner programming problems. With the computational power of computers increasing over the last decade, the use of meta-heuristics is become more and more popular; especially for problems which cannot be handled with the current MINLP solvers either due to the size of the problem or due to the degeneracy.

The deregulation of the gas market introduced additional modeling aspects and computational challenges: various (additional) stochastic elements have been added to the 'classical' problems. This underlying structure of the problems cannot be ignored by any serious model and we expect that future research will focus on stochastic models and, especially, on new techniques how to solve these (large-scale) practical problems when also integer and nonconvex, nonlinear functions are present.

CHAPTER 5
NATURAL GAS NETWORK EXPANSION PLANNING

Due to the increasing demands for natural gas, it is playing a more important role in the energy system, and its system expansion planning is drawing more attentions. In this chapter, we propose expansion planning models which include both natural gas transmission network expansion and LNG (Liquified Natural Gas) terminals location planning. These models take into account the uncertainties of demands and supplies in the future, which make the models stochastic integer programs with discrete subproblems. Also we consider risk control in our models by including probabilistic constraints, such as a limit on CVaR (Conditional Value at Risk). In order to solve the large-scale problems, especially with a large number of scenarios, we propose the embedded Benders decomposition algorithm, which applies Benders cuts in both first and second stages, to tackle the discrete subproblems. Numerical results show that our algorithm is efficient for large scale stochastic natural gas transportation system expansion planning problems.

5.1 Introduction

Natural gas, which once was considered the byproduct or spare gas of oil and coal mining, has become a very precious and important energy source in the world's energy system. It is a relatively cleaner energy source compared to coal and oil because it releases less green house gas. Especially after the introduction of combined cycle power turbines, which is much more efficient than the traditional electricity power generators using coal, natural gas is playing a more important role in the world's energy supply. From 1986 to 2006, the annual world's consumption of natural gas has been doubled to 102.2 trillion cubic feet from 52.9 trillion cubic feet according to EIA (Energy Information Administration) 2009 annual report. Its annual demand is forecasted to increase by 50% in 2030. Because of the increasing demands for natural gas, it is very important to study the natural gas system expansion planning, especially its

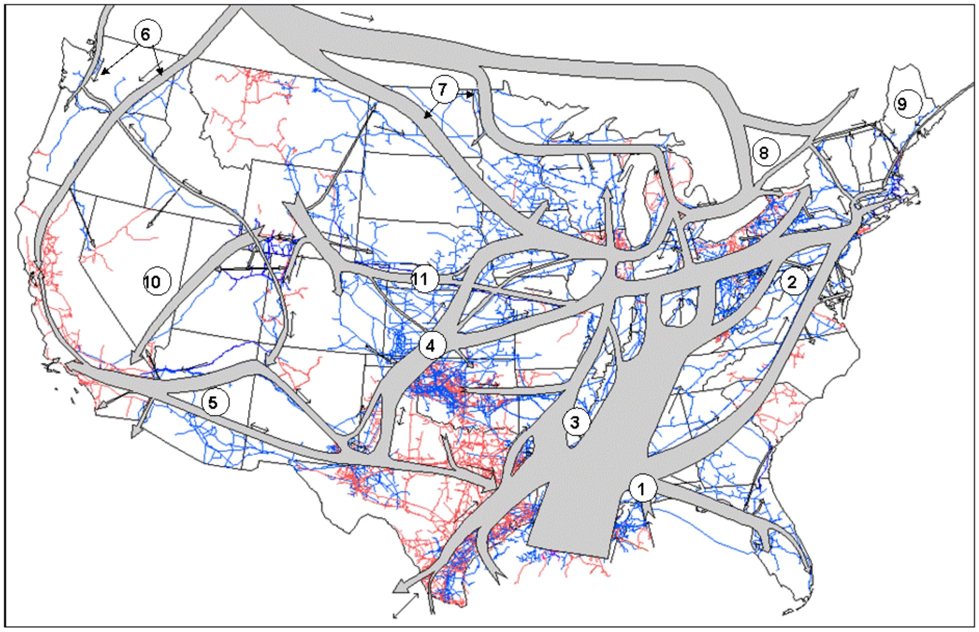

Figure 5-1. US natural gas transmission corridor from EIA 2008

transportation system. Traditionally, its transportation system is mainly composed of transmission pipelines. (The US transmission corridor of natural gas is shown in Figure 5-1 according to EIA 2008 report.) With the increasing intercontinental LNG (Liquified Natural Gas) shipment, the expansion planning on selecting the locations and sizes of the LNG terminals should also be considered an important part of the transportation system. The proposed and accomplished LNG terminals of US is shown in Figure 5-2 according to Federal Energy Regulation Commission.

As is discussed in [68], mathematical modeling and optimization have been extensively applied in natural gas industry and yielded a lot of great results. In this paper, we try to come up with a stochastic expansion planning model which considers both transmission pipeline network expansion planning and LNG terminal location planning. We model the expansion of a pipeline and the setup of a new size of LNG terminal by binary variables. The model is trying to minimize the total expansion cost and transmission cost while considering the whole transportation system. Also this model assumes generalized network flows as in [16]. In addition, we also propose a risk

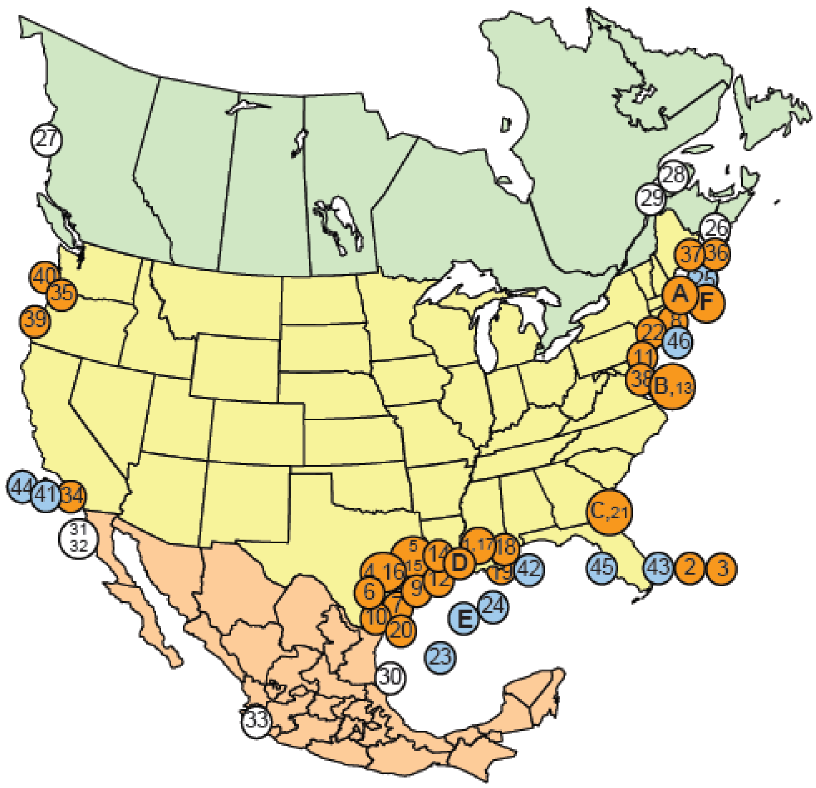

Figure 5-2. Existing and Proposed North American LNG Terminals

management model which includes CVaR (Conditional Value at Risk) risk constraints, in order to balance between the minimal cost and the risk of losing demands. Due to the existence of integer variables within both stages, the stochastic program is not a trivial problem to solve, especially when the number of scenarios is big.

The remaining part of this chapter is organized as follows. First, in section 5.2, we propose the stochastic model and the risk management model with CVaR constraints. In section 5.3 we explain our Embedded Benders Decomposition process and propose the algorithm. Section 5.4 shows the computational results and compares the solutions with different CVaR constraints.

5.2 Expansion Planning Models

We have already known that the world's natural gas supply will not last for ever because the reserves are dwindling and will not grow by themselves. So a very interesting and important question to ask is whether we need to expand our gas system.

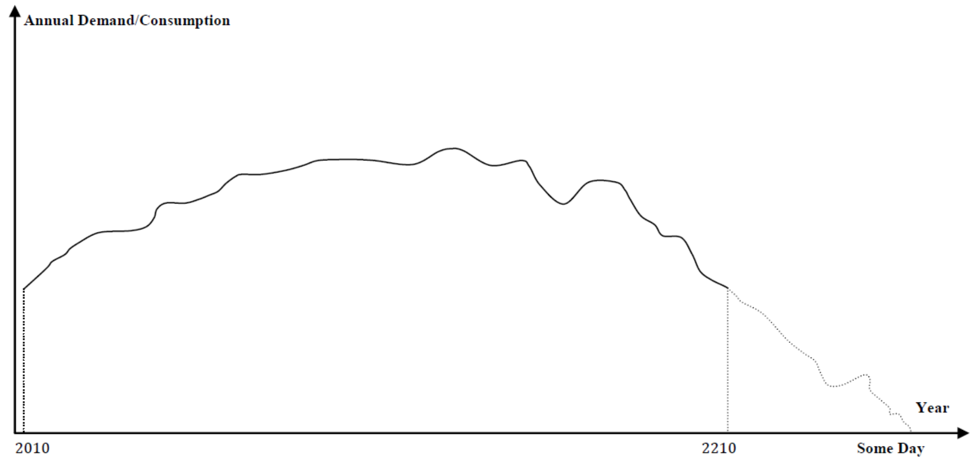

Figure 5-3. Natural gas long term consumption expectation

The up-shooting trend will not always hold in the future, and some time the demand will drop for sure. However, it seems the gas reserves can still support us for multiple decades or even a hundred years. (A natural gas long term consumption expecting is shown in Figure 5-3) Hence, it is important to do the expansion planning economically and reliably, which can cope with different future situations. The gas reserves are quite different in different regions of the world. According to EIA International Energy Outlook 2009, the world average RTP (Reserves-To-Production) ratio is about 63 years; Central and South America RTP is about 48 years; Russia and Africa RTP are 78 and 79 years respectively ; RTP of middle east is more than 100 years; US production rate is about 20 TCF (Trillion Cubic Feet) per year and its estimated reserves are about 1747.47 TCF, which make its RTP 87 years. The imbalance of natural gas reserves and economic growth in different regions make intercontinental transportation necessary. The main intercontinental transportation is LNG shipment. In the national level, it is important to analyze the whole natural gas system by considering pipeline networks and LNG locations together. A network example which considers both of them is shown in Figure 5-4, in which all transmission lines are expandable and those dashed lines denote possible new transmission lines, and nodes associated with “LNG” are possible LNG terminals.

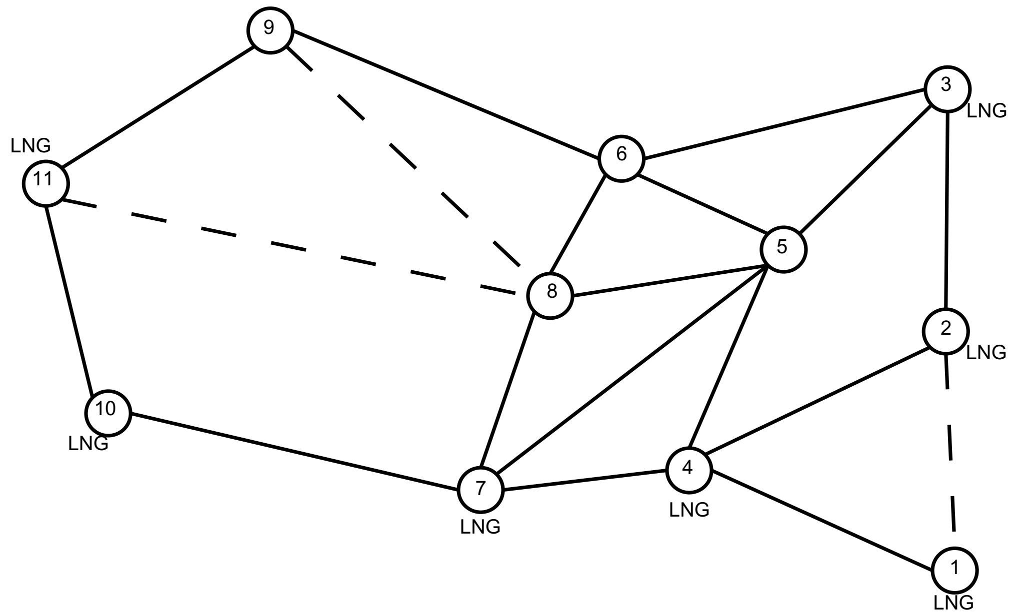

Figure 5-4. A natural gas transmission network example

5.2.1 The Stochastic Planning Model

Our modeling aims to formulate the problem in the system level while considering uncertainties of future supplies and demands. The objective is to minimize the total cost, which includes the costs of both arc expansions and LNG terminal expansions, and the transmission costs, while satisfying all demands. In this model, we assume discrete expansions, which are actually what is happening now. For example, the diameters of the gas pipelines and sizes of LNG containers are usually discrete when you try to buy them from the manufacturers. Figure 5-5 shows the discretized expansion costs of the gas pipeline. We use 0-1 integer variable, α_{ij}^k, to denote whether an expansion of size Δ_{ij}^k is made for arc (i,j), and then total cost of pipeline expansion is

$$Cost_{ARC} = \sum_{(i,j)\in A} \sum_{k\in K_{ij}} c_{ij}^k \alpha_{ij}^k$$

So does the LNG terminal opening cost as follows,

$$Cost_{LNG} = \sum_{i\in N_{LNG}} \sum_{k\in K_i} c_i^k \beta_i^k$$

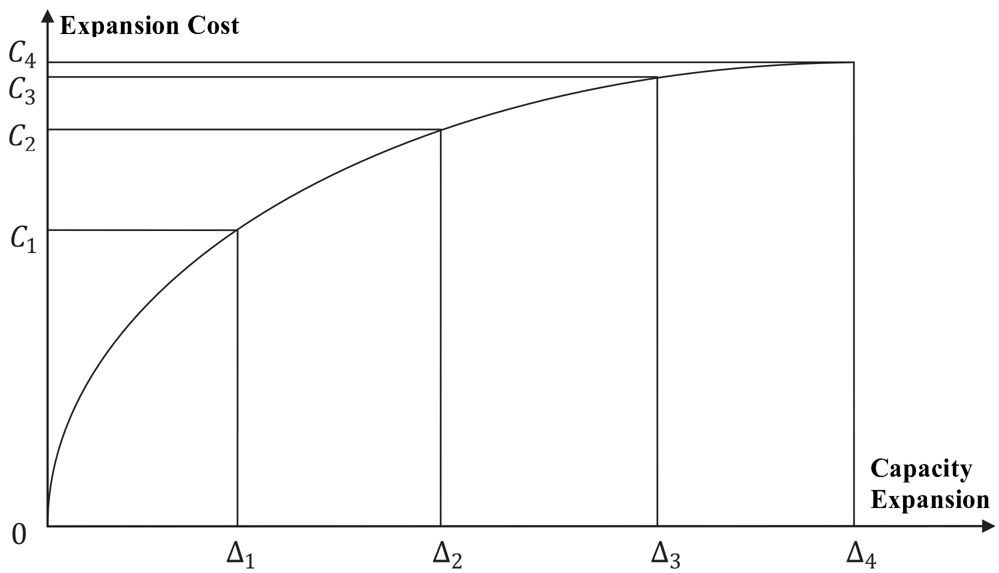

Figure 5-5. Discrete Expansion Costs

Table 5-1. EXPN Sets and Indices

N	The set of all nodes in the network
A	The set of all arcs in the network
A_i^+	The set of outgoing arcs from node i
A_i^-	The set of incoming arcs to node i
K_{ij}	The set of all possible expansion sizes on arc (i, j)
K_i	The set of all possible expansion sizes on arc i
N_{LNG}	Possible(approved) LNG terminals
Ξ	The set of all scenarios of the demand patterns
ξ, ζ	Denote a specific scenario

For the convenience, we make the following three assumptions, (A.1-A.3), which always hold throughout the whole paper.

A.1 Assume discrete distribution of uncertainties, $\Xi = \{\xi^1, \xi^2, \dots, \xi^r\}$, where r is a finite positive integer;

A.2 $(1 - l)^{|A|} \sum_{i \in N} SF^0 \geq \sum_{i \in N} d^0$, and $(1 - l)^{|A|} \sum_{i \in N} SF^1(\xi) \geq \sum_{i \in N} d^1(\xi),\ \forall \xi \in \Xi$, where $l = \max_{(i,j) \in A} l_{ij}$;

A.3 Making the maximum expansion on every node and arc is enough to satisfy all demands.

Table 5-2. Parameters

Δ_{ij}^{k}	The k^{th} expansion size of arc (i,j)
Δ_{i}^{k}	The k^{th} expansion size of LNG port i
$c_{ij}^{0,k}$, $c_{ij}^{1,k}$	Current and future costs of the expansion of size Δ_{ij}^{k} on arc (i,j)
$c_{i}^{0,k}$, $c_{i}^{1,k}$	Cost of the expansion of size Δ_{i}^{k} of LNG port i
h_{ij}^{0}, h_{ij}^{1}	Unit transportation cost of arc (i,j)
d_i^0	Current demand of the node i
$d_i^1(\xi)$	Future demand of node i under scenario ξ
l_{ij}	Transmission loss rate on arc (i,j)
SF_i^0	Current self supply limit of node i
SL_i^0	Current LNG supply limit of node i
$SF_i^1(\xi)$	Future self supply limit of node i under scenario ξ
$SL_i^1(\xi)$	Future LNG supply limit of node i under scenario ξ
$\underline{u}_{ij}$	The previous capacity of arc (i,j)
$\underline{v}_i$	The previous capacity of LNG port i
$Pr(\xi)$	Probability of scenario ξ

In order to facilitate the description of our models, Table 5-1 defines all the sets of arcs, nodes, scenarios, etc, and Table 5-2 defines all coefficients and parameters, while Table 5-3 defines all decision variables of both first and second stages.

Our stochastic planning model is to minimize the current cost plus the expected future cost which are shown in (5–1) and (5–2) respectively. Within each of them, there are three parts: arc (pipeline) expansion cost, LNG terminal expansion cost and transportation cost. (5–3) defines the flow balance constraints, where gas loss is considered by multiplying different factors on all the incoming flows, since in reality there are always leaking problems and compressor stations need to use some gas to maintain pressure of pipelines. We assume bidirectional flows on each arc and the capacity constraint of each arc is defined in (5–4). Since we include transportation cost in the objective function, for each arc the optimal solution will only have nonzero flow at most in one direction. (5–5) is the arc expansion constraints. Constraint (5–6) requires that LNG supply at any LNG port node cannot exceed its throughput capacity, while constraint (5–7) requires that LNG supply at any LNG port node also cannot exceed its LNG supply limit. Constraint (5–8) states that the total supply of every LNG

Table 5-3. Decision Variables

$\alpha_{ij}^{0,k}$	Binary variable to denote whether a Δ_{ij}^k expansion is made right now for arc (i,j)
$\alpha_{ij}^{1,k}(\xi)$	Binary variable to denote whether a Δ_{ij}^k expansion is made in the future for arc (i,j) under scenario ξ
$\beta_i^{0,k}$	Binary variable to denote whether a Δ_i^k expansion is made right now for LNG port i
$\beta_i^{1,k}(\xi)$	Binary variable to denote whether a Δ_i^k expansion is made in the future for LNG port i under scenario ξ
f_{ij}^0	Flow of arc (i,j) right now
$f_{ij}^1(\xi)$	Flow of arc (i,j) in the future under scenario ξ
s_i^0	Total Supply from node i right now
$s_i^1(\xi)$	Total Supply from node i in the future under scenario ξ
g_i^0	LNG supply from node i right now
$g_i^1(\xi)$	Future LNG supply from node i in the future
u_{ij}^0	The current Capacity of arc (i,j)
$u_{ij}^1(\xi)$	The future Capacity of arc (i,j) under scenario ξ
v_i^0	The current capacity of LNG port i
$v_i^1(\xi)$	The future capacity of LNG port i under scenario ξ

node should be less than its self supply limit plus its LNG supply. (5–9) is the LNG throughput capacity expansion constraints. At all the non-LNG nodes, the supply is bounded by its self supply limit, which is shown in (5–10). Constraints (5–11), (5–12) and (5–13) define the nonnegative continuous flow, capacity and supply variables, and binary expansion variables. Constraint (5–14)-(5–24) define the second stage feasible region, which almost replicates the first stage $|\Xi|$ times for all scenarios with different demands, supplies, and, most importantly, the decision variables. The whole extensive formulation of the stochastic planning problem is shown in the following mixed integer linear minimization program, [EXPN].

[**EXPN**]:

$$\text{Min} \quad \sum_{(i,j)\in A}\sum_{k\in K_{ij}} c_{ij}^{0,k}\alpha_{ij}^{0,k} + \sum_{i\in N_{LNG}}\sum_{k\in K_i} c_i^{0,k}\beta_i^{0,k} + \sum_{(i,j)\in A} h_{ij}^0 f_{ij}^0 \tag{5–1}$$

$$+\sum_{\xi\in\Xi} Pr(\xi)\left[\sum_{(i,j)\in A}\sum_{k\in K_{ij}} c_{ij}^{1,k}\alpha_{ij}^{1,k}(\xi) + \sum_{i\in N_{LNG}}\sum_{k\in K_i} c_i^{1,k}\beta_i^{1,k}(\xi) + \sum_{(i,j)\in A} h_{ij}^1 f_{ij}^1(\xi)\right] \tag{5–2}$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{(i,j)\in A_i^+} f_{ij}^0 - \sum_{(j,i)\in A_i^-} (1-l_{ji}) f_{ji}^0 = s_i^0 - d_i^0, \quad \forall i \in N, && (5\text{–}3)\\
& f_{ij}^0 + f_{ji}^0 \le u_{ij}^0, \quad \forall (i,j) \in A, && (5\text{–}4)\\
& u_{ij}^0 = \underline{u}_{ij} + \sum_{k\in K_{ij}} \Delta_{ij}^k \alpha_{ij}^{0,k}, \quad \forall (i,j) \in A, && (5\text{–}5)\\
& g_i^0 \le v_i^0, \quad \forall i \in N_{LNG}, && (5\text{–}6)\\
& g_i^0 \le SL_i^0, \quad \forall i \in N_{LNG}, && (5\text{–}7)\\
& s_i^0 \le g_i^0 + SF_i^0, \quad \forall i \in N_{LNG}, && (5\text{–}8)\\
& v_i^0 = \underline{v}_i + \sum_{k\in K_i} \Delta_i^k \beta_i^{0,k}, \quad \forall i \in N_{LNG}, && (5\text{–}9)\\
& s_i^0 \le SF_i^0, \quad \forall i \in N \setminus N_{LNG}, && (5\text{–}10)\\
& s_i^0, f_{ij}^0, u_{ij}^0, v_i^0, g_i^0(\xi) \ge 0, \forall (i,j) \in A, i \in N && (5\text{–}11)\\
& \alpha_{ij}^{0,k} \in \{0,1\}, \quad \forall k \in K_{ij}, (i,j) \in A, && (5\text{–}12)\\
& \beta_i^{0,k} \in \{0,1\}, \quad \forall k \in K_i, i \in N_{LNG}, && (5\text{–}13)\\
& (\textit{constraints to be continued})
\end{aligned}
$$

[EXPN] is a two stage mixed integer program, where binary variables are present in both stages. Because the second stage also includes binary variables, we can not generate Benders cuts directly as in the L-Shaped method. One method of getting valid Benders cuts from the second stage for the first stage is the Embedded Benders Decomposition which relaxes the second stage to a linear program and also use Benders cuts to approximate the convex hull of the second stage program. To enhance the convergence, we also add integer L-shaped optimality cuts [38] when necessary. The solution to the simple example is shown in Figure 5-6, which indicates only LNG terminal expansions are needed at the current point, with size 2 LNG terminal expansions at node 1 and 10, and size 3 LNG terminal expansions at node 2, 3 and 11.

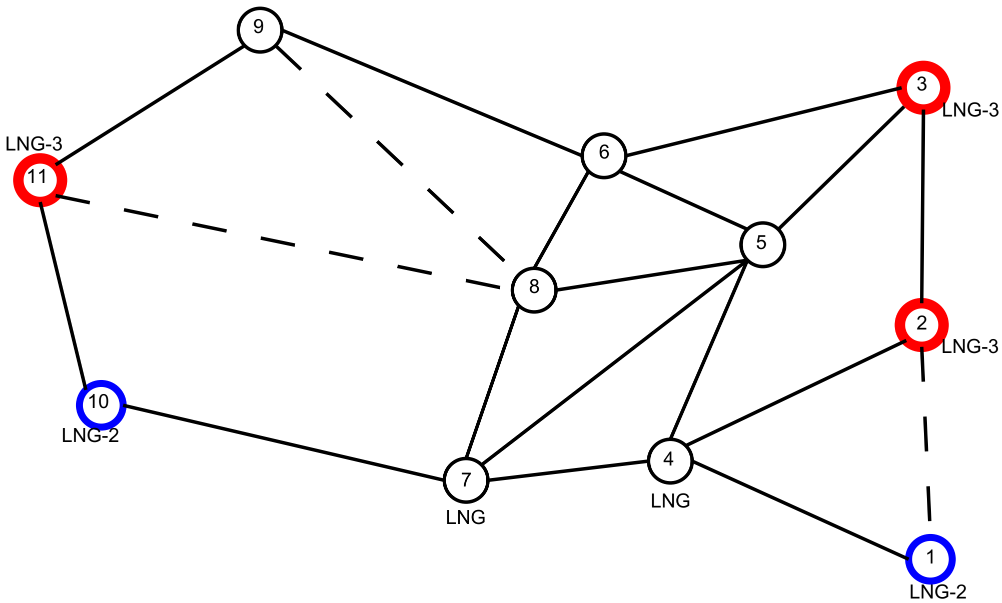

Figure 5-6. Solution of a Simple EXPN Example

The constraints of second stage are as follows,

$$(constraints\ continued)$$

$$\sum_{(i,j)\in A_i^+} f_{ij}^1(\xi) - \sum_{(j,i)\in A_i^-} (1-l_{ji}) f_{ji}^1(\xi) = s_i^1(\xi) - d_i^1(\xi), \quad \forall i \in N, \xi \in \Xi, \tag{5–14}$$

$$f_{ij}^1(\xi) + f_{ji}^1(\xi) \le u_{ij}^1(\xi), \quad \forall (i,j) \in A, \xi \in \Xi, \tag{5–15}$$

$$u_{ij}^1(\xi) = u_{ij}^0 + \sum_{k\in K_{ij}} \Delta_{i,j}^k \alpha_{ij}^{1,k}(\xi), \quad \forall (i,j) \in A, \xi \in \Xi, \tag{5–16}$$

$$g_i^1(\xi) \le v_i^1(\xi), \quad \forall i \in N_{LNG}, \xi \in \Xi, \tag{5–17}$$

$$g_i^1(\xi) \le SL_i^1(\xi), \quad \forall i \in N_{LNG}, \xi \in \Xi, \tag{5–18}$$

$$s_i^1(\xi) \le g_i^1(\xi) + SF_i^1(\xi), \quad \forall i \in N_{LNG}, \xi \in \Xi, \tag{5–19}$$

$$v_i^1(\xi) = v_i^0 + \sum_{k\in K_i} \Delta_i^k \beta_i^{1,k}(\xi), \quad \forall i \in N_{LNG}, \xi \in \Xi, \tag{5–20}$$

$$s_i^1(\xi) \le SF_i^1(\xi), \quad \forall i \in N \setminus N_{LNG}, \xi \in \Xi, \tag{5–21}$$

$$s_i^1(\xi), f_{ij}^1(\xi), u_{ij}^1(\xi), v_i^1(\xi), g_i^1(\xi) \ge 0, \forall (i,j) \in A, i \in N, \xi \in \Xi, \tag{5–22}$$

$$\alpha_{ij}^{1,k}(\xi) \in \{0,1\}, \quad \forall k \in K_{ij}, (i,j) \in A, \xi \in \Xi, \tag{5–23}$$

$$\beta_i^{1,k}(\xi) \in \{0,1\}, \quad \forall k \in K_i, i \in N_{LNG}, \xi \in \Xi, \tag{5–24}$$

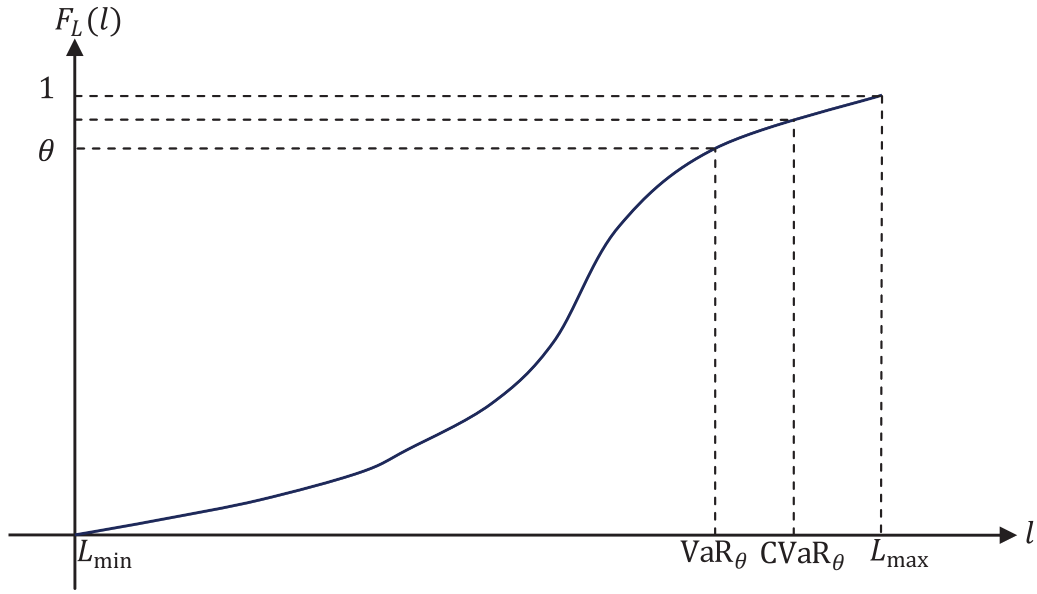

Figure 5-7. Value at Risk v.s. Conditional Value at Risk

5.2.2 The Planning Model with Risk Constraints

The solution to [EXPN] might have to sacrifice a lot to satisfy an extreme scenario which has big demands, which means we may pay for extremely "bad" things which are very unlikely to happen. In order to find such kind of scenarios and tolerate risks to a certain extent, we may need to include a measure which can help us locate those scenarios and tell how "bad" they are and how "unlikely" they are. A risk management model with chance constraints would take care of this situation while controlling the risk in an acceptable manner.

Value at Risk (VaR) and Conditional Value at Risk (CVaR) are two generally used risk measures in the literature due to their structural and computational easiness compared to variance. As is stated in [36], VaR has been widely used in financial areas and is also the standard risk measure of Bank for International Settlements. Mathematically speaking, VaR is the minimum value, such that the probability of random loss is greater than or equal to this value is less than a small predefined a percentage (the risky area), which is defined as follows,

$$\text{VaR}_\theta = inf \left\{ l \in \mathbb{R} : P(L(X,y) \geq l) \leq 1 - \theta \right\},$$

where $L(X,y)$ is the loss function of random variable X and decision variable y, θ is the confidence level. However, modeling VaR constraints needs to use integer variables, which makes some linear programming models difficult to solve. Instead CVaR constraints does not need to introduce integer variables and only involves linear constraints. In general, CVaR is the expected loss given the fact that the loss is greater than or equal to VaR, which is shown as follows,

$$\text{CVaR}_\theta = E \left\{ L(X,y) | L(X,y) \geq \text{VaR}_\theta \right\}.$$

As discussed in [3], CVaR constraints are tighter than VaR constraints since the risk constraints are generally of the format as $\text{VaR}_\theta \leq \bar{\phi}$ or $\text{CVaR}_\theta \leq \bar{\phi}$, and CVaR_θ is a bigger value than VaR_θ, as shown in Figure 5-7, in which $F_L(l)$ is the cumulative distribution function of random loss L, and $L_{\min}$ and $L_{\max}$ are respectively the minimum and maximum values that L can take.

In our risk management model, we would like to use CVaR as the risk measure because it not only provides a tighter bound but also consists of only linear constraints and continuous variables. The risk in this expansion problem is the shortage of gas supply to the customers. So we introduce a new variable, $\lambda_i(\xi)$, to denote the shortage in the future at node i under scenario ξ. So the original flow balance constraint (5–14) in the second stage (stochastic part) is changed to constraint (5–25), which includes the shortage variable $\lambda_i(\xi)$. Also, in order to differentiate nodes by priorities, we only allow a certain amount of shortage, $\bar{\lambda}_i$, for the nodes in N_R, a subset of all nodes. This is realized by constraints (5–26) and (5–27).

The expansion planning model with CVaR constraints, which shares the same objective function (5–1) + (5–2), same first stage constraints (5–3)-(5–13) and (5–15)-(5–24) with [EXPN], is formulated as follows,

[**EXPN-R**]:

$$
\begin{aligned}
\text{Min} \quad & (5\text{–}1) + (5\text{–}2) \\
\text{s.t.} \quad & (5\text{–}3) - (5\text{–}13), \\
& \sum_{(i,j)\in A_i^+} f_{ij}^1(\xi) - \sum_{(j,i)\in A_i^-} (1 - l_{ji}) f_{ji}^1(\xi) \\
& \qquad\qquad = s_i^1(\xi) - d_i^1(\xi) + \lambda_i(\xi), \quad \forall i \in N, \xi \in \Xi, && (5\text{–}25) \\
& (5\text{–}15) - (5\text{–}24), \\
& \lambda_i(\xi) = 0, \quad \forall i \in N \setminus N_R, \xi \in \Xi, && (5\text{–}26) \\
& \bar{\lambda}_i \geq \lambda_i(\xi) \geq 0, \quad \forall i \in N_R, \xi \in \Xi, && (5\text{–}27) \\
& \sum_{i\in N} \lambda_i(\xi) \leq \eta + w(\xi), \quad \forall \xi \in \Xi, && (5\text{–}28) \\
& \eta + \sum_{\xi\in\Xi} \frac{Pr(\xi)}{1-\theta} w(\xi) \leq \bar{\phi}, && (5\text{–}29) \\
& w(\xi) \geq 0, \quad \forall \xi \in \Xi, && (5\text{–}30)
\end{aligned}
$$

where constraints (5–28)-(5–30) are the risk constraints which are equivalent to $\text{CVaR}_\theta \leq \bar{\phi}$. Constraint (5–30) introduces the nonnegative continuous variable, $w(\xi)$, to denote the amount of loss greater VaR_θ for scenario ξ. Constraint (5–28) defines the bounds for the total losses of all scenarios, where the solution of η, after solving the problem, is actually VaR_θ for most of the time as discussed in [54]. Constraint (5–29) then finally defines the bound on CVaR_θ.

5.3 Embedded Benders Decomposition

The two problems proposed in Section 5.2 are both stochastic mixed integer linear programs, which also include integer variables in the second stage. When the number of scenarios are big, these problems will include a huge number of integer variables, which make the problem difficult to solve as a whole. Hence it is necessary to decompose the problems and solve the second stage problems separately. L-shaped method [63] and Benders Decomposition [9] are widely used to solve the two stage stochastic linear programs as described in [10]. However, they are not directly applicable to our problems because the second stage includes integer variables and solving mixed integer programs does not generally produce useful dual solutions.

In this chapter, we propose a different kind of valid global cuts to approximate the convex hull of the second stage discrete problem. In constraints (5–14)-(5–24), there are both linear and integer parts. Hence it is possible to further decompose the subproblem itself by Benders decomposition, and the Benders cuts within the subproblem can help cut off the combination of the first stage and second stage integer solution. Our approach tries to also use Benders cuts to convexify the subproblem. So we call our algorithm Embedded Benders Decomposition in which Benders cuts are generated for both master and subproblems. In this section, we will discuss how to implement our decomposition scheme to solve the stochastic expansion planning problem. The restricted master problem, [RMP], is as follows,

[**RMP**]:

$$\begin{aligned}
\text{Min} \quad & \sum_{(i,j)\in A}\sum_{k\in K_{ij}} c_{ij}^{0,k}\alpha_{ij}^{0,k} + \sum_{i\in N_{LNG}}\sum_{k\in K_i} c_i^{0,k}\beta_i^{0,k} + \sum_{(i,j)\in A} h_{ij}^0 f_{ij}^0 + \sum_{\xi\in\Xi} Pr(\xi)z(\xi) && \text{(5–31)}\\
\text{s.t.} \quad & (5\text{–}3) - (5\text{–}12),\\
& z(\xi) \geq \sum_{(i,j)\in A} \hat{x}_{ij}^t(\xi)u_{ij}^0 + \sum_{i\in N_{LNG}} \hat{y}_i^t(\xi)v_i^0 + r^t(\xi), \quad \forall t\in\mathcal{T}(\xi), \xi\in\Xi, && \text{(5–32)}
\end{aligned}$$

where t denotes the t^{th} cut of scenario ξ, and $\hat{x}_{ij}^t(\xi)$, $\hat{y}_i^t(\xi)$ and $r^t(\xi)$ are respectively the optimal dual multipliers and sum product of multipliers and right hand sides of the relaxed subproblem of scenario ξ, which will be shown later. This program is always feasible because of (A.2) and (A.3). Also, in the above [RMP] formulation, we show how the disaggregated cuts are added, and we also will talk about the aggregated cuts and compare these two kinds of cuts adding schemes after we finish the discussion of how the valid Benders cuts for the first stage are generated. (Note [RMP] itself can be further decomposed, since the network constraints are not related to the expansion decisions directly.)

After solve the [RMP] and obtain its solution, we can solve the second stage problem or the subproblem by fixing the first stage expansion decisions, $\hat{u}^0$ and $\hat{v}^0$. Because we also have expansion decision variables in the second stage, the subproblems are always feasible due to the assumptions, (A.2) and (A.3). This allows us to solve the subproblems of different scenarios separately because there is no coupling between scenarios except the first stage decisions. The subproblem corresponding to scenario ξ is shown as follows,

[**SP(**ξ**)**]:

$$
\begin{aligned}
\text{Min} \quad & \sum_{(i,j)\in A}\sum_{k\in K_{ij}} c_{ij}^{1,k}\alpha_{ij}^{1,k}(\xi) + \sum_{i\in N_{LNG}}\sum_{k\in K_i} c_i^{1,k}\beta_i^{1,k}(\xi) + h_{ij}^1 f_{ij}^1(\xi) \\
\text{s.t.} \quad & -\sum_{(i,j)\in A_i^+} f_{ij}^1(\xi) + \sum_{(j,i)\in A_i^-} (1-l_{ji}) f_{ji}^1(\xi) + s_i^1(\xi) = d_i^1(\xi), \quad \forall i \in N, \\
& -f_{ij}^1(\xi) - f_{ji}^1(\xi) + u_{ij}^1(\xi) \geq 0, \quad \forall (i,j) \in A, \\
& u_{ij}^1(\xi) - \sum_{k\in K_{ij}} \Delta_{ij}^k \alpha_{ij}^{1,k}(\xi) = \hat{u}_{ij}^0, \quad \forall (i,j) \in A, \\
& g_i^1(\xi) - v_i^1(\xi) \leq 0, \quad \forall i \in N_{LNG}, \\
& g_i^1(\xi) \leq SL_i^1(\xi), \quad \forall i \in N_{LNG}, \\
& s_i^1(\xi) - g_i^1(\xi) \leq SF_i^1(\xi), \quad \forall i \in N_{LNG}, \\
& v_i^1(\xi) - \sum_{k\in K_i} \Delta_i^k \beta_i^{1,k}(\xi) = \hat{v}_i^0, \quad \forall i \in N_{LNG}, \\
& s_i^1(\xi) \leq SF_i^1(\xi), \quad \forall i \in N \setminus N_{LNG}, \\
& s_i^1(\xi), f_{ij}^1(\xi), u_{ij}^1(\xi), v_i^1(\xi), g_i^1(\xi) \geq 0, \forall (i,j) \in A, i \in N, \\
& \alpha_{ij}^{1,k}(\xi) \in \{0,1\}, \quad \forall k \in K_{ij}, (i,j) \in A, \\
& \beta_i^{1,k}(\xi) \in \{0,1\}, \quad \forall k \in K_i, i \in N_{LNG}.
\end{aligned}
$$

This is a mixed integer linear program and will not directly provide useful Benders cuts for [RMP] in general. However, compared to the whole problem, this program is much easier to solve since it only involves one scenario and has much less both integer and continuous variables. After solve [SP(ξ)] and fix the integer variable at its optima, $\hat{\alpha}^1(\xi)$ and $\hat{\beta}^1(\xi)$, then we can solve the following [LP (ξ)] to generate embedded Benders cuts, the global cuts, for the subproblem of scenario ξ. Again, the program [LP (ξ)] is always feasible since [SP(ξ)] is always feasible and the binary variables are fixed as the optimal (feasible) solution of [SP(ξ)].

[**LP(ξ)**]:

$$\text{Min} \quad \sum_{(i,j)\in A} h_{ij}^1 f_{ij}^1(\xi) \tag{5–33}$$

$$\text{s.t.} \quad -\sum_{(i,j)\in A_i^+} f_{ij}^1(\xi) + \sum_{(j,i)\in A_i^-} (1-l_{ji}) f_{ji}^1(\xi) + s_i^1(\xi) = d_i^1(\xi), \quad \forall i \in N, \tag{5–34}$$

$$-f_{ij}^1(\xi) - f_{ji}^1(\xi) + u_{ij}^1(\xi) \geq 0, \quad \forall (i,j) \in A, \tag{5–35}$$

$$u_{ij}^1(\xi) = \sum_{k\in K_{ij}} \Delta_{ij}^k \hat{\alpha}_{ij}^{1,k}(\xi) + \hat{u}_{ij}^0, \quad \forall (i,j) \in A, \tag{5–36}$$

$$g_i^1(\xi) - v_i^1(\xi) \leq 0, \quad \forall i \in N_{LNG}, \tag{5–37}$$

$$g_i^1(\xi) \leq SL_i^1(\xi), \quad \forall i \in N_{LNG}, \tag{5–38}$$

$$s_i^1(\xi) - g_i^1(\xi) \leq SF_i^1(\xi), \quad \forall i \in N_{LNG}, \tag{5–39}$$

$$v_i^1(\xi) = \sum_{k\in K_i} \Delta_i^k \hat{\beta}_i^{1,k}(\xi) + \hat{v}_i^0, \quad \forall i \in N_{LNG}, \tag{5–40}$$

$$s_i^1(\xi) \leq SF_i^1(\xi), \quad \forall i \in N \setminus N_{LNG}, \tag{5–41}$$

$$s_i^1(\xi), f_{ij}^1(\xi), u_{ij}^1(\xi), v_i^1(\xi), g_i^1(\xi) \geq 0, \quad \forall (i,j) \in A, i \in N, \tag{5–42}$$

Now we have a pure linear program without expansion decisions, and then solving it can help generate the following global cut which is valid given any first stage expansion decision status, $\hat{u}^0$ and $\hat{v}^0$.

$$\pi \geq \sum_{(i,j)\in A} \hat{p}_{ij}^u \left[\left(\sum_{k\in K_{ij}} \Delta_{ij}^k \alpha_{ij}^{1,k}(\xi) \right) + \hat{u}_{ij}^0 \right]$$

$$+ \sum_{i \in N_{LNG}} \hat{p}_i^v \left[\left(\sum_{k \in K_i} \Delta_i^k \beta_i^{1,k}(\xi) \right) + \hat{v}_i^0 \right]$$

$$+ \sum_{i \in N} \left(\hat{q}_i^d d_i^1(\xi) + \hat{q}_i^{SF} SF_i^1(\xi) \right) + \sum_{i \in N_{LNG}} \hat{q}_i^{SL} SL_i^1(\xi) \quad (5\text{–}43)$$

where $\hat{p}_{ij}^u$, $\hat{p}_{ij}^v$, $\hat{q}_i^d$, $\hat{q}_i^{SL}$, $\hat{q}_i^{SF}$ are the optimal dual multipliers corresponding to (5–36), (5–40), (5–37), (5–38), (5–39) and (5–41) respectively. For convenience, this cut can be rewritten in vector format as follows,

$$\pi \geq a_l^T \alpha^1(\xi) + b_l^T \beta^1(\xi) + (P_l^u)^T \hat{u}^0 + (P_l^v)^T \hat{v}^0$$

$$+ (Q_l^d)^T d^1(\xi) + (Q_l^{SL})^T SL^1(\xi) + (Q_l^{SF})^T SF^1(\xi) \quad (5\text{–}44)$$

where l denote the l^{th} cut. Then we can include these global cuts to construct a relaxed version of the subproblems as follows,

[RSP(ξ)]:

$$\text{Min} \quad \sum_{(i,j) \in A} \sum_{k \in K_{ij}} c_{ij}^{1,k} \alpha_{ij}^{1,k}(\xi) + \sum_{i \in N_{LNG}} \sum_{k \in K_i} c_i^{1,k} \beta_i^{1,k}(\xi) + \pi \quad (5\text{–}45)$$

$$\text{s.t.} \quad \pi \geq a_l^T \alpha^1(\xi) + b_l^T \beta^1(\xi) + (P_l^u)^T \hat{u}^0 + (P_l^v)^T \hat{v}^0$$

$$+ (Q_l^d)^T d^1(\xi) + (Q_l^{SL})^T SL^1(\xi) + (Q_l^{SF})^T SF^1(\xi), \quad \forall l \in \mathcal{L}(\xi), \quad (5\text{–}46)$$

$$0 \leq \alpha_{ij}^{1,k}(\xi) \leq 1, \quad \forall k \in K_{ij}, (i,j) \in A, \quad (5\text{–}47)$$

$$0 \leq \beta_i^{1,k}(\xi) \leq 1, \quad \forall k \in K_i, i \in N_{LNG}, \quad (5\text{–}48)$$

where the second stage expansion decision variables are relaxed to be continuous while being bounded within $[0, 1]$.

Proposition 5.1. *(5–43) is valid Benders cuts for [RSP(ξ)] given any first stage solution $\hat{u}^0$ and $\hat{v}^0$.*

Proof. In [LP(ξ)], first stage solutions $\hat{u}^0$ and $\hat{v}^0$ only exist on the the right hand sides of constraints (5–36) and (5–40). [LP(ξ)]s with different first stage solutions share the same dual space even they have different primal feasible regions. This is because the objective coefficients and left hand side coefficients are the same for different problems.

The cut, (5–43), is constructed by using the dual solution to [LP(ξ)], and then it is valid for any given first stage decision. $\square$

This means that we still can use the previously generated embedded Benders cuts, (5–43), for the current first stage decision. So we could sequentially add the embedded Benders cuts to [RSP(ξ)]. But we need to change the right hand sides of the embedded Benders cuts by plugging in the current first stage solution. Subproblems of all scenarios have the same left hand sides because the uncertainties in the expansion planning model lie in the demands and supplies, which are all on the right hand sides of the constraints. So all [LP(ξ)]s share the same dual space, and then the dual optimal solution to any [LP(ξ)], $(a, b, P^u, P^v, Q^d, Q^{SL}, Q^{SF})$, can be used to construct embedded Benders cuts for [LP(ζ)], $\forall \zeta \in \Xi$, as follows,

$$\begin{aligned} \pi \;\geq\;& a^T\alpha^1(\zeta) + b^T\beta^1(\zeta) + (P^u)^T\hat{u}^0 + (P^v)^T\hat{v}^0 \\ & +(Q^d)^T d^1(\zeta) + (Q^{SL})^T SL^1(\zeta) + (Q^{SF})^T SF^1(\zeta). \end{aligned}$$

Hence we do not need to maintain a individual set of dual optimal solutions, $\mathcal{L}(\xi)$, for every scenario, but only need to maintain one set $\mathcal{L}$ for all scenarios, because the dual solutions can be used for all scenarios. Then we can solve the [RSP(ξ)] to derive Benders cuts for the first stage, which is shown as follows,

$$\begin{aligned} z(\xi) \;\geq\;& \sum_{l\in\mathcal{L}} \hat{\gamma}_l^t(\xi)\left[(P_l^u)^T u^0 + (P_l^v)^T v^0 + (Q_l^d)^T d^1(\xi) + (Q_l^{SL})^T SL^1(\xi) + (Q_l^{SF})^T SF^1(\xi)\right] \\ & + \sum_{(i,j)\in A}\sum_{k\in K_{ij}} \hat{\rho}_{ij}^{k,t}(\xi) + \sum_{i\in N_{LNG}}\sum_{k\in K_i} \hat{\sigma}_i^{k,t}(\xi) \qquad (5\text{–}49) \end{aligned}$$

where $\hat{\gamma}_l^t(\xi)$ is the optimal dual multiplier corresponding to the l^{th} embedded Benders cut in $\mathcal{L}$, and t denotes the t^{th} Benders cut for [RMP]. $\hat{\rho}_{ij}^t(\xi)$ and $\hat{\sigma}_i^t(\xi)$ are optimal dual multipliers corresponding to (5–47) and (5–48) respectively.

Proposition 5.2. *(5–49) is a valid Benders cut for [RMP].*

Proof. Suppose that $Z_{SP(\xi)}$ and $Z_{RSP(\xi)}$ are the optimal objective values of [SP(ξ)] and [RSP(ξ)] respectively. Given any first stage solution, $Z_{SP(\xi)} \geq Z_{RSP(\xi)}$, because (5–43) is a global cut for [RSP(ξ)] and the feasible region of [RSP(ξ)] contains [SP(ξ)]'s. Also, we have

$$
\begin{aligned}
Z_{RSP(\xi)} \quad \geq \quad & \sum_{l\in\mathcal{L}} \hat{\gamma}_l^t(\xi) \left[(P_l^u)^T u^0 + (P_l^v)^T v^0 + (Q_l^d)^T d^1(\xi) + (Q_l^{SL})^T SL^1(\xi) + (Q_l^{SF})^T SF^1(\xi)\right] \\
& + \sum_{(i,j)\in A} \sum_{k\in K_{ij}} \hat{\rho}_{ij}^{k,t}(\xi) + \sum_{i\in N_{LNG}} \sum_{k\in K_i} \hat{\sigma}_i^{k,t}(\xi).
\end{aligned}
$$

Hence, (5–49) is a valid Benders cut for [RMP]. □

If we move all items involving variables to the left hand side for constraint (5–46), we will get

$$
\begin{aligned}
-a_l^T \alpha^1(\xi) - b_l^T \beta^1(\xi) + \pi \quad \geq \quad & (P_l^u)^T \hat{u}^0 + (P_l^v)^T \hat{v}^0 \\
& + (Q_l^d)^T d^1(\xi) + (Q_l^{SL})^T SL^1(\xi) + (Q_l^{SF})^T SF^1(\xi), \quad \forall l \in \mathcal{L}
\end{aligned}
$$

where $\mathcal{L}(\xi)$ is replaced by $\mathcal{L}$, since any embedded Benders cut is valid for all restricted subproblems. It is interesting to note that the coefficients of $\alpha^1(\xi)$ and $\beta^1(\xi)$ are independent of scenarios. So all [RSP(ξ)]s have the same left hand side coefficients and objective function, which means they share the same dual space (dual feasible region). Hence solving one [RSP(ξ)] means that we obtain multiple cuts for all scenarios, and then we have following proposition.

Proposition 5.3. *For all $\zeta \in \Xi$,*

$$
\begin{aligned}
z(\zeta) \geq \quad & \sum_{l\in\mathcal{L}} \hat{\gamma}_l^t(\xi) \left[(P_l^u)^T u^0 + (P_l^v)^T v^0 + (Q_l^d)^T d^1(\zeta) + (Q_l^{SL})^T SL^1(\zeta) + (Q_l^{SF})^T SF^1(\zeta)\right] \\
+ \quad & \sum_{(i,j)\in A} \hat{\sum}_{k\in K_{ij}} \rho_{ij}^{k,t}(\xi) + \sum_{i\in N_{LNG}} \sum_{k\in K_i} \hat{\sigma}_i^{k,t}(\xi)
\end{aligned}
\tag{5–50}
$$

is a valid Benders cut for [RMP], where $\left(\hat{\gamma}_l^t(\xi), \hat{\rho}_{ij}^t(\xi), \hat{\sigma}_i^t(\xi)\right)$ are the dual optimal solution to [RSP(ξ)].

Because of Proposition 5.3, in constraint (5–32) we only need to maintain a single set of dual solutions, $\mathcal{T}$, instead of multiple sets for each individual scenario, $\mathcal{T}(\xi)$. In the [RMP], however, we still need $|\Xi|$ recourse variables, $z(\xi)$s, and $|\mathcal{T}| \times |\Xi|$ Benders cuts, which we refer to as disaggregated cuts. Actually each dual solution in $\mathcal{T}$ is corresponding to multiple disaggregated cuts in (5–32), which can be aggregated to one cut, (5–51). The aggregated cut is obtained by adding the weighted disaggregated cuts together, where the weight to a cut is its corresponding probability. It is shown as follows,

$$\begin{aligned} z \geq \quad & \sum_{l\in\mathcal{L}} \hat{\gamma}_l^t(\xi) \left[(P_l^u)^T u^0 + (P_l^v)^T v^0 + (Q_l^d)^T \bar{d}^1 + (Q_l^{SL})^T \bar{SL}^1 + (Q_l^{SF})^T \bar{SF}^1 \right] \\ & + \sum_{(i,j)\in A} \sum_{k\in K_{ij}} \hat{\rho}_{ij}^{k,t}(\xi) + \sum_{i\in N_{LNG}} \sum_{k\in K_i} \hat{\sigma}_i^{k,t}(\xi), \end{aligned} \tag{5–51}$$

where $\sum_{\zeta\in\Xi} Pr(\zeta)z(\zeta)$ is replaced by z, and $\bar{d}^1$, $\bar{SL}^1$ and $\bar{SF}^1$ are the expected demand and supply vectors, which are equal to $\sum_{\zeta\in\Xi} Pr(\zeta)d^1(\zeta)$, $\sum_{\zeta\in\Xi} Pr(\zeta)SL^1(\zeta)$ and $\sum_{\zeta\in\Xi} Pr(\zeta)SF^1(\zeta)$ respectively. Aggregation can reduce the numbers of recourse variables and Benders cuts greatly if there are a huge amount of scenarios. If the aggregated cuts (5–51) are used in [RMP], it only needs one recourse variable, z, and its objective function needs to be modified accordingly as follows,

$$\sum_{(i,j)\in A} \sum_{k\in K_{ij}} c_{ij}^{0,k} \alpha_{ij}^{0,k} + \sum_{i\in N_{LNG}} \sum_{k\in K_i} c_i^{0,k} \beta_i^{0,k} + \sum_{(i,j)\in A} h_{ij}^0 f_{ij}^0 + z. \tag{5–52}$$

According to the number of scenarios, we could choose different strategies to add valid Benders cuts in [RMP]. As is discussed in [10], the disaggregated scheme is chosen in the case of a small number of scenarios, and vice versa. For the model [EXPN], we prefer the aggregated scheme because the aggregated cuts need less variables, contain more information from all scenarios, and are very easy to generate due to the sharing of same dual space among [RSP(ξ)]s. For the [EXPN-R] model, we can do the same decomposition by separating the risk constraints, and changing the flow balance constraint as (5–25). Because constraint (5–29) bundles all the variables together, it is included the [RMP]. So do the variables, $w(\xi)$s, related to this constraint. The remaining risk constraints, (5–26, 5–27, 5–28), are added to [SP(ξ)]s. Then the same decomposition strategy applies.

As is described in section 5.3, Benders cuts are used in both the master problem and the subproblems, and Benders cuts in the subproblems help to construct the Benders cuts for the master problem, which is the reason that we name this algorithm Embedded Benders Decomposition. In practice, Benders decomposition could converge very slowly. Then, in order to speed up convergence, in the master problem we add another type of cuts, integer L-shaped "optimality" cuts, which is proposed by [38]. An integer L-shaped "optimality" cut is as follows,

$$z \geq (Q(\hat{x}) - L)\left(\sum_{j \in T} x_j - \sum_{j \in F} x_j - |T| + 1\right) + L$$

where $Q(x)$ is the recourse function and L is a lower bound for the second stage problem, and $T = \{j|\hat{x}_j = 1\}$ and $F = \{j|\hat{x}_j = 0\}$, if x is the first stage decision variable and $\hat{x}$ is the current solution. This follows from the fact that the right hand side will be equal to $Q(\hat{x})$ if $x = \hat{x}$, and less than L otherwise since $\left(\sum_{j \in T} x_j - \sum_{j \in F} x_j - |T| + 1\right) \leq 0$ if $x \neq \hat{x}$. We refer interested readers to [38] for the detailed proof. Without having to define the two sets, T and F, after rearranging terms the cut can be expressed by the following equivalent inequality,

$$(Q(\hat{x}) - L)\sum_j (1 - 2\hat{x}_j)x_j + z \geq Q(\hat{x}) - (Q(\hat{x}) - L)\sum_j \hat{x}_j.$$

Because the second stage problems, SP(ξ)s, are actually affected only by the binary solutions, $\hat{\alpha}^0$ and $\hat{\beta}^0$ because $\hat{u}^0$ and $\hat{v}^0$ are determined if these binary variables are chosen. This means we can use the integer L-shaped "optimality" cuts for solving our models because first stage continuous decisions (flows) does not affect the second

Step 0.	Set $\mathrm{UB} = \infty$, $\mathrm{LB} = 0$, $\mathcal{L} = \emptyset$, $\mathcal{T} = \emptyset$, $\hat{z} = 0$; Solve [RMP] without taking into account recourse variable, z; Get the optimal objective, Z_{RMP}, and solution $(\hat{u}^0, \hat{v}^0)$;
Step 1.	For $\xi \in \Xi$ Solve [SP(ξ)] with $(\hat{u}^0, \hat{v}^0)$; Get the optimal objective, $Z_{SP}(\xi)$, and solution $(\hat{\alpha}^1(\xi), \hat{\beta}^1(\xi))$; Solve [LP($\xi$)] with $(\hat{u}^0, \hat{v}^0)$ and $(\hat{\alpha}^1(\xi), \hat{\beta}^1(\xi))$; Compute $(a, b, Q^d, Q^{SL}, Q^{SF}, P^u, P^v)$ after solving LP(ξ); $\mathcal{L} \leftarrow \mathcal{L} \cup (a, b, Q^d, Q^{SL}, Q^{SF}, P^u, P^v)$; Solve [RSP($\xi$)] with $(\hat{u}^0, \hat{v}^0)$, obtain optimal dual solution $(\hat{\gamma}, \hat{\rho}, \hat{\sigma})$; Construct a new aggregated cut as in (5–51) and add it to $\mathcal{T}$; End For $\mathrm{U} = Z_{RMP} - \hat{z} + \sum_{\xi \in \Xi} Pr(\xi) Z_{SP}(\xi)$; $\mathrm{UB} \leftarrow \min(\mathrm{UB}, \mathrm{U})$;
Step 2.	Add a new integer L-shaped cut to [RMP] as in (5–53);
Step 3.	Solve [RMP] and obtain optimal objective, Z_{RMP}, and solution $(\hat{z}, \hat{u}^0, \hat{v}^0)$; $\mathrm{LB} \leftarrow \max(\mathrm{LB}, Z_{RMP})$;
Step 4.	If $\mathrm{UB} - \mathrm{LB} \leq \epsilon$, then stop; Otherwise go to Step 1.

Figure 5-8. Embedded Benders Decomposition Algorithm for EXPN

stage. An integer L-shaped "optimality" cut for [RMP] is as follows,

$$\begin{aligned} &\left[Q(\hat{\alpha}^0, \hat{\beta}^0) - L\right] \left[\sum_{(i,j)\in A} \sum_{k \in K_{ij}} \left(1 - 2\hat{\alpha}_{ij}^{0,k}\right) \alpha_{ij}^{0,k} + \sum_{i \in N_{LNG}} \sum_{k \in K_i} \left(1 - 2\hat{\beta}_i^{0,k}\right) \beta_i^{0,k}\right] + z \geq \\ &\qquad Q(\hat{\alpha}^0, \hat{\beta}^0) - \left[Q(\hat{\alpha}^0, \hat{\beta}^0) - L\right] \left[\sum_{(i,j)\in A} \sum_{k \in K_{ij}} \hat{\alpha}_{ij}^{0,k} + \sum_{i \in N_{LNG}} \sum_{k \in K_i} \beta_i^{0,k}\right] \end{aligned} \tag{5–53}$$

The best the case, in the sense of lowest cost, for the second stage problems is that no expansion is needed. Even there is no expansion cost, there is always transportation cost. Hence, a lower bound L is the minimal transportation cost of the second stage, which can be calculated as follows,

$$L = \sum_{\xi \in \Xi} Pr(\xi) \mathbf{h}^T \mathbf{f}_\xi^*,$$

where $\mathbf{f}_\xi^*$ is the optimal solution of the network flow problem of scenario ξ.

An initial solution of the first stage decisions can be obtained by solving the [RMP] without including the recourse variable z and any Benders cut. The lower bound is actually the optimal objective value of [RMP]. The upper lower bound can be obtained by adding up the total cost of a feasible solution. The embedded Benders decomposition algorithm is shown in Figure 5-8.

5.4 Numerical Examples

In this section, we present numerical results of our algorithm on serval problems with different sizes. We code our embedded Benders decomposition algorithm in Microsoft Visual C++ while calling CPLEX 10 (Concert Technology) to solve the decomposed problems. All programs are run in Microsoft Windows XP Professional 2002 SP2 on a Dell Desktop with Intel Pentium 4 CPU 3.40 GHz and 2 GB of RAM.

We test three groups of instances, each of which has different numbers of LNG nodes, total nodes, arcs, arc expansion and LNG terminal expansion sizes. Also, we assume all arcs are expandable, and same possible arc and node expansion capacities at different arcs and LNG nodes. So let $|K_{ij}|$ denotes the number of possible expansion sizes of all arcs, and $|K_i|$ denotes the number of possible expansion sizes of all LNG nodes. Then the number of binary variables in the extensive formulation, [EXPN], is $(|K_i| \times |N_{LNG}| + |K_{ij}| \times |A|) \times (1 + |\Xi|)$, which means $(|K_i| \times |N_{LNG}| + |K_{ij}| \times |A|)$ binary variables in each decomposed problem. In each group, we randomly generate different amounts of scenarios for a specific instance. The numbers of scenarios range from 10 to 10 thousand. When the number of scenarios is big, this extensive formulation is not an easy problem. While dealing with the extensive formulation, [EXPN], directly, CPLEX-MIP solver does not efficiently solve the instances with a big number of scenarios, e.g., 10k scenarios, with either exceeding the 2 hour computational time limit or running out of computer memory. However, our EBD algorithm can solve these instances with a big number of scenarios in a timely manner. In Table 1, we define three groups of instances. And then computational results are shown in Table 5-5, 5-6 and 5-7, where times are counted in seconds. In the three tables, we list that total computational times, and computing times for RMP, SP, LP and RSP. As can be

Table 5-4. Groups of Instances EXPN

Group No.	$\lvert N_{LNG} \rvert$	$\lvert N \rvert$	$\lvert A \rvert$	$\lvert K_{ij} \rvert$	$\lvert K_i \rvert$
1	2	4	5	2	2
2	3	7	12	3	3
3	7	11	20	5	5

Table 5-5. Computational times for instances group 1

$\lvert \Xi \rvert$	Total	RMP	SP	LP	RSP
10	5.157	0.549	3.203	0.971	0.434
20	11.699	0.579	7.765	2.182	1.173
50	2.876	0.031	2.045	0.499	0.253
100	5.781	0.047	3.889	1.029	0.566
200	12.574	0.035	8.273	2.610	1.189
300	18.652	0.031	11.717	3.048	1.613
500	29.155	0.046	18.477	5.923	2.626
700	36.198	0.32	24.421	6.511	4.045
1k	58.660	0.047	34.632	9.343	5.455
2k	115.244	0.047	68.916	18.672	13.059
10k	419.715	0.31	286.709	125.629	3.209

seen in the tables, computing time almost increase linearly with respect to the number of scenarios, which means our EBD method is well suited to problems with a huge number of scenarios. Also, the EBD algorithm spends a big portion of time to solve SP and LP. Hence it is possible to further reduce computing time if we do not calculate new IC and OF cuts for each scenario in Step 1 because looping through all scenarios takes a lot of time, especially when we have a huge number of scenarios, e.g. 10k or more. More advanced implementation could help to achieve this and improve the overall performance.

In addition to testing the computational time of our algorithm, we also conduct numerical examples on the risk management model, [ESPN-R], to find out how the risk constraints affect the performance (optimal objective value) of the model. Figure 5-9 shows how the optimal objective function changes when we vary the confidence level, θ, and the upper bound of the risk measure, Conditional Value at Risk. As is seen from the figure, the optimal objective value decreases as the upper bound, $\bar{\phi}$, on CVaR increases. Four lines are drawn according to different confidence levels, and

Table 5-6. Computational times for instances group 2

$\lvert\Xi\rvert$	Total	RMP	SP	LP	RSP
10	1.608	0.156	0.904	0.345	0.203
20	2.639	0.125	1.748	0.504	0.262
50	6.113	0.109	3.874	1.383	0.745
100	11.696	0.126	7.815	2.422	1.333
200	23.221	0.110	15.305	4.974	2.832
300	37.490	0.109	25.633	7.002	4.746
500	52.650	0.108	38.809	10.152	3.581
700	78.342	0.94	51.567	18.222	8.459
1k	100.656	0.109	68.513	23.712	8.322
2k	190.851	0.093	134.326	45.323	8.179
10k	934.013	0.109	685.224	240.201	8.479

Table 5-7. Computational times for instances group 3

$\lvert\Xi\rvert$	Total	RMP	SP	LP	RSP
3	.577	0.124	0.280	0.109	0.64
10	1.609	0.109	1.033	0.301	0.156
20	3.449	0.125	2.172	0.699	0.453
50	7.222	0.094	4.863	1.440	0.825
100	15.216	0.093	10.517	2.848	1.758
200	27.932	0.109	18.767	5.844	3.212
300	41.837	0.094	28.996	8.722	4.025
500	73.950	0.093	50.584	14.977	8.296
700	127.578	0.095	85.583	26.902	14.998
1k	145.097	0.094	98.226	26.963	19.814
2k	295.885	0.094	215.374	60.724	19.693
10k	1290.880	0.94	996.116	276.143	18.526

the line with higher confidence level bounds from below the one with lower confidence level, which says that we need to pay more if want we want to be more secure (higher confidence level). Another interesting fact to note from the figure is that the four lines become more deviated from each other when the upper bound $\bar{\phi}$ increases. This is may be explained by a simple example. If $\bar{\phi} = 0$, then the optimal objective value of any confidence level should be the same, because the right hand side of constraint (5–29) is 0, and then θ will not be able to affect the optimal solution of η and w_is. With $\bar{\phi}$ gradually increasing, the effectiveness of θ on the optimal objective value keeps increasing. It, directly reading from the figure, looks like that the optimal objective value is a convex

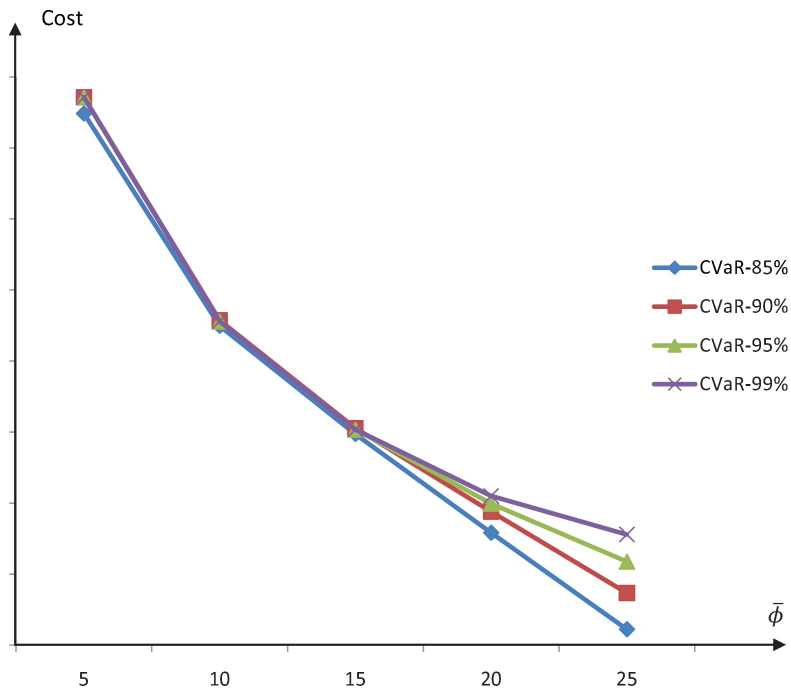

Figure 5-9. Minimal Cost V.S. Limit of CVaR

function of $\bar{\phi}$. This is true if we are dealing with pure linear models. However, this may not hold for our models since we also have discrete decision variables.

CHAPTER 6
CONCLUSIONS

This dissertation discusses stochastic integer programming and its applications in the energy systems. First this dissertation gives a brief review about stochastic programming, stochastic integer programming, and solution methods. Then this dissertation proposes the Embedded Benders' Decomposition (EBD) method for both deterministic and stochastic mixed integer programming, which include integer decision variables in the second stage. This method takes advantage of Benders' decomposition and applies the Benders' cuts in both first and second stages. All Benders' cuts generated for the second stage are reusable along the iterations of the algorithm since these cuts are valid for the second stage given any first stage decision.

Next, this dissertation studies the two-stage stochastic security constrained unit commitment (SSCUC) problem by applying the EBD method. The SSCUC problem includes both day-ahead scheduling of coal fired generators and real-time quick-start generators scheduling. Computational results show that EBD is very well suited for the SSCUC problems, especially when dealing with a lot of scenarios. After that, a detailed review of optimization models and techniques applied in natural gas industry is presented. Also, this dissertation proposes a mathematical programming model for natural gas transmission system expansion planning, which, to our best knowledge, is the first model that combines transmission line expansion and LNG terminal expansion together, and considers uncertainties in demands and supplies, and risk controlling. Numerical results indicate that CVaR is a very good risk measure in the sense of taking risk to reduce cost. Because expansions are modeled by integer variables, the model is a two stage stochastic mixed integer programs where integer variables exist in both stages. SMIP of this type is a very challenging problem. EBD algorithm is also applied to solve the expansion planning problem. Benders cuts are implemented in both stages, but they act as different roles. The Benders cuts for the second stage are reusable given any first stage solution, and solving the subproblems of one scenario can help generate multiple cuts for all scenarios due to their sharing of dual feasible space. The numerical results show that the EBD algorithm is also very well suited for the expansion planning problems, especially when there are a huge number of scenarios. Future research would focus on how to further improve the EBD algorithm by reducing the computational time on RSP, LP and RSP. Also, more sophisticated techniques to generate strong Benders cuts could be used to improve the convergence rate of the algorithm.

There are two types of future research we would like to pursue, theories and applications. On the theoretical side, we would like to find a finitely convergent method for general stochastic mixed integer programs based on the results of Embedded Benders Decomposition. Also, the development of fast convergent methods for multistage stochastic integer programs based on Benders decomposition and polyhedral theory is fascinating. On the application side, network based stochastic unit commitment problems are very important in reality, since generators are located in a decentralized power grid. Network-based models can provide more insights about how to coordinate and integrate all resources within the power grid. The natural gas contract optimization problem is another very interesting application problem on which we can apply the multistage EBD if we can enhance the convergence. More and advanced studies of polyhedral properties of subproblem in each stage are crucial to develop advanced algorithms for the multistage problems.

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BIOGRAPHICAL SKETCH

Qipeng Zheng was born in a heavy industry town, which is a major manufacturer of China's aluminum production, in Zibo, Shandong, in 1979. Upon his graduation from the local high school in 1997, he went to Beijing for college in North China University of Technology, majoring in Industrial Automation. After obtaining his bachelor's degree in 2001, he devoted 8 months in a startup company to developing software for embedded systems. In 2002, he started pursuing his master's degree in the Department of Automation at Tsinghua University, China. With his master's degree, he came to Gainesville, Florida, to pursue his doctorate in the Department of Industrial and Systems Engineering at University of Florida, in summer 2005.