Gas Transmission Network Observability

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Abstract— **This paper presents an efficient method for dealing with the problems of topological observability in steady-state natural gas transmission networks. In the practical world, these type of instances are very large, in terms of the number of decision variables. In this paper, we present a study of the properties of gas pipeline networks, and exploit them to develop a technique that can be used to reduce significantly problem dimension, without disrupting problem structure. The correctness of this method is established by the lemmas and theorems provided in the text. The proposed method has been successfully applied to a real-life gas transmission network.**

I. INTRODUCTION

As natural gas pipeline systems have grown larger and more complex, the importance of optimum operation and planning of these facilities has increased. The investment costs and operation expenses of pipeline networks are so large that even small improvements in system utilization can involve substantial amounts of money.

The natural gas industry services include producing, moving, and selling gas. Our main interest in this study is focused on the transportation of gas through a pipeline network. Moving gas is divided into two classes: transmission and distribution. Transmission of gas means moving a large volume of gas at high pressures over long distances from a gas source to distribution centers. In contrast, gas distribution is the process of routing gas to individual customers. For both transmission and distribution networks, the gas flows through various devices including pipes, regulators, valves, and compressors. In a transmission network, gas pressure is reduced due to friction with the pipe wall as the gas travels through the pipe. Some of this pressure is added back at compressor stations, which raises the pressure of the gas passing through them.

Depending on how the gas flow changes with respect to time, we distinguish between systems in steady state and transient state. A system is said to be in steady state when the values characterizing the flow of gas in the system are independent of time. In this case, the system constraints, particularly the ones describing the gas flow through the pipes, can be described using algebraic non-linear equations. In contrast, transient analysis requires the use of partial differential equations to describe such relationships. This makes the problem considerably harder to solve from the optimization perspective. In fact, optimization of transient models is one of the most challenging areas of opportunity for future research. In the case of transient optimization, variables of the system, such as pressures and flows, are functions of time. In this work, we focus on steady-state gas transmission network problems with the objective of minimizing the number of measurement points.

Gas network topologies are in a relatively well developed stage. This problem is represented by a network, where arcs correspond to pipelines and compressor stations, and nodes correspond to their physical interconnection points. The decision variables are the mass flow rates through every arc, and the gas pressure level at every node.

The main contribution of our work is to provide a way to significantly reduce the size of the problem instances at preprocessing without disrupting problem structure. In fact, our approach has been successfully incorporated in recent work such as Wu et al. [15], Kim [3], and Kim, Rios-Mercado, and Boyd [4]. For a more complete review on algorithms for pipeline optimization the reader is referred to the work of Carter [1] and Rios-Mercado [9].

The rest of the paper is organized as follows. In Section 2, we introduce the problem and present the mathematical formulation. This is followed by Sections 3 and 4, where we present the relevant results related to graph theory and the pipeline network flow equations, respectively. In Section 5, we develop the main theoretical results about uniqueness and existence of solutions using techniques from nonlinear functional analysis [11]. We continue in Section 6 with the description of the proposed network reduction method and show how to apply it in the two basic cases of network topologies. Simulation on the real-life gas network is studied in section 7. Then we conclude in Section 8.

II. MODEL DESCRIPTION

This problem involves the following constraints:

(i) mass flow balance equation at each node;

(ii) gas flow equation through each pipe;

(iii) pressure limit constraints at each node;

(iv) operation limits in each compressor station.

The first two are also called steady-state network flow equations. We emphasize that while the mass flow balance equations (i) are linear, the pipe flow equations (ii) are nonlinear; this has been well documented in [14,15]. For medium and high pressure flows, when taking into account the fact that a change of the flow direction of the gas stream may take place in the network, the pipe flow equation takes the following form:

$$p_i^2 - p_j^2 = c_{ij} u |u|^{\alpha}, \qquad (1)$$

where p_i and p_j are pressures at the end nodes of pipe (i, j), u is the mass flow rate through the pipe, α is a constant $(\alpha \approx 1)$, and the pipe resistance c_{ij} is a positive quantity depending on the pipe physical attributes.

The steady-state network flow equations can be stated in a very concise form by using incidence matrices.Let us consider a network with n nodes, l pipes, and m compressor stations. Each pipe is assigned a direction which may or may not coincide with the direction of the gas flow through the pipe. Let A_l be the $n \times l$ matrix whose elements are given by

$$a_{ij}^{l} = \begin{cases} 1, & \textit{if pipe } j \textit{ goes} \text{ into } \textit{node } i; \\ -1, & \textit{if pipe } j \textit{ goes} \text{ into } \textit{node } i; \textit{and} \\ 0, & \textit{otherwise.} \end{cases}$$

A_l is called the node-pipe incidence matrix.

Similarly, let A_m be the $n \times m$ matrix whose elements are given by

$$a_{ik}^{m} = \begin{cases} 1, & \textit{if node } i \textit{ is the} \text{ discharge} \\ & \textit{node of station } k; \\ -1, & \textit{if node } i \textit{ is the suction} \\ & \textit{node of station } k; \textit{and} \\ 0, & \textit{otherwise.} \end{cases}$$

A_m is called the node-station incidence matrix. The matrix formed by appending A_m to the right hand side of A_l will be denoted as A, i.e., $A = (A_l A_m)$,which is an $n \times (l + m)$ matrix.

Let $\mathbf{u} = (u_1, \ldots, u_l)^T$ and $\mathbf{v} = (v_1, \ldots, v_m)^T$ be the mass flow rate through the pipes and stations, respectively. Let $\mathbf{w} = (\mathbf{u}^T, \mathbf{v}^T)^T$. A component u_j or v_k is positive if the flow direction coincides with the assigned pipe or station direction, negative, otherwise. Let p_i be the pressure at node i, $\mathbf{p} = (p_1, \ldots, p_n)^T$. $\mathbf{s} = (s_1, \ldots, s_n)^T$ be the source vector, where the source s_i at node i is positive (negative) if the node is a supply (delivery) node. A node that is neither a supply or delivery node is called a transition node and has s_i equal to zero. We assume, without loss of generality, the sum of the sources to be zero:

$$\sum_{i=1}^{n} s_i = 0 \qquad (2)$$

The network flow equations can now be stated as the following:

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ A_l^T \mathbf{p}^2 = \phi(\mathbf{u}) \end{cases} \qquad (3)$$

Where $\mathbf{p}^2 = (p_1^2, \ldots, p_n^2)^T$, and $\phi(\mathbf{u}) = (\phi_1(u_1), \ldots, \phi_l(u_l))^T$, with the pipe flow equation at pipe j equal to $\phi_j(u_j) = c_j u_j |u_j|^{\alpha}$.

III. GRAPH THEORY CONCEPTS

In this section we present some concepts from graph theory that will be used to develop our techniques. The results presnted here can be found in most of the books on graph theory, e.g. [2,5].

A *graph* $G = (V, E)$ is a structure consisting of a finite set of elements V called *vertices* or *nodes* and a set E of unordered pairs of nodes called *edges*. A *directed graph* or *digraph* is defined similarly, except that each edge is an ordered pair, giving it direction from one node to another. To make the distinction, we call the ordered pair *arc*. For both graphs and digraphs, an arc or edge from node i to node j is denoted by (i, j). An arc (i, i) is called a *loop*. Throughout this work, we will make the assumption that no loops are present. A *walk* of a graph G is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the two vertices immediately preceding and following it. A walk in which no vertex appears more than once is called a *path*. A path beginning and ending with the same vertex is called a *cycle*. Similarly, in a digraph, a *directed walk* is an alternating sequence of vertices and arcs, where v_{i-1}, e_i, v_i being in the sequence implies arc $e_i = (v_{i-1}, v_i)$. A *directed path* is a directed walk in which no vertex appears more than once, and a *directed cycle* is a directed path beginning and ending with the same vertex. A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. A *tree* is a connected graph with no cycles. A *spanning tree* T of a graph G, is a tree consisting of all the vertices in G. For a given spanning tree T of a graph G, any edge in G which is not in the tree T is called a *chord*. A basic result from graph theory states that adding a chord to a spanning tree T makes T no longer acyclic. This cycle formed in T is called a *fundamental cycle*.

Here are some basic results regarding spanning trees and fundamental cycles of a graph. Proofs of Theorems 1 and 2 can be found in [2].

Theorem 1 Let n and e be the numbers of vertices and edges, respectively, in a graph G. Let T be a spanning tree of G. Then

(a) The number of edges in a spanning tree T is $n-1$, the number of chords corresponding to the spanning tree T is $e-n+1$.

(b) The number of the fundamental cycles corresponding to the spanning tree T is $e-n+1$. Every other cycle in G is a linear combination of the fundamental cycles.

Note that the set of fundamental cycles corresponding to a given tree T is independent, since each contains an edge not in any of the others. Suppose G is a digraph with n vertices and e arcs. The node-arc incident matrix A of G is an $n \times e$ matrix, defined by

$$a_{ij} = \begin{cases} 1, & \textit{if arc } j \textit{ is incident out of vertex } i; \\ -1, & \textit{if arc } j \textit{ is incident out of vertex } i; \textit{and} \\ 0, & \textit{otherwise.} \end{cases}$$

By deleting one row from the matrix A, the remainder matrix is called *reduced incident matrix*, denoted by A_f, which is $n-1$ by e, whose $n-1$ row vectors are linearly independent. The vertex corresponding to the deleted row is called *reference vertex*. The incident matrix of a digraph completely determines the digraph. For digraph G there is an associated undirected graph G', consisting of the same set of vertices and edges, that is, for each arc (i, j) in G there is an edge (i, j) in G'. In other words, the undirected version of G ignores the direction of the arcs. Each cycle in G', after being arbitrarily assigned an orientation, can be represented by a vector whose components are 1,-1,0 according to whether and how the edge is included in the circuit. A *cycle matrix* B is a matrix where each row (column) corresponds to a cycle (arc) vector, and it is defined by

$$b_{ij} = \begin{cases} 1, & \textit{if cycle } i \textit{ contains arc } j \\ & \textit{and their orientations coincide}; \\ -1, & \textit{if cycle } i \textit{ contains arc } j \\ & \textit{but their orientations are opposite}; \textit{and} \\ 0, & \textit{otherwise.} \end{cases}$$

As implied by Theorem 1, only $e-n+1$ fundamental cycle vectors with respect to a spanning tree are independent. A cycle matrix consisting of $e-n+1$ fundamental cycle vectors is called *reduced cycle matrix*, and it is denoted by B_f, which is $(e-n+1) \times e$ matrix.

Theorem 2 Let G be a digraph, A_f and B_f be the reduced incident and cycle matrices, respectively, using the same order of edges. Then

$$A_f B_f^T = B_f A_f^T = 0$$

IV. THE PIPELINE NETWORK FLOW EQUATIONS

Now let us consider a gas pipeline network subsystem which consists of nodes and pipes only, that is with no compressor stations. We arbitrarily assign a direction for every pipe and view it as a digraph. Let G be such a digraph with n vertices and e edges.Following notation from Section 2, $\mathbf{w} = (w_1, \ldots, w_e)^T$ denotes the flow vector with w_j the mass flow rate through the j th edge. The flow w_j is positive if the directions of the flow and the edge coincide, negative otherwise. Let $\mathbf{s} = (s_1, \ldots, s_n)^T$ be the source vector satisfying (2). Given that no compressor stations are considered, the compressor flows u_j can be ignored so that system (3) can now be restated as:

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ A^T\mathbf{p}^2 = \phi(\mathbf{w}) \end{cases} \quad (4)$$

In many network flow problems functions $\{\phi_j\}$ describing the relationship between arc flows and node variables at end points of the arc are nonlinear. In the case of gas transmission networks, the most commonly used functions are of the following form:

$$\phi_j(w_j) = c_j w_j |w_j|, \quad 1 \le j \le d$$

with $c_j > 0$. In some cases, ϕ_j 's could also be of the form:

$$\phi_j(w_j) = c_j w_j |w_j|^{\alpha}, \quad 1 \le j \le d$$

where $\alpha \ge 0$.

Now suppose a source vector $\mathbf{s}$ is given satisfying the zero sum condition (2) and a reference vertex has been selected whose pressure is also given (which is a necessary condition to solve system (4)). The number of the unknowns is $e+n-1$ and the number of flow equations is $e+n$. Since $rank(A) = n-1$, only $n-1$ node flow balance equations are linearly independent. Let A_f be the reduced incident matrix with respect to the selected vertex; let B_f be the reduced cycle matrix with respect to some spanning tree. Since $B_f A^T = 0$ (from 9,Theorem 2), system (4) is equivalent to:

$$\begin{cases} A_f\mathbf{w} = \mathbf{s}_f \\ B_f\phi(\mathbf{w}) = \mathbf{0} \\ A^T\mathbf{p}^2 = \phi(\mathbf{w}) \end{cases} \quad (5)$$

Where $\mathbf{s}_f$ is an $(n-1)$ -vector formed by removing from $\mathbf{s}$ the source term corresponding to the selected reference vertex. The advantage of system (5) is that the first two sets of equations:

$$\begin{cases} A_f\mathbf{w} = \mathbf{s}_f \\ B_f\phi(\mathbf{w}) = \mathbf{0} \end{cases} \quad (6)$$

contain only the flow vector $\mathbf{w}$. Notice that system (6) consists of e equations and e unknowns. If it has a unique solution, the flow vector $\mathbf{w}$ can be solved separately from the pressure vector $\mathbf{p}$, and the pressure vector $\mathbf{p}$ can be directly computed from the third equation of system (5) if the pressure at a reference vertex is given. We now address the question on whether system (6) has a unique solution.

V. UNIQUENESS AND EXISTENCE OF THE SOLUTION

In this section, we show that system (6) has a unique solution. A direct corollary of this result is that system (4) has a unique solution if the source vector $\mathbf{s}_f$ and the pressure value at a reference node are given. We begin with some definitions. Let H be a Hilbert space with a scalar product $(.,.)$, and let $\|\,.\,\|$ denote the associated norm, i.e. $\|x\| = \sqrt{(x,x)}$ for any $x \in H$.

Definition 1. A mapping $\phi : H \to H$ is said strongly monotonic if there exists a constant $a > 0$, such that, for every $x, y \in H$ we have

$$(\phi(x) - \phi(y), x - y) \ge a(x - y, x - y)$$

Definition 2. A mapping $\phi : H \to H$ is said strictly monotonic if for every $x, y \in H$ we have

$$(\phi(x) - \phi(y), x - y) \ge 0$$

and equality holds if and only if $x = y$.

Definition 3. A mapping $\phi : H \to H$ is said to be a basin if for every $x_0 \in H$, the set

$$X_{x_0} = \{x \in H : (\phi(x), x - x_0) \le 0\}$$

is bounded.

Now we prove some basic results related to the above concepts.

Lemma 1 If $\phi : H \to H$ is strongly monotonic, ϕ is a strictly monotonic basin.

However, a mapping ϕ that is a strictly monotonic basin is not necessarily strongly monotonic. Here is an example.

Lemma 2 Let $H = \mathbb{R}^d$ with the Euclidean scalar product, where d is a positive integer. Let ϕ: $\mathbb{R}^d \to \mathbb{R}^d$ be a mapping as follows: for every $\mathbf{x} = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d$,

$$\phi(\mathbf{x}) = (\phi_1(x_1), \phi_2(x_2), \ldots, \phi_d(x_d))^T$$

where

$$\phi_j(x_j) = c_j x_j |x_j|^{\alpha}, \quad 1 \le j \le d$$

with $c_j > 0$ and $\alpha \ge 0$ .Then ϕ is a strictly monotonic basin.

Remark: ϕ is not strongly monotonic if $\alpha > 0$. If $\alpha = 0$, we obtain the following.

Corollary 1 The identity function ϕ: $\mathbb{R}^d \to \mathbb{R}^d$, $\phi(\mathbf{x}) = \mathbf{x}$ is a strictly monotonic basin.

The following lemma can be found in [11].

Lemma 3 Let H be a Hilbert space. If $\phi: H \to H$ is continuous and strongly monotonic, then ϕ maps H onto H.

Let $r > 0$, $t \geq 0$, be two integers, and $d = r + t$. We say an $r \times d$ matrix A and a $t \times d$ matrix B are perpendicular to each other if they satisfy the following hypothesis.

Hypothesis P.

1. $rank(A) = r, rank(B) = t$.
2. $AB^T = BA^T = 0$

Theorem 3 (Uniqueness) Let matrices A and B be perpendicular to each other. Suppose ϕ: $\mathbb{R}^d \to \mathbb{R}^d$ is strictly monotonic, then, for every $\mathbf{s} \in \mathbb{R}^r$,the solution to the system of equations

$$\begin{cases} A\mathbf{w} = \mathbf{s} \\ B\phi(\mathbf{w}) = \mathbf{0} \end{cases} \quad (7)$$

is unique.

Lemma 4 Suppose ϕ: $\mathbb{R}^d \to \mathbb{R}^d$ is continuous and strongly monotonic. Then, for every $\mathbf{s} \in \mathbb{R}^r$,system (7) has a solution.

Theorem 4 Let matrices A and B be perpendicular to each other. Let ϕ: $\mathbb{R}^d \to \mathbb{R}^d$ be continuous. Suppose

(i) ϕ is strictly monotonic;

(ii) ϕ is a basin.

Then system (7) has a solution for every $\mathbf{s} \in \mathbb{R}^r$.

Corollary 2 System (6) has a unique solution for every $\mathbf{s}_f \in \mathbb{R}^{n-1}$.

Systems of nonlinear equations can have very strange behaviors. Even a single nonlinear equation could have no solution or more than one solution. Interestingly, some systems of nonlinear equations which arise from industrial and engineering problems have a unique solution as do (6) and (7) proposed in this paper.

For gas pipeline network flow problems, the presented result is quite interesting itself. One additional fact is that, since the function ϕ involved in gas pipeline network problems is monotonic, solving the system (6) by Newton's method is very stable, fast, and accurate. These facts lead us to introduce the Network Reduction Method for networks consisting of nodes, pipes, and compressor stations.

We will show in the next section that this method can greatly reduce the size of the problem, without modifying its mathematical structure.

VI. THE NETWORK REDUCTION METHOD

The main result obtained in the previous section is that, when all the sources (that is, the mass flow rates at all the nodes of the network going into or out of the network) are given, all the flows in the pipes are completely determined, while the pressures at the nodes can be determined if the pressure at one (reference) node is given. It must be pointed out that this result is based on two facts:

1. Each node has a mass flow balance equation.
2. Each pipe has a pipe flow equation defining the relation between the flow rate and the pressures at the two end nodes.

This result is valid in networks consisting of pipes only. Let us take a step further and consider now a network consisting of pipes and compressor stations too. The mass flow balance equations must still be satisfied at each node, and a pipe flow equation must be satisfied at each edge representing a pipe; however, for each edge representing a station there is no equation relating the flow rate through the station and the pressures at its suction and discharge sides. Flow rate, suction pressure, and discharge pressure of a station are actually independent of each other, and there are only certain inequalities these variables must satisfy. Hence, the result obtained in the previous section can not be directly applied to such networks. In this section, we will introduce the Network Reduction Method for networks consisting of nodes, pipes, and compressor stations. In the sequel, we refer to the latter simply as "stations."

Let us first start by introduing the concept of a reduced network. By removing all the stations' arcs from a network, which consists of nodes, pipes, and compressor stations, we are left with several *disconnected* components, each of them called a *subnetwork*, consisting of only nodes and pipes. By construction, there are no stations in any subnetwork. On the other hand, if we view each subnetwork as a single (big) node for the network, i.e., shrinking each subnetwork to a node, and placing back the compressor arcs we had previously removed, we get a new network which consists only of the (big) nodes, each representing a subnetwork, and the stations. There are no pipes in this network because all the pipes are encapsulated in the (big) nodes. This new network is called a *reduced network* (where each node represents a subnetwork, and each edge represents a station). It is easy to see that there is a unique (connected) reduced network associated to a given (original) network. The structure of the undirected graph associated with the reduced network can be either a tree or a graph with cycles, depending on the configuration of the compressor stations in the network.

For real-world instances of pipeline networks, we have found that the topology of a reduced network is much less complicated than that of the original network. Although a network may have a number of cycles, especially cycles in pipes, its associated reduced network is, most of the time, a tree. Even if the associated reduced network is not a tree, the number of cycles in the reduced network can be significantly less than that in the original network. We distinguish two cases in terms of the network topology.

VI.I. CASE 1: REDUCED GRAPH IS A TREE

In this section we assume that the reduced graph is a tree. In this case, since each node in the reduced network represents a subnetwork, we can define the source value at this node as the sum of the source values at all the nodes included in this subnetwork. In this sense, the sources at all the nodes in the reduced network are fixed. Since the reduced network is a tree, all the flow rates through the edges of the reduced network are uniquely determined (by Corollary 2, with $B = 0$). Since each edge in the reduced network represents a station in the original network, it means that the flow rates through all the stations are known.

Now let us look at the subnetworks. We can see that, for each subnetwork, the sources at all the nodes, including the nodes connecting to stations are all known. By Theorem 4, we conclude that the flow rates through all the pipes in the subnetwork can be uniquely determined. Moreover, the pressures at all the nodes in the subnetwork are uniquely determined by the pressure at one node, the reference node. These pressures will also be increased or decreased as the pressure at the reference node is increased or decreased, respectively. Hence, we have the following fundamental theorem of the network reduction method.

Theorem 5 Suppose that

(i) The pipeline network consists of only nodes, pipes, and stations;

(ii) the sources at all the nodes are given; and

(iii) the associated reduced graph is a tree.

Then

1. Flow rates through all the pipes and stations are known.

2. For each subnetwork, pressure p at any node is related to the pressure p_r at a reference node by

$$p^2 - p_r^2 = c,$$

Where

$$c = \sum_{j \in J} c_j u_j \left| u_j \right|^{\alpha}$$

is a constant, where J is an index set of pipes in a path connecting the node and the reference node, c_j and α are constants, u_j is the flow rate in the j th pipe which is known.

Note that the constant c is independent of the selection of the path because the flow rate u_j 's are solved from the equations such that summation $\sum_{j \in J} c_j u_j \left| u_j \right|^{\alpha}$ along any cycle in a subnetwork is zero. Hence, if a network is divided into b subnetworks, the total number of independent variables in the network is b, i.e., the pressure variables p_r at the b reference nodes.

In this case, the number of measurement points is equal to the number of subnetworks, since we need to know the pressure value at the reference nodes.

VI.II. CASE 2: REDUCED GRAPH HAS CYCLES

Since the reduced network is a digraph with cycles, the flow rates cannot be uniquely determined, although the network reduction method can still be successfully used. In this case, the mass flow rate v through the stations satisfies a simple system of linear equations:

$$\mathcal{A}\mathbf{v} = \mathbf{S}, \qquad (8)$$

Where $\mathcal{A}$ is the node-edge incidence matrix for the super-network and $\mathbf{S}$ is the vector of sources at the nodes in the reduced network. The i th element of $\mathbf{S}$ is the sum of the sources at all the nodes in the i th subnetwork.

Theorem 6 The number of independent variables in system (8) is equal to the number of fundamental cycles in the associated reduced network.

Therefore the number of measurement points, we need, is equal to the number of fundamental cycles. Also it is necessary to have one measurement point for each reference node in each subnetwork. So the total number of measurement points is the number of fundamental cycles plus the number of subnetworks in the reduced graph.

VII. SIMULATION RESULTS

The algorithm is simulated for data's of gas transmission network in Belgium.

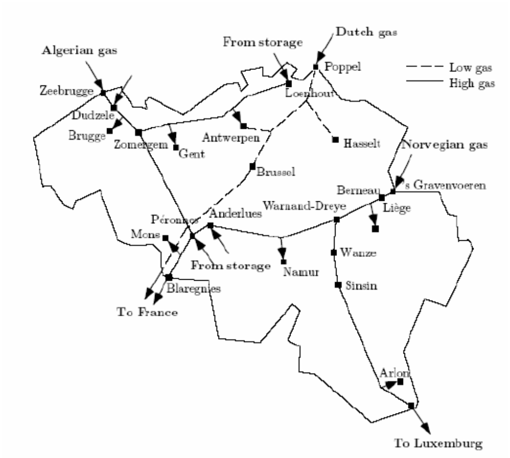

Figure 1: Schematic Belgium gas network

We can see that, for this real-life network, the reduced graph has one cycle. So we need one measurement point for one of the compressor stations. Also the reduced graph has 3 subnetworks so we need 3 measurement points for 3 reference nodes, each of them for one subnetwork.

VIII. CONCLUSIONS

We have proposed a reduction technique for gas pipeline optimization problems. The justification of the technique was based on a novel combination of graph theory and nonlinear functional analysis. The reduction technique can decrease the problem size by more than an order of magnitude in practice, without disrupting its mathematical structure.

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