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The repeated *p*-dispersion problem

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ABSTRACT

The grey pattern problem is to select a pattern of p points in a square that is replicated in adjacent squares so that they are spread out as uniformly as possible. The goal is to cover a large area with many squares of the same pattern of p points. In the original formulation a special objective function is designed. In this paper we suggest the criterion of maximizing the minimum distance between points in the same square and in the eight adjacent squares, four with a common side and four with a common vertex. We prove properties of the proposed objective, and propose alternate formulations of the model. Extensive computational experiments are reported on instances using Euclidean distances and Manhattan distances with good results.

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1. Introduction

The grey pattern problem was originally proposed by Ulichney (1987) to select a pattern of p points out of n points in a rectangle of sizes n_1 by n_2, so that $n = n_1 n_2$, that will look as smooth as possible. The rectangle is replicated with identical patterns to cover the whole plane. Taillard (1995) created an objective function which is a sum of reciprocal squared distances between all selected points and the other selected points in the rectangle as well as in the eight adjacent rectangles as depicted in Figure 1. He proposed to solve it as a Quadratic Assignment Problem (QAP, Koopmans and Beckmann 1957; Burkard et al. 1997; Drezner et al. 2015). For details see Drezner et al. (2022, 2015), Misevicius et al. (2021), and Taillard (1995).

The objective function proposed by Taillard (1995) is based on the distances between points $X_i = (x_i, y_i)$ and $X_j = (x_j, y_j)$ for $i \neq j$ in the square and the equivalent ones in the eight adjacent squares. Converting to a square of side 1, the (squared) distance d_{ij}^2 between points i and j is:

$$d_{ij}^2 = \min_{u,v \in \{-1,0,1\}} \left\{ (x_i - x_j + u)^2 + (y_i - y_j + v)^2 \right\} \tag{1}$$

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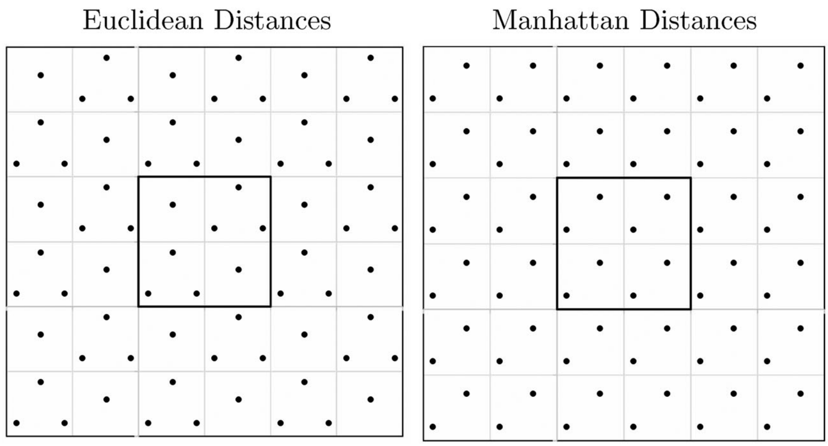

Figure 1. Optimal $p = 8$ configurations.

and the objective of the continuous grey pattern problem is:

$$\min_{0 \leq X_i, X_j \leq 1} \left\{ 2 \sum_{1 \leq i < j \leq p} \frac{1}{d_{ij}^2} \right\} . \tag{2}$$

The continuous version of the problem in a square, using expression (2), was proposed and investigated in Drezner and Kalczynski (2017). Similar to the discrete grey pattern problem, the continuous grey pattern problem is useful when a grey pattern is needed in a rectangle with a large number of points which is impractical to solve. A grey pattern configuration is found in a relatively small square and many small squares with identical configurations are placed in the big rectangle. For example, suppose that we need to construct a 40% grey pattern in a square with 40,000 points. We first find a configuration of 40 points in a 10 by 10 points square. We then divide the 'big' square into 400 small squares of 100 points each (20 in each row and 20 in each column) and place the 40-point configuration in each square. If we have 45,000 points in a rectangle of sides 2 and 1, we can use the 40-point configuration in 450 small squares 30 per row and 15 per column.

The practical applications of the original grey pattern problem are primarily in digital colour halftone generation (Kang 1999; Misevičius et al. 2013; Lau and Arce 2018; Drezner et al. 2022). The problem also applies to pixel arrangement (physical or virtual) in flat display panels as well as dot, spot, sensor, or photovoltaic cell arrangements. Some configurations may also help reduce the Moiré effect (Rong and Kuiper 1993) which degrades the quality of the image (Lin et al. 2015). It can help design pixel configurations for large-scale media facades (Halskov and Ebsen 2013) or sensor matrices. It may also be suitable in the computer-aided design software for the designers of abstract and/or computer visual arts (Wong and Memon 2003; Kuznetsov 2021).

In this paper we propose to replace the special function designed by Taillard (1995), which is suitable for the QAP, by the objective of maximizing the minimum distance between points in a square and the eight adjacent squares with identical patterns. In the objective function (2), distances to points that are not adjacent to one another affect the value of the objective function. We select an objective function that considers only distances between closest selected points, and ignores the distances to farther ones. The objective function (2) is replaced by:

$$\max_{0 \le X_i, X_j \le 1} \left\{ \min_{1 \le i < j \le p} \left\{ d_{ij} \right\} \right\} . \tag{3}$$

We investigate here problems based on the original Euclidean d_{ij} distances and also the Manhattan (ℓ_1) distances: ($\min_{u,v \in \{-1,0,1\}} \left\{ |x_i - x_j + u| + |y_i - y_j + v| \right\}$, Francis et al. 1992; Love et al. 1988).

In Figure 1, the optimal solution (found in this paper) is depicted for $p = 8$ points in a square surrounded by eight squares with identical patterns. Suppose that the minimum distance between points is D. If the points are circles with a given radius r, then by applying Euclidean distances, the maximum minimum distance between the circles' circumferences is $D - 2r$. Therefore, in this case maximizing the minimum distance between points is equivalent to maximizing the minimum distance between the circles. If the points are squares with sides s parallel to the axes, the minimum Manhattan distance between the squares is $D - 2s$ in most cases, but for example, if the two squares have the same y-coordinate the distance is D – s. If the points have a rhombus shape (a square rotated in 45°), which is the unit ball for the Manhattan distances (Love et al. 1988), the distances between perimeters are $D - \sqrt{2}s$.

The $p = 8$ objective is the same as the $p = 7$ objective (Tables 1 and 4). This means that the removal of any point in the configurations in Figure 1 yields an optimal $p = 7$ configuration. Such a removal will create a 'hole' in the $p = 7$ configuration. For example, removal of the point near the centre of the square creates a hole and visually it seems possible to move some of the five or six points surrounding it inward and allow for a larger minimum distance. However, this is not possible because both $p = 7$, 8 have the same optimal minimum distance.

It is interesting that the best configuration, likely optimal, (found in Drezner and Kalczynski 2017) for the continuous grey pattern formulation (2) is *identical* to the Manhattan configuration in Figure 1 even though the distances in (2) are Euclidean. The objective value of (2) is 336 and the minimum distance between points is $\frac{\sqrt{2}}{4} \approx 0.354$ (see Table 4). The Euclidean configuration has a worse continuous grey pattern objective of 344.42 even though the minimum distance is better ($\frac{\sqrt{3}-1}{2} \approx 0.366$), which means that the points are more spread out. This confirms that formulation (2) may result in a smaller minimum distance because the distances to farther points distort the objective of points being spread out.

This problem is similar to the p-dispersion problem in a square (Kuby 1987; Drezner and Erkut 1995). The p-dispersion problem is to locate p points in a bounded a region, such as a square, in order to maximize the minimum distance between them. Distances to points outside the square are not considered. The p-dispersion in a square using Euclidean distances is equivalent to circle packing in a square (packing p circles in a square with the largest possible radius, Maranas et al. 1995; Nurmela and Oestergard 1999; Locatelli and Raber 2002; Szabo et al. 2007;

Lopez and Beasley 2011). The p-dispersion problem is equivalent to finding the smallest square box that can accommodate p circles (such as cans or bottles) of a given size. That is, rather than finding p circles with the largest radius in a given box, we find the smallest box that accommodates p circles of a given radius.

2. Basic formulation and properties

We propose to solve the continuous problem replacing the objective proposed in Taillard (1995) by maximizing the minimum distance between each point in a square of side 1 and the points in the same square and the eight adjacent squares. This problem can be formulated as a non-linear program.

Let (x_i, y_i) for $i = 1, ..., p$ be the unknown locations of p points. The nine points related to point i, eight in adjacent squares, are $(x_i + u, y_i + v)$ for $u, v \in \{-1, 0, 1\}$. The point itself is obtained for $u = v = 0$.

The value of the objective function, to be maximized, is:

$$\textbf{Euclidean Distances}: \quad F^2 = \min_{1 \le i < j \le p} \left\{ \min_{u,v \in \{-1,0,1\}} \left\{ (x_i - x_j + u)^2 + (y_i - y_j + v)^2 \right\} \right\} \tag{4}$$

$$\textbf{Manhattan Distances}: \quad F = \min_{1 \le i < j \le p} \left\{ \min_{u,v \in \{-1,0,1\}} \left\{ |x_i - x_j + u| + |y_i - y_j + v| \right\} \right\} \tag{5}$$

Note that the order of the min's can be reversed for both distance measures.

2.1. Properties

Inspecting the examples in Figure 1, it is intuitive that moving all the points by the same distance in the same direction does not affect the distances and the value of the objective function. If, for example, all points are moved to the right, all distances are the same and if a point moves outside the square, the equivalent point in the left square moves into the square thus retaining the number of points in the square. We formally prove this intuitive property.

Define the following transformation for given $-1 \le a, b \le 1$

$$x'_i(a) = \begin{cases} x_i + a & | \; 0 \le x_i + a \le 1 \\ x_i + a - 1 & | \; x_i + a > 1 \\ x_i + a + 1 & | \; x_i + a < 0 \end{cases} \qquad y'_i(b) = \begin{cases} y_i + b & | \; 0 \le y_i + b \le 1 \\ y_i + b - 1 & | \; y_i + b > 1 \\ y_i + b + 1 & | \; y_i + b < 0 \end{cases} \tag{6}$$

Lemma 1. *The objective function (4) or (5) is the same for $0 \le a \le 1$, $b = 0$ when x_i is replaced by $x'_i(a)$ for all $i = 1, ..., p$.*

Proof. Since $x_i, a \geq 0$, $x_i + a \geq 0$. For $i \neq j$, if both $x'_i(a) = x_i + a$ and $x'_j(a) = x_j + a$, then $x_i - x_j$ does not change. If both are equal to $x_i + a - 1$ and $x_j + a - 1$ then $x_i - x_j$ does not change. If they are different, for example, $x'_i(a) = x_i + a$ and $x'_j(a) = x_j + a - 1$, then $x'_i(a) - x'_j(a) = x_i - x_j + 1$ and $x'_i(a) - x'_j(a) + u = x_i - x_j + u + 1$. The three terms for $u = -1, 0, 1$ are transformed to $u=0$, 1, 2. However, $u=2$ cannot be the minimum because $-1 \leq x_i - x_j \leq 1$, and thus $x_i - x_j + 2 \geq 1$, while the value for $u=0$ is ≤ 1. A similar argument is valid if i and j are exchanged. □

Lemma 2. *Lemma 1 is true for* $-1 \leq a \leq 1$.

Proof. The case $-1 \leq a \leq 0$ can be proved by arguments similar to those of the proof of Lemma 1. However, it is simply true because $x_i = x'_i(a) - a$ for $0 \leq -a \leq 1$, and thus the roles of x_i and $x'_i(a)$ are reversed. □

Lemma 3. *The objective function (4) or (5) is the same for* $-1 \leq b \leq 1$, $a = 0$ *when* y_i *is replaced by* $y'_i(b)$ *for all* $i = 1, ..., p$.

Proof. The case $0 \leq b \leq 1$ is proved the same way as Lemma 1 switching the roles of a and b. The case $-1 \leq b \leq 0$ can be proved by arguments similar to those of the proof of Lemma 2. Since $y_i = y'_i(b) - b$ for $0 \leq -b \leq 1$, the roles of y_i and $y'_i(b)$ are reversed. □

Theorem 1. *The objective function (4) or (5) is the same for* $-1 \leq a, b \leq 1$ *when* x_i *is replaced by* $x'_i(a)$ *and* y_i *is replaced by* $y'_i(b)$ *for all* $i = 1, ..., p$.
Proof. Follows Lemmas 2 and 3 by sequentially transforming x and y. □

Theorem 2. *In addition to transformation (6), the following transformations for every* $1 \leq i \leq p$, *and all sequential applications of them, maintain the value of the objective function (4) or (5):*

(a) *Any permutation of the indices.*
(b) $x'_i = 1 - x_i$; $y'_i = y_i$ *(reflection by the vertical line through the centre of the square).*
(c) $x'_i = x_i$; $y'_i = 1 - y_i$ *(reflection by the horizontal line through the centre of the square).*
(d) $x'_i = y_i$; $y'_i = x_i$ *(reflection by the diagonal line through the centre of the square).*

Proof. (a) A permutation just changes the order in the outer min in (4) or (5). Let the transformed (u, v) be (u', v'). (b) $0 \leq x'_i \leq 1$ and $u' = -u$. (c) Similar to (b). (d) Use $u' = v$ and $v' = u$. □

As a consequence of Theorems 1 and 2 we can show that there is an optimal solution (among an infinite number of optimal solutions) that satisfies the following properties:

Property 1. *There is an optimal solution for which* $x_1 = y_1 = 0$.

Proof. For $a = -x_1$ and $b = -y_1$, the transformed points are $x_1'(a) = 0$ and $y_1'(b) = 0$ by (6). The objective function does not change by Theorem 1. □

Property 2. *There is an optimal solution that satisfies Property 1 and satisfies* $x_2 \le y_2$, $x_3 \le y_3 \ldots x_j \le y_j$ *for* $j = \lfloor \frac{p}{2} \rfloor + 1$.

Proof. First we transform the solution so that $x_1 = y_1 = 0$. Of the remaining $p - 1$ points $2 \le i \le p$ at least half either satisfy $x_i \le y_i$, or $y_i \le x_i$. If the majority satisfies $y_i \le x_i$, apply transformation (d) in Theorem 2, so that after the transformation the majority satisfies $x_i \le y_i$. Then, by property (a) in Theorem 2, we can sort these points to be points $2, 3, \ldots, j$ which satisfy the property. □

The following theorem is obvious:

Theorem 3. *The optimal minimal distance* d *for* $p = p_2 > p_1$ *cannot exceed the optimal minimum distance for* $p = p_1$.

Proof. Removing $p_2 - p_1$ points from the configuration with a minimum distance d for $p = p_2$, yields a configuration for $p = p_1$ points with a minimum distance of at least d. Therefore, the optimal configuration for $p = p_1$ cannot have a minimum distance smaller than d. □

Let $D(p)$ be the best known solution (the minimum distance) for locating p points.

Lemma 4. *For locating* $p = 2k^2$ *points with integer* k *and Manhattan distances, there is a configuration with* $D(p) = \frac{1}{k}$.

Proof. Consider the following configuration with a minimum distance of $D(p) = \frac{1}{k}$: a row of k points on the x-axis ($y = 0$) at $x = 0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$; a second row of k points for $y = \frac{1}{2k}$ and $x = \frac{1}{2k}, \frac{3}{2k}, \frac{5}{2k}, \ldots, \frac{2k-1}{2k}$. This configuration of $2k$ points is replicated for increasing values of y by $\frac{1}{k}$ (e.g. the second set of $2k$ points has $y = \frac{1}{k}$ and $\frac{3}{2k}$), and the same values of x, for a total of k configurations of $2k$ points each with a total of $2k^2$ points. □

For illustration, a configuration for $p = 50$ ($k = 5$) is depicted in Figure 2.

Theorem 4. $D(p) \ge \frac{1}{\lceil \sqrt{\frac{1}{2}p} \rceil}$. *(*$\lceil x \rceil$ *is* x *rounded up) for Manhattan distances.*

Proof. Define $k = \lceil \sqrt{\frac{1}{2}p} \rceil$. Since $2\lceil \sqrt{\frac{1}{2}p} \rceil^2 = 2k^2 \ge p$, by Lemma 4 there is a configuration of $2k^2$ points with a minimum distance of $\frac{1}{k}$. The theorem follows Theorem 3: for $p \le 2k^2$ there is a configuration of distance $\frac{1}{k}$ by removing any $2k^2 - p$ points from the $2k^2$ configuration. □

For example, a lower bounds for $p = 2$ is $L = 1$, for $p = 8$ it is $L = \frac{1}{2}$, for $p = 18$ it is $L = \frac{1}{3}$. All these configurations are optimal as reported later in Table 4.

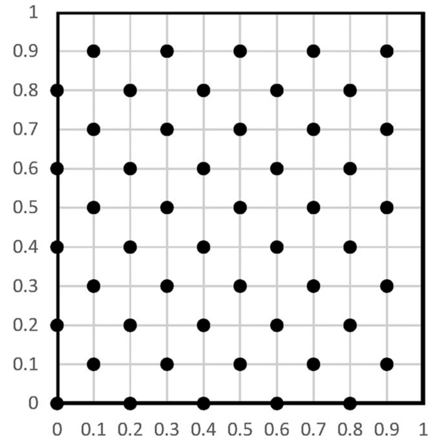

Figure 2. A configuration of 50 points.

Theorem 5. *$D(k^2p) \geq \frac{1}{k}D(p)$, for an integer k, for both Euclidean and Manhattan Distances.*

Proof. Generate a configuration of k by k squares. In each square place the configuration of p points, getting a configuration of k^2p points in a square with side equal k, and with a minimum distance $D(p)$ between all points and points in adjacent squares. Then divide all coordinates by k and obtain a configuration in a square of side 1 and a minimum distance between all points and points in adjacent squares of $\frac{1}{k}D(p)$. This is a lower bound for $D(k^2p)$. □

Since $D(2) = 1$ for Manhattan distances (see Table 4), Theorem 4 is a special case of Theorem 5 for Manhattan distances by substituting $p = 2$ in Lemma 4. For example, the configuration for $p = 50$ in Figure 2, consists of 25 little squares each having the optimal $p = 2$ configuration which is a point on the lower left vertex and a point at the centre of the square. The case of $p = 100$ gives by Theorem 4 a lower bound $D(100) = \frac{1}{8} = 0.125$. Applying Theorem 5 for $p = 25$, $k = 2$ we get a lower bound $D(100) = \frac{1}{2}D(25) = 0.137097$ because $D(25) = 0.274194$ (see Table 5).

2.2. A configuration example using Euclidean distances

Six of the seven points in one of the optimal configurations for $p = 7$ using Euclidean distances, depicted in Figure 3, is a configuration of a square and two equilateral triangles. If the sides of the square and the triangles are equal to x, then the vertical line satisfies $x + 2\frac{\sqrt{3}}{2}x = 1$ yielding $x = \frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2} = 0.366025$. The seventh point should be located in the 'hole' on the top right quarter of the square.

The feasible area for the seventh point, to maintain the value of the objective function, is outside circles of radius $\frac{\sqrt{3}-1}{2}$ depicted in Figure 4.

There is an area outside all circles and any point in that area is suitable for the optimal location of the seventh point. We selected the centre of that area for the seventh point and got the configuration in Figure 5 after shifting the configuration right

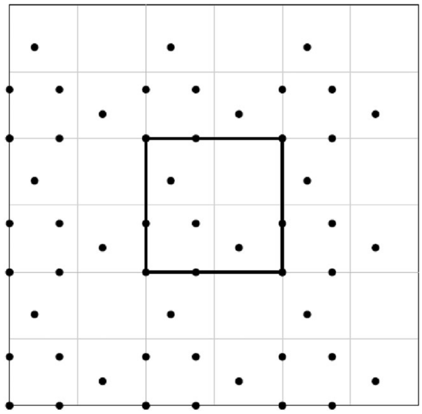

Figure 3. Six of the seven points in the $p = 7$ optimal pattern.

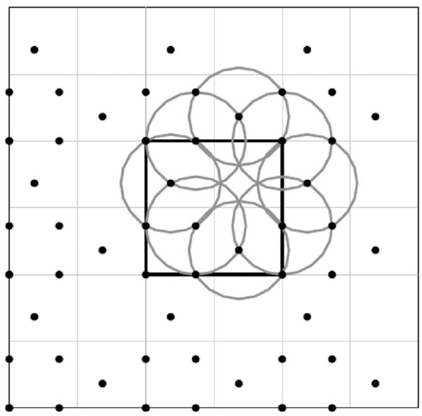

Figure 4. The non-feasible circles.

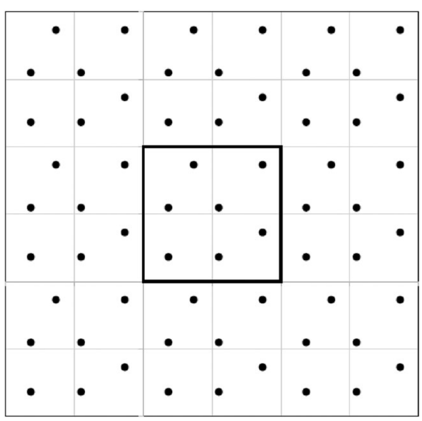

Figure 5. One possible $p = 7$ optimal configuration.

and up. The Gurobi (Gurobi Optimization Incorporated 2018) optimal configuration reported in Table 1, had the seventh point in that area but not at the middle point.

3. Non-linear formulations

There are many possible non-linear and mixed binary formulations that can be implemented in available solvers. Different formulations are proposed for Euclidean and Manhattan distances. Let L be the minimum distance (squared distance for Euclidean) between points in the same square and the eight adjacent squares. The objective is to maximize L.

3.1. Euclidean distances

$$\max\{\ L\ \} \tag{7}$$

Subject to :

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{8}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{9}$$

$$(x_i - x_j - 1)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{10}$$

$$(x_i - x_j)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{11}$$

$$(x_i - x_j)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{12}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{13}$$

$$(x_i - x_j - 1)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{14}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{15}$$

$$(x_i - x_j - 1)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{16}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 1 \leq i \leq p$$

By property (a) in Theorem 2, the points can be sorted so that $x_1 \leq x_2 \leq \cdots \leq x_p$. So, $(x_i - x_j - 1)^2 > (x_i - x_j)^2$ and constraints with $(x_i - x_j - 1)^2$ are not required, getting

$$\max\{\ L\ \} \tag{17}$$

Subject to :

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{18}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{19}$$

$$(x_i - x_j)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{20}$$

$$(x_i - x_j)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{21}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{22}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{23}$$

$$x_k \leq x_{k+1} \quad \text{for } 1 \leq \mathrm{k} \leq p - 1 \tag{24}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 1 \leq i \leq p$$

In addition, we can reduce the number of variables and constraints by setting $x_1 = y_1 = 0$ by Property 1 and get:

Euclidean Formulation (Euc)

$$\max\{ L \} \tag{25}$$

Subject to :

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{26}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j)^2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{27}$$

$$(x_i - x_j)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{28}$$

$$(x_i - x_j)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{29}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j + 1)^2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{30}$$

$$(x_i - x_j + 1)^2 + (y_i - y_j - 1)^2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{31}$$

$$x_j^2 + y_j^2 \geq L \quad \text{for} \quad 2 \leq j \leq p \tag{32}$$

$$(1 - x_j)^2 + y_j^2 \geq L \quad \text{for} \quad 2 \leq j \leq p \tag{33}$$

$$x_j^2 + (1 - y_j)^2 \geq L \quad \text{for} \quad 2 \leq j \leq p \tag{34}$$

$$(1 - x_j)^2 + (1 - y_j)^2 \geq L \quad \text{for} \quad 2 \leq j \leq p \tag{35}$$

$$x_k \leq x_{k+1} \quad \text{for } 2 \leq \text{k} \leq p - 1 \tag{36}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 2 \leq i \leq p$$

These non-linear formulations can be heuristically solved in a multi-start approach using non-linear solvers such as SNOPT (Gill et al. 2005). These formulations are non-convex non-linear programs with reverse convex constraints and can also be solved in a multi-start approach by Multi-start Sequential Linear Programming (MSLP, Drezner and Kalczynski 2020). A feasible random starting solution (such as p points in the square and $L=0$) is generated. Each constraint is replaced by its tangent plane at the starting solution and the resulting linear program solved. The result is used as a starting solution for the next iteration until convergence. Drezner and Kalczynski (2020) proved that for this type of objective and constraints the procedure terminates at a local optimum.

3.2. Manhattan distances

The non-linear formulation is:

$$\max\{ L \} \tag{37}$$

Subject to :

$$|x_i - x_j| + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{38}$$

$$|x_i - x_j + 1| + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{39}$$

$$|x_i - x_j - 1| + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{40}$$

$$|x_i - x_j| + |y_i - y_j + 1| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{41}$$

$$|x_i - x_j| + |y_i - y_j - 1| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{42}$$

$$|x_i - x_j + 1| + |y_i - y_j + 1| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{43}$$

$$|x_i - x_j - 1| + |y_i - y_j + 1| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{44}$$

$$|x_i - x_j + 1| + |y_i - y_j - 1| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{45}$$

$$|x_i - x_j - 1| + |y_i - y_j - 1| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{46}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 1 \leq i \leq p$$

We simplify this formulation. First, because $0 \leq x_i, x_j \leq 1$, $|x_i - x_j + 1| = x_i - x_j + 1$, and $|x_i - x_j - 1| = 1 - x_i + x_j$, and the same for the y's. Therefore, the formulation is equivalent to:

$$\max\{\ L\ \} \tag{47}$$

Subject to :

$$|x_i - x_j| + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{48}$$

$$x_i - x_j + 1 + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{49}$$

$$x_j - x_i + 1 + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{50}$$

$$|x_i - x_j| + y_i - y_j + 1 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{51}$$

$$|x_i - x_j| + y_j - y_i + 1 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{52}$$

$$x_i - x_j + y_i - y_j + 2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{53}$$

$$x_j - x_i + y_i - y_j + 2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{54}$$

$$x_i - x_j + y_j - y_i + 2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{55}$$

$$x_j - x_i + y_j - y_i + 2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{56}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 1 \leq i \leq p$$

By Theorem 2 condition (a), one of the optimal solutions satisfies $x_1 \leq x_2 \leq \cdots \leq x_p$ (we will add the constraint $x_1 = y_1 = 0$ at the end of the derivation). With this condition, $x_j \geq x_i$. In addition, $x_j - x_i + 1 > x_j - x_i$, and three constraints can be removed leading to:

$$\max\{\ L\ \} \tag{57}$$

Subject to :

$$x_j - x_i + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{58}$$

$$x_i - x_j + 1 + |y_i - y_j| \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{59}$$

$$x_j - x_i + y_i - y_j + 1 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{60}$$

$$x_j - x_i + y_j - y_i + 1 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{61}$$

$$x_i - x_j + y_i - y_j + 2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{62}$$

$$x_i - x_j + y_j - y_i + 2 \geq L \quad \text{for} \quad 1 \leq i < j \leq p \tag{63}$$

$$x_k \leq x_{k+1} \text{ for } k = 1, \ldots, p-1 \tag{64}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 1 \leq i \leq p$$

By Property 1, a solution can be transformed to an equivalent one so that $x_1 = y_1 = 0$. Consequently, for $i = 1$ we remove two variables $x_i = y_i = 0$ and replace $|y_1 - y_j|$ by y_j. When finding an optimal solution for $x_1 = y_1 = 0$, only $2(p-1)$ x, y variables are needed. We get:

$$\max\{ L \} \tag{65}$$

Subject to :

$$x_j - x_i + |y_i - y_j| \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{66}$$

$$x_i - x_j + 1 + |y_i - y_j| \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{67}$$

$$x_j - x_i + y_i - y_j + 1 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{68}$$

$$x_j - x_i + y_j - y_i + 1 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{69}$$

$$x_i - x_j + y_i - y_j + 2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{70}$$

$$x_i - x_j + y_j - y_i + 2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{71}$$

$$x_k \leq x_{k+1} \text{ for } k = 2, \ldots, p-1 \tag{72}$$

$$x_j + y_j \geq L \text{ for } 2 \leq j \leq p \tag{73}$$

$$y_j + 1 - x_j \geq L \text{ for } 2 \leq j \leq p \tag{74}$$

$$x_j + 1 - y_j \geq L \text{ for } 2 \leq j \leq p \tag{75}$$

$$2 - x_j - y_j \geq L \text{ for } 2 \leq j \leq p \tag{76}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 2 \leq i \leq p$$

One way to accommodate the only non-linear terms $|y_i - y_j|$, is to define additional variables r_{ij} for $i \neq j$. We add the constraints $r_{ij} \leq y_i - y_j + 1$, $r_{ij}r_{ji} = 0$, then $|y_i - y_j|$ can be replaced by $r_{ij} + r_{ji} - 1$, because $r_{ij} + r_{ji} - 1$ achieves its maximum possible value of $|y_i - y_j|$. The resulting formulation is:

Manhattan Formulation 1 (Man1)

$$\max\{ L \} \tag{77}$$

Subject to :

$$x_j - x_i + r_{ij} + r_{ji} - 1 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{78}$$

$$x_i - x_j + r_{ij} + r_{ji} \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{79}$$

$$x_j - x_i + y_i - y_j + 1 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{80}$$

$$x_j - x_i + y_j - y_i + 1 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{81}$$

$$x_i - x_j + y_i - y_j + 2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{82}$$

$$x_i - x_j + y_j - y_i + 2 \geq L \quad \text{for} \quad 2 \leq i < j \leq p \tag{83}$$

$$0 \leq r_{ij} \leq y_i - y_j + 1 \text{ for } 2 \leq i \neq j \leq p \tag{84}$$

$$r_{ij}r_{ji} = 0 \text{ for } 2 \leq i < j \leq p \tag{85}$$

$$x_j + y_j \geq L \text{ for } 2 \leq j \leq p \tag{86}$$

$$y_j + 1 - x_j \geq L \text{ for } 2 \leq j \leq p \tag{87}$$

$$x_j + 1 - y_j \geq L \text{ for } 2 \leq j \leq p \tag{88}$$

$$2 - x_j - y_j \geq L \text{ for } 2 \leq j \leq p \tag{89}$$

$$x_k \leq x_{k+1} \text{ for } k = 2, \ldots, p-1 \tag{90}$$

$$0 \leq x_i \leq 1; \quad 0 \leq y_i \leq 1 \quad \text{for } 2 \leq i \leq p$$

3.2.1. Binary formulations

The only non-linear constraints in formulation (77) are $r_{ij}r_{ji} = 0$. They can be linearized by introducing binary variables θ_{ij} for $2 \le i < j \le p$. Since $r_{ij}, r_{ji} \le 2$, constraint $r_{ij}r_{ji} = 0$ can be replaced by $r_{ij} \le 2\theta_{ij}$ and $r_{ji} \le 2(1-\theta_{ij})$. If $\theta_{ij} = 1$, then $r_{ji} = 0$, and if $\theta_{ij} = 0$, then $r_{ij} = 0$, thus forcing $r_{ij}r_{ji} = 0$. This results in a mixed binary linear program that can be solved to optimality for problems of reasonable size by CPLEX or Gurobi.

Manhattan Formulation 2 (Man2)

$$\max\{\ L\ \} \tag{91}$$

Subject to :

$$x_j - x_i + r_{ij} + r_{ji} - 1 \ge L \quad \text{for} \quad 2 \le i < j \le p \tag{92}$$

$$x_i - x_j + r_{ij} + r_{ji} \ge L \quad \text{for} \quad 2 \le i < j \le p \tag{93}$$

$$x_j - x_i + y_i - y_j + 1 \ge L \quad \text{for} \quad 2 \le i < j \le p \tag{94}$$

$$x_j - x_i + y_j - y_i + 1 \ge L \quad \text{for} \quad 2 \le i < j \le p \tag{95}$$

$$x_i - x_j + y_i - y_j + 2 \ge L \quad \text{for} \quad 2 \le i < j \le p \tag{96}$$

$$x_i - x_j + y_j - y_i + 2 \ge L \quad \text{for} \quad 2 \le i < j \le p \tag{97}$$

$$0 \le r_{ij} \le y_i - y_j + 1 \ \text{for}\ 2 \le i \ne j \le p \tag{98}$$

$$r_{ij} \le 2\theta_{ij}; \quad r_{ji} \le 2(1-\theta_{ij}) \ \text{for}\ 2 \le i < j \le p \tag{99}$$

$$x_j + y_j \ge L \ \text{for}\ 2 \le j \le p \tag{100}$$

$$y_j + 1 - x_j \ge L \ \text{for}\ 2 \le j \le p \tag{101}$$

$$x_j + 1 - y_j \ge L \ \text{for}\ 2 \le j \le p \tag{102}$$

$$2 - x_j - y_j \ge L \ \text{for}\ 2 \le j \le p \tag{103}$$

$$x_k \le x_{k+1} \ \text{for}\ k = 2, \ldots, p-1 \tag{104}$$

$$0 \le x_i \le 1; \quad 0 \le y_i \le 1 \quad \text{for}\ 2 \le i \le p \tag{105}$$

$$\theta_{ij} \in \{0,1\} \quad \text{for } 2 \le i < j \le p \ .$$

Another option is to formulate it without the r_{ij} variables and without the constraints associated with them, getting a much smaller mixed binary program, but with products of binary and continuous variables.

$$\begin{aligned} |y_i - y_j| &= \max\{y_i - y_j, y_j - y_i\} = \theta_{ij}(y_i - y_j) + (1 - \theta_{ij})(y_j - y_i) \\ &= y_j - y_i + 2\theta_{ij}(y_i - y_j) \end{aligned}$$

Since the objective function is a maximization, the solver will 'choose' the θ_{ij} that yields the maximum, which is the absolute value of $y_i - y_j$. The formulation is:

Manhattan Formulation 3 (Man3)

$$\max\{\ L\ \} \tag{106}$$

Subject to :

$$x_j - x_i + y_j - y_i + 2\theta_{ij}(y_i - y_j) \ \ge\ L \quad \text{for} \quad 2 \le i < j \le p \tag{107}$$

$$x_i - x_j + 1 + y_j - y_i + 2\theta_{ij}(y_i - y_j) \ \ge\ L \quad \text{for} \quad 2 \le i < j \le p \tag{108}$$

$$x_j - x_i + y_i - y_j + 1 \ \ge\ L \quad \text{for} \quad 2 \le i < j \le p \tag{109}$$

$$x_j - x_i + y_j - y_i + 1 \ \ge\ L \quad \text{for} \quad 2 \le i < j \le p \tag{110}$$

$$x_i - x_j + y_i - y_j + 2 \ \ge\ L \quad \text{for} \quad 2 \le i < j \le p \tag{111}$$

$$x_i - x_j + y_j - y_i + 2 \ \ge\ L \quad \text{for} \quad 2 \le i < j \le p \tag{112}$$

$$x_j + y_j \ \ge\ L \text{ for } 2 \le j \le p \tag{113}$$

$$y_j + 1 - x_j \ \ge\ L \text{ for } 2 \le j \le p \tag{114}$$

$$x_j + 1 - y_j \ \ge\ L \text{ for } 2 \le j \le p \tag{115}$$

$$2 - x_j - y_j \ \ge\ L \text{ for } 2 \le j \le p \tag{116}$$

$$x_k \ \le\ x_{k+1} \text{ for } k = 2, \ldots, p-1 \tag{117}$$

$$0 \le x_i \le 1; \quad 0 \le y_i \le 1 \quad \text{for } 2 \le i \le p \tag{118}$$

Table 1. Results for smaller problems with Euclidean distances.

		Gurobi		SNOPT (100 runs)			MSLP (100 runs)		
p	Best found	†	Time‡	†	*	Time‡	†	*	Time‡
2	0.707107	0%	0.0	0%	100	0.5	0%	100	0.9
3	0.517638	0%	0.1	0%	83	0.7	0%	72	1.5
4	0.517638	0%	0.2	0%	79	1.2	0%	76	1.8
5	0.447214	0%	0.6	0%	57	0.7	0%	67	2.3
6	0.400406	0%	3.8	0%	39	1.1	0%	58	2.8
7	0.366026	0%	656.7	0%	59	1.4	0%	85	3.2
8	0.366026	0%	286.2	0%	33	1.7	0%	58	4.3
9	0.343724	0%	1,504.9	0%	39	2.4	0%	49	5.3
10	0.316228	0%	730,930.0	0%	4	2.6	0%	2	5.7
11	0.305449	0.71%	18,000.0	0%	7	3.1	0%	19	6.9
12	0.300463	0.37%	18,000.0	0%	56	4.0	0%	66	8.5
13	0.278147	2.15%	18,000.0	0%	6	5.2	0%	7	9.6
14	0.276179	4.88%	18,000.0	0%	23	5.4	0%	23	11.2
15	0.274875	8.05%	18,000.0	0%	7	6.8	0%	8	13.3
16	0.258818	3.90%	18,000.0	0%	34	9.6	0%	25	16.5
17	0.245116	5.03%	18,000.0	0%	1	11.0	0.16%	3	17.1
18	0.243049	7.78%	18,000.0	0%	3	14.0	0%	6	21.9
19	0.232119	4.83%	18,000.0	0.01%	2	17.1	0%	2	22.4
20	0.231495	8.77%	18,000.0	0%	2	21.7	0%	2	29.1

†Percent below best found distance.
‡Run time in seconds for all runs.
*Number of times out of 100 that the best observed solution was found.

$$\theta_{ij} \in \{0, 1\} \quad \text{for } 2 \leq i < j \leq p$$

4. Computational experiments

The MIP formulations were solved by Gurobi 9.1 (Gurobi Optimization Incorporated 2018) on a virtualized Windows environment with 16 vCPUs and 128GB of vRAM. The physical server used was a 2 CPU (8 cores each) PowerEdge R720 Intel E5-2650 CPUs with 128 GB RAM using shared storage on MD3620i *via* 10GB interfaces. We used the default MIP solver settings. We also tested the procedures on CPLEX (CPLEX and IBM ILOG 2019), but it performed poorly, and therefore CPLEX's results are not reported.

4.1. Results for Euclidean distances

The results using Gurobi 9.1 (Gurobi Optimization Incorporated 2018), and the heuristic multi-start approaches SNOPT (Gill et al. 2005) and MSLP (Drezner and Kalczynski 2020) are reported in Tables 1–3.

Gurobi found the optimal solutions for $p \leq 10$. Run time for $p = 10$ was about eight and a half days. Runs for $p > 10$ were stopped after 5 h and the best result recorded. The results for $p > 20$ reported in Table 2 are not usable.

The MSLP approach performed better than SNOPT especially for larger values of p. For $p \leq 20$ and 100 starting solutions (Table 1) SNOPT performed slightly better. For $p > 20$ and 100 starting solutions MSLP performed better both in solution quality and run times. For example, for $p = 100$ MSLP was more than 8 times faster.

Table 2. Results for larger problems with Euclidean distances.

		Gurobi		SNOPT (100 runs)			MSLP (100 runs)		
p	Best found	†	Time‡	†	*	Time‡	†	*	Time‡
25	0.206758	11.51%	18,000.0	0%	3	60.3	0%	5	47.1
30	0.194366	26.48%	18,000.0	0%	31	114.0	0%	49	75.8
35	0.174456	27.76%	18,000.0	0%	6	215.9	0%	3	109.2
40	0.163689	36.58%	18,000.0	0%	1	476.6	0%	1	173.8
45	0.157623	39.40%	18,000.0	0.03%	1	631.2	0%	1	249.4
50	0.146632	21.08%	18,000.0	0.14%	1	991.7	0.17%	1	352.2
55	0.142902	24.90%	18,000.0	0%	1	1,502.1	0%	1	481.9
60	0.137455	19.02%	18,000.0	0.01%	1	2,232.2	0.01%	1	674.7
65	0.129337	32.07%	18,000.0	0.13%	1	2,948.7	0.10%	1	776.4
70	0.125200	48.49%	18,000.0	0.10%	2	3,949.8	0.06%	1	1,055.1
75	0.122927	92.37%	18,000.0	0.10%	1	5,338.6	0%	1	1,359.8
80	0.117924	99.15%	18,000.0	0%	1	6,528.5	0%	3	901.2
85	0.114787	91.55%	18,000.0	0.72%	1	7,905.7	0.13%	1	1,096.1
90	0.111548	99.97%	18,000.0	0.06%	1	9,650.8	0%	1	1,246.3
95	0.107285	99.97%	18,000.0	0.31%	1	11,393.7	0.44%	1	1,538.1
100	0.105071	99.97%	18,000.0	0.26%	1	13,291.6	0.24%	1	1,570.3

†Percent below best found distance.
‡Run time in seconds for all runs.
*Number of times out of 100 that the best observed solution was found.

Table 3. Results from 1000 random starting solutions.

		SNOPT			MSLP		
p	Best found	†	*	Time‡	†	*	Time‡
2	0.707107	0%	1000	3.7	0%	1000	9.7
3	0.517638	0%	886	6.6	0%	736	14.9
4	0.517638	0%	780	10.1	0%	708	18.2
5	0.447214	0%	534	8.0	0%	569	22.0
6	0.400406	0%	442	11.1	0%	567	27.6
7	0.366026	0%	622	12.9	0%	853	31.8
8	0.366026	0%	368	19.1	0%	533	42.6
9	0.343724	0%	411	21.8	0%	516	52.1
10	0.316228	0%	28	26.0	0%	37	57.3
11	0.305449	0%	64	32.0	0%	241	69.9
12	0.300463	0%	503	38.9	0%	580	84.3
13	0.278147	0%	44	50.6	0%	38	98.6
14	0.276179	0%	194	57.5	0%	236	110.6
15	0.274875	0%	73	70.7	0%	119	132.9
16	0.258818	0%	267	91.6	0%	258	159.6
17	0.245116	0%	12	108.6	0%	21	173.3
18	0.243049	0%	33	140.3	0%	48	207.2
19	0.232119	0%	7	170.4	0%	6	223.6
20	0.231495	0%	6	222.6	0%	6	277.6
25	0.206758	0%	31	587.9	0%	25	468.7
30	0.194366	0%	293	1,109.9	0%	469	740.4
35	0.174456	0%	38	2,087.3	0%	27	1,132.3
40	0.163689	0%	5	3,745.7	0%	1	1,658.6
45	0.157623	0%	2	6,399.5	0%	1	2,400.5
50	0.146632	0%	1	10,039.6	0%	1	3,423.2
55	0.142902	0%	4	15,616.0	0%	12	4,746.1
60	0.137455	0%	3	22,636.3	0%	2	6,596.0
65	0.129337	0.09%	1	30,268.9	0%	1	8,224.2
70	0.125200	0.05%	1	40,339.2	0%	3	10,358.8
75	0.122927	0%	3	53,248.5	0%	5	13,613.5
80	0.117924	0%	22	65,156.7	0%	29	9,327.6
85	0.114787	0%	2	81,011.3	0%	1	11,665.2
90	0.111548	0%	3	97,656.6	0%	1	12,953.7
95	0.107285	0.16%	1	115,263.4	0%	1	16,493.5
100	0.105071	0.16%	1	135,102.6	0%	2	15,479.2

†Percent below best found distance.
‡Run time in seconds for all runs.
*Number of times out of 1000 that the best observed solution was found.

Table 4. Optimal results for Manhattan distances by Gurobi.

p	Optimum	†	Man1‡	Man2‡	Man3‡	Best*	Euclid.**	%***
2	1.000000	$\frac{1}{1}$	0.17	0.00	0.02	0.707107	0.707107	0%
3	0.666667	$\frac{2}{3}$	0.14	0.03	0.05	0.517638	0.471405	8.93%
4	0.666667	$\frac{2}{3}$	0.17	0.03	0.05	0.517638	0.471405	8.93%
5	0.600000	$\frac{3}{5}$	0.16	0.04	0.06	0.447214	0.447214	0%
6	0.500000	$\frac{1}{2}$	0.17	0.05	0.06	0.400406	0.353553	11.70%
7	0.500000	$\frac{1}{2}$	0.20	0.05	0.08	0.366026	0.353553	3.41%
8	0.500000	$\frac{1}{2}$	0.28	0.06	0.09	0.366026	0.353553	3.41%
9	0.428571	$\frac{3}{7}$	0.40	0.24	0.36	0.343724	0.319438	7.07%
10	0.400000	$\frac{2}{5}$	0.58	0.32	0.42	0.316228	0.282843	10.56%
11	0.400000	$\frac{2}{5}$	0.88	0.36	0.51	0.305449	0.282843	7.40%
12	0.400000	$\frac{2}{5}$	2.80	0.63	0.73	0.300463	0.282843	5.86%
13	0.384615	$\frac{5}{13}$	6.26	5.27	6.53	0.278147	0.277350	0.29%
14	0.357143	$\frac{5}{14}$	81.40	16.94	21.36	0.276179	0.257539	6.75%
15	0.333333	$\frac{1}{3}$	105.49	31.19	39.11	0.274875	0.235702	14.25%
16	0.333333	$\frac{1}{3}$	304.78	72.62	82.51	0.258818	0.235702	8.93%
17	0.333333	$\frac{1}{3}$	295.00	149.46	185.73	0.245116	0.235702	3.69%
18	0.333333	$\frac{1}{3}$	1057.90	150.39	188.58	0.243049	0.235702	3.02%
19	0.300000	$\frac{3}{10}$	\$\$	381,469.00	\$\$	0.232119	0.223607	3.67%
20	0.300000\$	$\frac{3}{10}$	\$\$	\$\$	\$\$	0.231495	0.223607	3.41%

†Result as a fraction.
‡Run time in seconds.
*Best found Euclidean solution (Table 1).
**Euclidean distance of the Manhattan solution.
***Percent of Euclidean distances below best Euclidean.
\$A configuration with minimum distance of 0.3 is optimal by Theorem 3.
\$\$Stopped after 5 h with a solution of 0.3.

Table 5. Results for Manhattan distances for large problems (stopped after 5 h).

p	Best‡	*	Man1†	Man2†	Man3†
25	0.274194	$\frac{17}{62}$	0.54%	0%	0%
30	0.250000	$\frac{1}{4}$	0%	0%	0%
35	0.222222	$\frac{2}{9}$	0%	0%	0%
40	0.217391	$\frac{5}{23}$	4.83%	0.68%	0%
45	0.200000	$\frac{1}{5}$	0%	0%	0%
50	0.200000	$\frac{1}{5}$	9.09%	0%	4.76%
55	0.179487	$\frac{7}{39}$	1.77%	0%	0%
60	0.166667	$\frac{1}{6}$	0%	0%	0%
65	0.166667	$\frac{1}{6}$	7.69%	0%	0%
70	0.166667	$\frac{1}{6}$	11.93%	4.00%	5.49%
75	0.153846	$\frac{2}{13}$	7.14%	0%	1.68%
80	0.150000	$\frac{3}{20}$	8.50%	3.47%	3.47%
85	0.142857	$\frac{1}{7}$	12.11%	0%	0%
90	0.142857	$\frac{1}{7}$	16.42%	4.32%	4.55%
95	0.142857	$\frac{1}{7}$	24.27%	6.67%	6.67%
100	0.137097	$\frac{17}{124}$	24.00%	7.13%	7.13%

*Result as a fraction.
‡Best known solution including the results by Theorems 4 and 5.
†Percent below best found solution.

In Table 3 the results for SNOPT and MSLP from 1000 starting solutions are reported. MSLP found the best known solutions in all runs and SNOPT missed four of them. Run times by MSLP are faster for larger problems. The largest problem required 4.3 h while SNOPT solved it in 37.5 h.

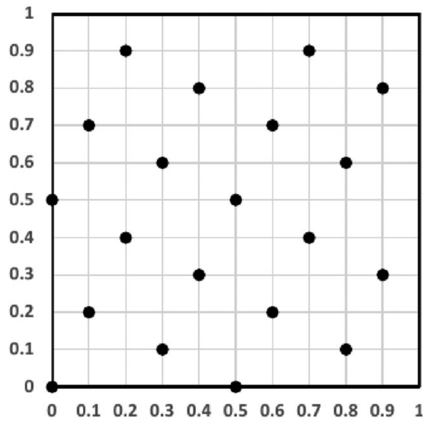

Figure 6. Optimal configuration of 20 points (Manhattan distances).

Table 6. Results for Manhattan distances by SNOPT from 100 starting solutions.

p	†	‡	Time*
2	0%	100	2.1
3	0%	100	2.4
4	0%	65	2.8
5	0%	14	3.3
6	0%	65	3.8
7	0%	41	4.6
8	0%	6	5.1
9	6.67%	55	23.9
10	0%	21	6.6
11	0%	3	31.9
12	0%	1	8.3
13	13.33%	43	9.2
14	6.67%	21	9.9
15	0%	10	10.5
16	0%	3	11.8
17	7.69%	1	13.6
18	12.35%	1	15.9
19	4.76%	3	16.0
20	4.76%	1	15.9
25	8.82%	3	30.1
30	11.11%	2	50.6
35	10.00%	8	74.6
40	13.85%	1	115.0
45	10.89%	1	178.4
50	13.79%	1	245.2
55	12.95%	1	367.4
60	10.45%	2	546.3
65	12.46%	1	815.7
70	16.71%	1	1,028.1
75	14.22%	1	1,427.5
80	12.77%	1	1,958.7
85	13.50%	1	2,477.5
90	17.27%	1	3,859.9
95	16.67%	1	4,597.2
100	16.20%	1	5,855.2

†Percent of the best found solution below best known solution.
‡Number of times out of 100 that the best solution was found.
*Run time in seconds for all 100 runs.

4.2. Results for Manhattan distances

In Table 4 the results obtained by the three Manhattan formulations for $2 \leq p \leq 20$ are reported, All results are optimal. The results up to $p = 18$ were optimally solved in a reasonable run time. The $p = 19$ instance was solved optimally by Man2 in about 106 h. An objective of 0.3 for $p = 20$ was obtained after 5 h by all three formulations. Also, by Theorem 5 for $k = 2; p = 5$ there is a feasible solution with an objective of 0.3. Since the $p = 19$ optimal solution is also 0.3, it must be optimal for $p = 20$ by Theorem 3. See Figure 6 for the optimal configuration. The square is divided into four small squares, each having the optimal $p = 5$ configuration: a point at a corner, and four points in a tilted square. We also compared the configurations as possible ones of Euclidean distances, but except for $p = 2$ and 5, the best found Euclidean configurations have a better objective function value.

In Table 5 results for larger instances for $25 \leq p \leq 100$ are reported. Learning from our experience of running the $p = 19$ and 20 instances, all runs were stopped after 5 h and the best result reported. In five instances results by Theorems 4 and 5 were better than results obtained by solving the non-linear formulations. These are:

For $p = 70$ $D(70) \geq \frac{1}{6}$ by Theorem 4.

For $p = 80$ $D(80) \geq \frac{1}{2}D(20) = 0.15$ by Theorem 5.

For $p = 90$ $D(90) \geq \frac{1}{7}$ by Theorem 4

For $p = 95$ $D(95) \geq \frac{1}{7}$ by Theorem 4

For $p = 100$ $D(100) \geq \frac{1}{2}D(25) = \frac{0.274194}{2} = 0.137097$ by Theorem 5.

Formulations Man2 and Man3 performed about equally well and clearly better than Man1. In Table 6 the results by SNOPT from 100 randomly generated starting solutions are reported. Run times are reasonable but the values of the objective function are much inferior to the other formulations, solved by Gurobi, that were stopped after 5 h.

5. Conclusions

We propose a variation of the grey pattern problem, which maximizes a measure of the dispersion of selected points in the plane. The grey pattern problem is to select a pattern of p points in a square that is replicated in adjacent squares so that they are spread out as uniformly as possible. Rather than minimizing a specially designed objective function proposed by Taillard (1995), our proposed objective is to maximize the minimum distance between pairs of points located in the same square and the eight adjacent squares. We applied the model with Euclidean and Manhattan distances.

The problem is not convex, and has many local optima. It is formulated as a non-linear program. Properties that simplify the formulation are proved. Extensive experiments with instances of up to 100 points are tested. For Euclidean distances the optimal solution was found by Gurobi for $p \leq 10$ and for Manhattan distances for $p \leq 20$.

Larger instances using Euclidean distances were solved by SNOPT (Gill et al. 2005) and MSLP (Drezner and Kalczynski 2020). MSLP performed considerably better than SNOPT for larger values of p.

Solving optimally the $p = 19$ instance by Gurobi for Manhattan distances required more than four days of computer time. Hence, for large instances ($p \geq 20$) using Manhattan distances Gurobi was stopped after 5 h, and the best found results were reported. In five instances, properties of the solutions proved by Theorems 4 and 5 showed that the results obtained by Gurobi in 5 h were not optimal because better solutions were obtained by applying these theorems.

For future research, we plan to extend the dispersion models presented here to the discrete version of the grey pattern problem.

Disclosure statement

No potential conflict of interest was reported by the authors.

Data availability

No external data exists.

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