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Discrete Optimization

Closest assignment constraints in discrete location problems

Inmaculada Espejo[a], Alfredo Marín[b,*], Antonio M. Rodríguez-Chía[a]

[a] *Departamento de Estadística e Investigación Operativa, Universidad de Cádiz, Spain*
[b] *Departamento de Estadística e Investigación Operativa, Universidad de Murcia, Spain*

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ABSTRACT

The objective of this paper is to identify the most promising sets of *closest assignment constraints* in the literature of Discrete Location Theory, helping the authors in the field to model their problems when clients must be assigned to the closest plant inside an Integer Programming formulation. In particular, constraints leading to weak Linear Programming relaxations should be avoided if no other good property supports their use. We also propose a new set of constraints with good theoretical properties.

1. Introduction

In the following, we state the general framework of a discrete location problem. Consider a set $A=\{1,\dots,n\}$ of points (clients and potential plants), distances $d_{ij} \geqslant 0$ from point i to point j, costs c_{ij} of assigning client i to plant j, and a fixed number of plants to be located, $p > 1$. As a part of a decision process, the set of p locations of the plants to be opened must be decided, and then each client must be assigned (allocated) to one of the plants, minimizing a cost. Part of this cost will be the sum of the assignment costs c_{ij} for all pairs client-assigned plant. A concrete model will probably include additional costs and/or additional constraints about how to assign clients to plants, how to design the set of plants or whatever.

Very frequently, to obtain the best (optimal) solution for a discrete location problem, an Integer Programming formulation is proposed. If this is the case, most of the times binary variables y_j, $j\in A$, taking value 1 iff a plant is opened at point j, and binary variables x_{ij}, $i,j\in A$, taking value 1 iff client i is assigned to plant j, will be included in the model (and perhaps some other variables z_k, $k\in K$ and constraints $t\in T$, where K and T are two arbitrary sets of indexes). Taking c_{ij}, c_j^y and c_k^z the costs associated with variables x_{ij}, y_j and z_k, respectively, and α_{ij}^t, β_j^t and γ_k^t the coefficients in constraint $t\in T$ for variables x_{ij}, y_j and z_k respectively, for any $i,j\in A$, $k\in K$, as well as $K' \subseteq K$ the set of indices of z-variables which must take integer values, the formulation of a general discrete location problem will be in the shape of

$$\min \quad \sum_{i\in A}\sum_{j\in A} c_{ij}x_{ij} + \sum_{j\in A} c_j^y y_j + \sum_{k\in K} c_k^z z_k, \tag{1}$$

$$\text{s.t.} \quad \sum_{j\in A} x_{ij} = 1 \quad \forall i\in A \tag{2}$$

$$x_{ij} \leqslant y_j \quad \forall i,j\in A, \tag{3}$$

$$\sum_{j\in A} y_j = p, \tag{4}$$

$$\sum_{i\in A}\sum_{j\in A} \alpha_{ij}^t x_{ij} + \sum_{j\in A} \beta_j^t y_j + \sum_{k\in K} \gamma_k^t z_k \leqslant \delta^t, \quad \forall t\in T \tag{5}$$

$$x_{ij} \geqslant 0 \quad \forall i,j\in A, \tag{6}$$

$$0 \leqslant y_j \leqslant 1 \quad \forall j\in A, \tag{7}$$

$$x_{ij}\in\mathbb{Z} \quad \forall i,j\in A, \tag{8}$$

$$y_j\in\mathbb{Z} \quad \forall j\in A, \tag{9}$$

$$z_k\in\mathbb{Z} \quad \forall k\in K' \subseteq K. \tag{10}$$

Observe that (2)–(4) are the three classical sets of location constraints when the number of plants is fixed to p.

The p-median problem (Hakimi, 1965 and ReVelle and Swain, 1970) is the simplest problem in this family. It is

$$\begin{aligned} \min \quad & \sum_{i\in A}\sum_{j\in A} d_{ij}x_{ij} \\ \text{s.t.} \quad & (2)-(4),\ (6)-(9) \end{aligned}$$

i.e., the assignment costs are the own distances between points (possibly multiplied by a constant P_i indicating the importance – e.g., population or demand– of client i). Then, once the plants are

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* Corresponding author. Tel.: +34 868 883627.
E-mail address: amarin@um.es (A. Marín).

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chosen, each client i will be assigned to the *closest plant*, the one at a minimum distance from i.

In a generalized version of the p-median problem, when the costs c in the objective function do not match the distances d, after choosing the set of plants each client will be assigned to the *cheapest* plant, i.e., the plant which contributes the minimum cost to the objective function, regardless of the distance this plant is from the client. Moreover, in the case where additional constraints affect the x-variables, assignment of clients to plants can also be independent of the distances d. In those models in which clients must be assigned to their closest plant, it is necessary to incorporate additional constraints (inequalities) to the formulation. These have been named *closest assignment constraints*, from now on CAC. Throughout the last forty years, several models in which CAC are required have been studied and formulated as Integer Programming problems. Once and again, authors have generated new sets of CAC, ignoring most of the times the inequalities previously used in the literature. To the best of our knowledge, no attempt of theoretically comparing all these CAC has been carried out up to date, although Gerrard and Church (1996) dealt with some properties of the CAC posed in the literature at that time. The most recent paper on the topic is Lei and Church (2010), where closest assignment constraints in the context of multi-level assignments are discussed.

It is worth mentioning that closest assignment has also been needed in a recently introduced family of location problems, the so-called *ordered median problems* (see e.g. Kalcsics et al., 2010). Nevertheless, the formulations used to model these problems present a different structure of variables and constraints and are out of the scope of this work. Another very active field of research in which CAC are needed is that of interdiction models (and the related facility hardening models), see for instance Church et al. (2004) and the recent paper by Liberatore et al. (2011). In these models, it is considered that p interdicted plants will not be available and clients must be assigned to the closest available plant.

Closest assignment constraints are also needed in discrete location models where the number of plants is not previously fixed, but to be decided as part of the optimization process. Although a fixed number of plants is initially considered, through the paper we will also pay attention to these other closely related models.

The objective of this paper is to identify the most promising sets of CAC, helping the authors in the Discrete Location field to model their problems. In particular, CAC leading to weak Linear Programming relaxations should be avoided if no other good property supports their use. Among these good properties, there are two which have been partially studied in the literature. The first one is whether or not the constraints are valid in case of tied distances with respect to the same client. The second one is whether or not the addition of CAC makes it possible to relax the integrality of a subset of integer variables. We will show which of the CAC satisfy each of these properties.

With this aim, in Section 2 we review all the sets of constraints used up to now in the literature. In Section 3 we prove the dominance of some sets of constraints over other sets and give examples to show that no other dominances exist in general. Based on one of the proofs of Section 3, in Section 4 we build a new, non-dominated set of CAC and we classify it. Some interesting properties of some CAC are shown in Section 5. Since one of the CAC studied in the literature was not valid in case of ties between distances, we study how to generalize it in Section 6. We close the article with some conclusions in Section 7.

2. Closest assignment constraints in the literature

The earliest work in which CAC were considered in an Integer Programming framework was Rojeski and ReVelle (1970). It was in the context of the budget constrained median problem where these authors proposed the following constraints:

$$x_{ij} + \sum_{a:d_{ia}<d_{ij}} y_a \geqslant y_j \quad \forall i,j \in A. \quad \mathcal{RR}.$$

How constraints $\mathcal{RR}$ work was explained in Plastria (2002) making use of the following construction.

Construction 2.1. If w_i, $i \in I$, are binary variables and z is a non negative variable upperly bounded by H, the logical implication $(w_i = 0 \ \ \forall i \in I \Rightarrow z = 0)$ is exactly expressed by the constraint $z \leqslant H\sum_{i\in I} w_i$.

In the case of $\mathcal{RR}$, the logic is as follows: If a plant is located at j $(1 - y_j = 0)$ and there is no other plant closer to client i than j $(y_a = 0$ $\forall a : d_{ia} < d_{ij})$, then client i must be assigned to plant j $(1 - x_{ij} = 0)$. Consequently, it suffices to replace, in Construction 2.1, H by 1, z by $1 - x_{ij}$ and $\{w_i : i \in I\}$ by $\{1 - y_j\} \cup \{y_a : d_{ia} < d_{ij}\}$ and constraints $\mathcal{RR}$ follow.

This logical reasoning is only valid in the absence of ties. Then, we will assume for the moment that, for a given client $i \in A$, distances d_{ij}, $j \in A$, are all different.

Later on, Wagner and Falkson (1975) introduced an alternative set of constraints for ensuring closest center behavior in an integer-linear location-allocation model, as follows:

$$\sum_{a:d_{ia}>d_{ij}} x_{ia} + y_j \leqslant 1 \quad \forall i,j \in A. \quad \mathcal{WF}$$

Here, the logical reasoning is as follows: If a plant is located at point j $(1 - y_j = 0)$, then the assignment of client i to any plant more distant from i than j is not allowed $(\sum_{a:d_{ia}>d_{ij}} x_{ia} = 0)$. Then, by replacing, in Construction 2.1, H by 1, z by $\sum_{a:d_{ia}>d_{ij}} x_{ia}$ and $\{w_i : i \in I\}$ by $1 - y_j$, $\mathcal{WF}$ follow.

The third linear set of constraints designed to force closest assignment was utilized by Church and Cohon (1976) for siting energy facilities, and afterwards by Hanjoul and Peeters (1987) in plant location models:

$$\sum_{a:d_{ia}\leqslant d_{ij}} x_{ia} \geqslant y_j \quad \forall i,j \in A. \quad \mathcal{CC}$$

The reasoning is: If client i is neither assigned to j nor to a plant that is not further from i than j $\left(\sum_{a:d_{ia}\leqslant d_{ij}} x_{ia} = 0\right)$, there cannot be a plant located at site j $(y_j = 0)$. Then, by replacing, in Construction 2.1, H by 1, z by y_j and $\{w_i : i \in I\}$ by $\{x_{ia} : d_{ia} \leqslant d_{ij}\}$, $\mathcal{CC}$ follow.

$\mathcal{RR}$ and $\mathcal{CC}$ are possibly the most widely referenced CAC (see Church and Roberts (1983), Hanjoul et al. (1990), Scaparra and Church (2008) and Texeira and Antunes (2008), among others).

Competitive location on a network was studied by Dobson and Karmarkar (1987). They introduced the following CAC:

$$x_{ij} + y_a \leqslant 1 \quad \forall i,j,a \in A: \ d_{ia} < d_{ij}. \quad \mathcal{DK}$$

The idea of Constraints $\mathcal{DK}$ is that client i is not assigned to plant j $(x_{ij} = 0)$ if there is another plant closer to i than j $(1 - y_a = 0)$. Hence, for each point a such that $d_{ia} < d_{ij}$, replace, in Construction 2.1, H by 1, z by x_{ij} and $\{w_i : i \in I\}$ by $1 - y_a$ to obtain $\mathcal{DK}$.

Gerrard and Church (1996) studied some structural properties of constraints $\mathcal{RR}, \mathcal{WF}, \mathcal{DK}$ and $\mathcal{CC}$. We will detail below which of our results were previously noted by these authors.

More recently, Cánovas et al. (2007) studied the Simple Plant Location Problem with Order. They considered an improvement of inequalities $\mathcal{WF}$ given by

$$\sum_{a:d_{ia}>d_{ij}} x_{ia} + \sum_{a:d_{ia}\leqslant d_{ij}, d_{ka}>d_{kj}} x_{ka} + y_j \leqslant 1 \quad \forall i,j,k \in A. \quad \mathcal{CGLM}$$

Note that additional x_{ka}-variables are added to the left hand side of $\mathcal{WF}$ which cannot take value 1 if either y_j or x_{ia}: $d_{ij} < d_{ia}$ take value 1.

To view how these constraints work, in Fig. 1 we have represented nodes as points in the real plane and used Euclidean distances. The two circumferences in the picture are centered at node k and i and have radius d_{kj} and d_{ij}, respectively. When $y_j = 1$, allocations drawn in the figure (represented by arrows with origins at i or k) are not possible because of closest assignment, i.e., in the worst case node i and k are allocated to node j (these nodes or some of them could be assigned to a closer node). These allocations are associated to variables x_{ia} and x_{ka} in the corresponding constraint $\mathcal{CGLM}$. When $y_j = 0$, the inequality is valid because all nodes pointed from k are closer to i than all nodes pointed from i and then, if k is allocated to some of these nodes, i cannot be allocated to the nodes which are further.

These constraints can still be obtained by replacing, in Construction 2.1, H by 1, z by $\sum_{a:d_{ia}>d_{ij}} x_{ia} + \sum_{a:d_{ia}\leqslant d_{ij}, d_{ka}>d_{kj}} x_{ka}$ (after noticing that at most one of these variables can take value 1) and $\{w_i : i \in I\}$ by $1 - y_j$.

Belotti et al. (2007) studied the obnoxious p-median problem, where inequalities $\mathcal{WF}$ were used to enforce closest assignment. Among other families of valid inequalities, these authors introduced the following:

$$p \sum_{a:d_{ia}>d_{ij}} x_{ia} \leqslant \sum_{a:d_{ia}>d_{ij}} y_a \quad \forall i,j \in A. \quad \mathcal{BLMN}$$

These constraints also enforce closest assignment. To see it, fix a client i and assume $y_j = 1$ and $y_a = 0$ $\forall a$: $d_{ia} < d_{ij}$. Then, from $\mathcal{BLMN}$, $p\sum_{a:d_{ia}>d_{ij}} x_{ia} \leqslant \sum_{a:d_{ia}>d_{ij}} y_a = p-1$ implying $\sum_{a:d_{ia}>d_{ij}} x_{ia} = 0$. Since (3) imply $\sum_{a:d_{ia}<d_{ij}} x_{ia} = 0$ it follows, from (2), $x_{ij} = 1$. $\mathcal{BLMN}$ can also be obtained, by Construction 2.1, replacing H by p, z by $\sum_{a:d_{ia}\leqslant d_{ij}} y_a$ and $\{w_i : i \in I\}$ by $\{x_{ia}$: $d_{ia} \leqslant d_{ij}\}$, and then using equalities (2) and (4).

Berman et al. (2009) provided the following CAC in the context of equitable location:

$$\sum_{a\in A} d_{ia}x_{ia} + (M - d_{ij})y_j \leqslant M \quad \forall i,j \in A, \quad \mathcal{BDTW}$$

where M is a large constant (it could be chosen $M = \max_i \sum_{a\in A} d_{ia}$). Here, if a plant is opened at j ($y_j = 1$), the assignment distance of customer i given by $\sum_{a\in A} d_{ia}x_{ia}$ is at most equal to d_{ij}.

Finally, Marín (2011) has proposed the following CAC:

$$q_{ij} \sum_{a:d_{ia}\geqslant d_{ij}} x_{ia} + \sum_{a:d_{ia}<d_{ij}} y_a \leqslant q_{ij} \quad \forall i,j \in A, \quad \mathcal{M}$$

where $q_{ij} = \min\{p, |\theta_{ij}|\}$ and $\theta_{ij} = \{a : d_{ia} < d_{ij}\}$.

These constraints can again be obtained by replacing, in Construction 2.1, H by q_{ij}, z by $\sum_{a:d_{ia}<d_{ij}} y_a$ and $\{w_i : i \in I\}$ by $\{x_{ia} : d_{ia} < d_{ij}\}$.

Note that constraints $\mathcal{M}$ can be adapted to the problems without a fixed number of plants in the following way:

$$|\theta_{ij}| \sum_{a:d_{ia}\geqslant d_{ij}} x_{ia} + \sum_{a:d_{ia}<d_{ij}} y_a \leqslant |\theta_{ij}|, \quad \forall i,j \in A. \quad \mathcal{M}'$$

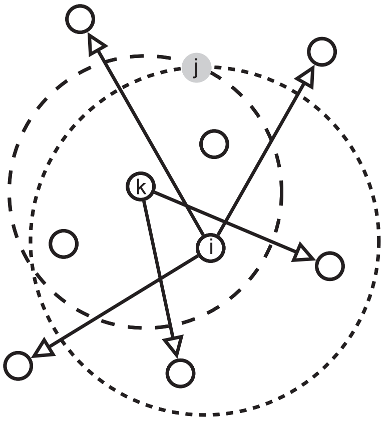

Fig. 1. Example of how constraints $\mathcal{CGLM}$ work.

3. Dominance relations between constraints

In order to compare all the CAC introduced in the previous section, we say that a set of linear constraints $A_x x + A_y y \leqslant b$ dominates another set of linear constraints $A'_x x + A'_y y \leqslant b'$ if and only if

$$\left\{(x,y) \in \mathbb{R}_+^{n^2} \times \mathbb{R}_+^n : A_x x + A_y y \leqslant b, \quad (2)-(4),\ (7)\right\} \subseteq$$
$$\left\{(x,y) \in \mathbb{R}_+^{n^2} \times \mathbb{R}_+^n : A'_x x + A'_y y \leqslant b', \quad (2)-(4),\ (7)\right\}.$$

Or equivalently, assuming that constraints (2)–(4), (7) hold, to prove that a set of linear constraints $A_x x + A_y y \leqslant b$ dominates another set of linear constraints $A'_x x + A'_y y \leqslant b'$ it suffices to show that $A_x x + A_y y \leqslant b$ implies $A'_x x + A'_y y \leqslant b'$.

We say that both sets of constraints are equivalent if they dominate one another.

In the following we state some dominance relations between the different CAC introduced above and summarized in Table 1.

Proposition 3.1. *Constraints $\mathcal{WF}$ and $\mathcal{CC}$ are equivalent.*

Proof. Using (2), constraints $\mathcal{WF}$ can be rewritten as

$$1 - \sum_{a:d_{ia}\leqslant d_{ij}} x_{ia} + y_j \leqslant 1 \Longleftrightarrow \sum_{a:d_{ia}\leqslant d_{ij}} x_{ia} \geqslant y_j. \qquad \square$$

Proposition 3.2. *Constraints $\mathcal{WF}$ dominate constraints $\mathcal{RR}$.*

Proof. Rewriting again $\mathcal{WF}$ as

$$1 - \sum_{a:d_{ia}\leqslant d_{ij}} x_{ia} + y_j \leqslant 1 \Longleftrightarrow x_{ij} + \sum_{a:d_{ia}<d_{ij}} x_{ia} \geqslant y_j$$

and now using (3), it follows

$$x_{ij} + \sum_{a:d_{ia}<d_{ij}} y_a \geqslant y_j. \qquad \square$$

Proposition 3.3. *Constraints $\mathcal{WF}$ dominate constraints $\mathcal{DK}$.*

Proof. Given values i, j and a satisfying $d_{ia} < d_{ij}$, the corresponding $\mathcal{DK}$ constraint $x_{ij} + y_a \leqslant 1$ is dominated by the $\mathcal{WF}$ constraint $\sum_{k:d_{ik}>d_{ia}} x_{ik} + y_a \leqslant 1$, since both constraints share the same right hand side and each coefficient in the left hand side of the former is less than or equal to the corresponding coefficient in the later. $\square$

Proposition 3.4. *Constraints $\mathcal{DK}$ dominate constraints $\mathcal{BDTW}$.*

Proof. We will prove that, for a fixed pair $i,\ j \in A$, $\mathcal{DK}$ constraints

$$x_{ia} + y_j \leqslant 1 \quad \forall a : d_{ia} > d_{ij}, \tag{11}$$

dominate $\mathcal{BDTW}$ constraint

$$\sum_{a\in A} d_{ia}x_{ia} + (M - d_{ij})y_j \leqslant M \tag{12}$$

for M a large amount. Summing constraints (11) multiplied by d_{ia}, we get

$$\sum_{a:d_{ia}>d_{ij}} d_{ia}x_{ia} + \left(\sum_{a:d_{ia}>d_{ij}} d_{ia}\right) y_j \leqslant \sum_{a:d_{ia}>d_{ij}} d_{ia}. \tag{13}$$

Table 1
Different CAC from the literature, summarized.

Year	Name	Constraints	Cardinality
1970	$\mathcal{RR}$	$x_{ij}+\sum_{a:d_{ia}<d_{ij}}y_a \geqslant y_j\ \forall i,j\in A$	$\mathcal{O}(n^2)$
1975	$\mathcal{WF}$	$\sum_{a:d_{ia}>d_{ij}}x_{ia}+y_j\leqslant 1\ \forall i,j\in A$	$\mathcal{O}(n^2)$
1976	$\mathcal{CC}$	$\sum_{a:d_{ia}\leqslant d_{ij}}x_{ia}\geqslant y_j\ \forall i,j\in A$	$\mathcal{O}(n^2)$
1987	$\mathcal{DK}$	$x_{ij}+y_a\leqslant 1\ \forall i,j,a\in A\text{: } d_{ia}<d_{ij}$	$\mathcal{O}(n^3)$
2007	$\mathcal{CGLM}$	$\sum_{a:d_{ia}>d_{ij}}x_{ia}+\sum_{a:d_{ia}\leqslant d_{ij},d_{ka}>d_{kj}}x_{ka}+y_j\leqslant 1\ \forall i,j,k\in A$	$\mathcal{O}(n^3)$
2007	$\mathcal{BLMN}$	$p\sum_{a:d_{ia}>d_{ij}}x_{ia}\leqslant\sum_{a:d_{ia}>d_{ij}}y_a\ \forall i,j\in A$	$\mathcal{O}(n^2)$
2009	$\mathcal{BDTW}$	$\sum_{a\in A}d_{ia}x_{ia}+(M-d_{ij})y_j\leqslant M\ \forall i,j\in A$	$\mathcal{O}(n^2)$
2011	$\mathcal{M}'$	$\lvert\theta_{ij}\rvert\sum_{a:d_{ia}\geqslant d_{ij}}x_{ia}+\sum_{a:d_{ia}<d_{ij}}y_a\leqslant\lvert\theta_{ij}\rvert\ \forall i,j\in A$	$\mathcal{O}(n^2)$
2011	$\mathcal{M}$	$q_{ij}\sum_{a:d_{ia}\geqslant d_{ij}}x_{ia}+\sum_{a:d_{ia}<d_{ij}}y_a\leqslant q_{ij}\ \forall i,j\in A$	$\mathcal{O}(n^2)$

Adding $\sum_{a:d_{ia}\leqslant d_{ij}}d_{ia}x_{ia}+(M-d_{ij}-\sum_{a:d_{ia}>d_{ij}}d_{ia})y_j$ to both sides of (13), it follows

$$\sum_{a\in A}d_{ia}x_{ia}+(M-d_{ij})y_j\leqslant\sum_{a:d_{ia}>d_{ij}}d_{ia}+\sum_{a:d_{ia}\leqslant d_{ij}}d_{ia}x_{ia}+\left(M-d_{ij}-\sum_{a:d_{ia}>d_{ij}}d_{ia}\right)y_j. \tag{14}$$

Since $y_j\leqslant 1$, the right hand side of (14) is less than or equal to

$$\sum_{a:d_{ia}\leqslant d_{ij}}d_{ia}x_{ia}+M-d_{ij}.$$

Using now (2), which implies $\sum_{a:d_{ia}\leqslant d_{ij}}d_{ia}x_{ia}\leqslant d_{ij}$, the right hand side of (14) is bounded by M and (12) follows. □

Proposition 3.5. *Constraints $\mathcal{WF}$ dominate constraints $\mathcal{M}'$.*

Proof. Consider a fixed pair $i,j\in A$ and a subset of constraints $\mathcal{WF}$ given by

$$\sum_{b:d_{ib}>d_{ia}}x_{ib}+y_a\leqslant 1\ \forall a:d_{ia}<d_{ij}. \tag{15}$$

Summing constraints (15) we get

$$\sum_{a:d_{ia}<d_{ij}}\ \sum_{b:d_{ib}>d_{ia}}x_{ib}+\sum_{a:d_{ia}<d_{ij}}y_a\leqslant\lvert\{a\in A:d_{ia}<d_{ij}\}\rvert\iff$$
$$\sum_{a:d_{ia}<d_{ij}}\ \sum_{b:d_{ib}\geqslant d_{ij}>d_{ia}}x_{ib}+\sum_{a:d_{ia}<d_{ij}}\ \sum_{b:d_{ij}>d_{ib}>d_{ia}}x_{ib}+\sum_{a:d_{ia}<d_{ij}}y_a\leqslant\lvert\theta_{ij}\rvert. \tag{16}$$

The first sum in (16) is

$$\sum_{a:d_{ia}<d_{ij}}\ \sum_{b:d_{ib}\geqslant d_{ij}>d_{ia}}x_{ib}=\sum_{a:d_{ia}<d_{ij}}\ \sum_{b:d_{ib}\geqslant d_{ij}}x_{ib}=\sum_{b:d_{ib}\geqslant d_{ij}}\ \sum_{a:d_{ia}<d_{ij}}x_{ib}=\sum_{b:d_{ib}\geqslant d_{ij}}\lvert\theta_{ij}\rvert x_{ib}.$$

In the second sum in (16), variables x_{ib} corresponding with points closer to i than j are added as many times as points a exist closer to i than b, i.e., $\lvert\theta_{ib}\rvert$ times. Then from (16) we get

$$\sum_{b:d_{ib}\geqslant d_{ij}}\lvert\theta_{ij}\rvert x_{ib}+\sum_{b:d_{ij}>d_{ib}}\lvert\theta_{ib}\rvert x_{ib}+\sum_{a:d_{ia}<d_{ij}}y_a\leqslant\lvert\theta_{ij}\rvert. \tag{17}$$

Constraint (17) has the same right hand side as the constraint in $\mathcal{M}'$ associated with the same pair i, j, and each coefficient in (17) is greater than or equal to the corresponding coefficient in $\mathcal{M}'$. Therefore, $\mathcal{M}'$ are dominated by $\mathcal{WF}$. □

It follows from the previous propositions that $\mathcal{CC}$ dominates $\mathcal{RR}$. This fact was observed in Hanjoul and Peeters (1987) and Texeira and Antunes (2008). Scaparra and Church (2008) compared these two sets of constraints in a computational framework.

Proposition 3.6. *Constraints $\mathcal{CGLM}$ dominate constraints $\mathcal{WF}$ and $\mathcal{CC}$.*

Proof. Each coefficient in $\mathcal{CGLM}$ is greater than or equal to the corresponding coefficient in $\mathcal{WF}$. Since $\mathcal{WF}$ and $\mathcal{CC}$ are equivalent, the result follows. □

Proposition 3.7. *Constraints $\mathcal{M}$ dominate constraints $\mathcal{BLMN}$.*

Proof. For $i,j\in A$ fixed, using (4) and multiplying both sides of constraints $\mathcal{BLMN}$ by $\frac{q_{ij'}}{p}$, where $j'=\min\{k:d_{ik}>d_{ij}\}$, these constraints can be rewritten as

$$p\sum_{a:d_{ia}>d_{ij}}x_{ia}\leqslant p-\sum_{a:d_{ia}\leqslant d_{ij}}y_a\iff q_{ij'}\sum_{a:d_{ia}>d_{ij}}x_{ia}+\frac{q_{ij'}}{p}\sum_{a:d_{ia}\leqslant d_{ij}}y_a\leqslant q_{ij'}. \tag{18}$$

We distinguish two cases:

1. $p\leqslant\lvert\theta_{ij'}\rvert$: Then, we have that $q_{ij'}=p$ and inequalities (18) are
$$q_{ij'}\sum_{a:d_{ia}>d_{ij}}x_{ia}+\sum_{a:d_{ia}\leqslant d_{ij}}y_a\leqslant q_{ij'},$$
or equivalently
$$q_{ij'}\sum_{a:d_{ia}\geqslant d_{ij'}}x_{ia}+\sum_{a:d_{ia}<d_{ij'}}y_a\leqslant q_{ij'},$$
exactly coinciding with $\mathcal{M}$ for a pair of indices i,j'. The case where $d_{ij}=\max_{k\in A}\{d_{ik}\}$, the corresponding constraint $\mathcal{BLMN}$ provides the inequality $\sum_{a\in A}y_a\leqslant p$, implied by (4).
2. $p>\lvert\theta_{ij'}\rvert$: Then, the x-coefficients and the right hand side in (18) are equal to the corresponding coefficients in $\mathcal{M}$, whereas the y-coefficients in (18) are $\frac{q_{ij'}}{p}=\frac{\min\{p,\lvert\theta_{ij'}\rvert\}}{p}=\frac{\lvert\theta_{ij'}\rvert}{p}$, strictly less than 1, which is the corresponding coefficient in $\mathcal{M}$. □

Observe that in the above dominance results, we have only used constraint (4) in Proposition 3.7. Therefore, all other relationships are still valid for the case where the number of facilities to be located are not fixed in advance.

We complete the classification with some counterexamples showing no other dominance relations between inequalities exist.

Example 3.1. Consider an instance with $n=3$, $p=2$ and $(d_{ij})=\begin{pmatrix}0&2&1\\2&0&3\\1&3&0\end{pmatrix}$. A fractional point which satisfies constraints $\mathcal{BDTW}$ as well as (2)–(4) is $\left(y_j^*\right)=(1/2,1/2,1)$, $\left(x_{ij}^*\right)=\begin{pmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&1\end{pmatrix}$. Furthermore,

Table 2
Sketch of examples of non dominance between constraints.

Constraints	(d_{ij})	(c_{ij})	p	$\mathcal{RR}$	$\mathcal{WF},\mathcal{CC}$	$\mathcal{DK}$	$\mathcal{CGLM}$	$\mathcal{BLMN}$	$\mathcal{BDTW}$	$\mathcal{M}$	$\mathcal{M}'$
				i,j (,k)							
$\mathcal{RR}$	$\begin{matrix}0&3&1&7\\3&0&2&6\\1&2&0&8\\7&6&8&0\end{matrix}$	$\begin{matrix}0&5&2&3\\5&0&4&6\\2&4&0&1\\3&6&1&0\end{matrix}$	3	-	3,2	3,4,2	3,2,4	3,2	3,2	3,4	3,4
$\mathcal{WF},\mathcal{CC}$	$\begin{matrix}0&7&5&3\\7&0&4&8\\5&4&0&6\\3&8&6&0\end{matrix}$	$\begin{matrix}0&3&5&4\\3&0&6&1\\5&6&0&2\\4&1&2&0\end{matrix}$	2	-	-	-	2,1,1	1,3	-	1,2	-
$\mathcal{DK}$	$\begin{matrix}0&3&5&6\\3&0&8&9\\5&8&0&7\\6&9&7&0\end{matrix}$	$\begin{matrix}0&5&2&3\\5&0&4&6\\2&4&0&1\\3&6&1&0\end{matrix}$	2	1,2	1,2	-	1,2,1	1,2	-	1,3	1,3
$\mathcal{CGLM}$	$\begin{matrix}0&3&8&6\\3&0&7&5\\8&7&0&2\\6&5&2&0\end{matrix}$	$\begin{matrix}0&8&5&1\\8&0&3&6\\8&7&0&2\\6&5&2&0\end{matrix}$	2	-	-	-	-	2,4	-	2,3	-
$\mathcal{BLMN}$	$\begin{matrix}0&3&4&6\\3&0&7&9\\4&7&0&2\\6&9&2&0\end{matrix}$	$\begin{matrix}0&3&9&7\\3&0&4&2\\9&4&0&8\\7&2&8&0\end{matrix}$	3	4,3	4,3	4,2,3	4,3,2	-	4,3	4,1	4,1
$\mathcal{BDTW}$	$\begin{matrix}0&2&1\\2&0&3\\1&3&0\end{matrix}$	$\begin{matrix}0&1&3\\1&0&2\\3&2&0\end{matrix}$	2	1,3	1,3	1,2,3	1,3,2	1,3	-	1,2	1,2
$\mathcal{M}$	$\begin{matrix}0&5&6&2\\5&0&1&3\\6&1&0&4\\2&3&4&0\end{matrix}$	$\begin{matrix}0&1&6&5\\1&0&4&3\\6&4&0&2\\5&3&2&0\end{matrix}$	2	4,2	4,2	4,3,2	4,2,3	-	4,2	-	-
$\mathcal{M}'$	$\begin{matrix}0&1&3&7\\1&0&4&6\\3&4&0&8\\7&6&8&0\end{matrix}$	$\begin{matrix}0&4&7&2\\4&0&9&3\\7&9&0&8\\2&3&8&0\end{matrix}$	2	1,3	1,3	1,4,3	1,3,4	1,3	1,3	1,4	-

(x^*,y^*) is an optimal solution of the linear relaxation of the generalized p-median problem, augmented with constraints $\mathcal{BDTW}$, given by

$$\begin{aligned}\min\quad & x_{12}+3x_{13}+x_{21}+2x_{23}+3x_{31}+2x_{32}\\ \text{s.t.}\quad & (2)\text{–}(4),\ \mathcal{BDTW},\quad 0\leqslant y_j\leqslant 1\ \forall j,\quad x_{ij}\geqslant 0\ \forall i,j.\end{aligned}$$

When $i = 1$, $j = 2$ and $a = 3$, the left hand side of $\mathcal{DK}$ in (x^*,y^*) takes value $x^*_{12}+y^*_3 = 1.5$, greater than 1. This constraint is not satisfied by (x^*,y^*).

Therefore constraints $\mathcal{BDTW}$ do not dominate constraints $\mathcal{DK}$.

Following the scheme of the above example, a complete set of counterexamples is given in Table 2. Each row shows the data for which the optimal solution of the generalized p-median problem augmented with the set of constraints indicated in the first column violates at least one constraint of each family which is not dominated by this set.

In each of the columns named $\mathcal{RR}$ to $\mathcal{M}'$, either two or three indices defining a violated constraint of the corresponding family are given. To find these indices we have taken advantage of the dominance relationship between families of constraints previously analyzed, in the sense that the indices defining a violated constraint of a family $\mathcal{X}$ are still valid to define a violated constraint of another family $\mathcal{Y}$, whenever $\mathcal{Y}$ dominates $\mathcal{X}$.

Note that it was possible in all the cases to find examples with at most $n = 4$, symmetric and with $d_{ii} = c_{ii} = 0\ \forall i \in A$. Moreover, all the matrices shown in Table 2 satisfy triangle inequality.

These examples demonstrate that no dominance relation exists between sets of inequalities, but the ones previously shown and the ones got from them using transitivity, thus completing the classification of CAC in the literature. Most of the sets of constraints are dominated by another set. Only $\mathcal{CGLM}$ and $\mathcal{M}$ remain non-dominated at this point. Note that the cardinality of $\mathcal{CGLM}$ is $\mathcal{O}(n^3)$ but a subset of cardinality $\mathcal{O}(n^2)$ which still dominates $\mathcal{WF}$ can be easily generated by fixing k to any value. $\mathcal{CGLM}$ do not make use of constraint (4) and can be used in models without fixed number of plants, like the Simple Plant Location Problem, whereas $\mathcal{M}$ takes advantage of (4) and can be combined with $\mathcal{CGLM}$ for solving discrete location problems with fixed number of plants.

4. A new set of constraints

Although inequalities $\mathcal{M}'$ are dominated by $\mathcal{WF}$, they were obtained weakening inequalities $\mathcal{M}$, which were designed to be applied in the presence of constraints (4). Now, using the ideas behind constraints (17) in the proof of Proposition 3.5 we can reinforce constraints $\mathcal{M}$ as follows:

$$\sum_{\substack{a:d_{ia}<d_{ij}\\ |\theta_{ij}|-|\theta_{ia}|\leqslant p}} (|\theta_{ia}|+(p-|\theta_{ij}|)^-)x_{ia}+q_{ij}\sum_{a:d_{ia}\geqslant d_{ij}} x_{ia}+\sum_{a:d_{ia}<d_{ij}} y_a\leqslant q_{ij}\quad \forall i,j,\ \mathcal{EMR}$$

where $z^- := \min\{0,z\}$.

To see the effect of these constraints, fix a pair (i,j) and assume $x_{ib} = 1$ for some b such that $d_{ib}\geqslant d_{ij}$. Then, from (2) the remaining x-variables in $\mathcal{EMR}$ will take value 0 and the effect will be the same as the effect of $\mathcal{M}$, i.e., closest assignment. Assume now $x_{ib} = 1$ for some b such that $d_{ib} < d_{ij}$ and $|\theta_{ij}|-|\theta_{ib}|\leqslant p$. Again the remaining x-variables in $\mathcal{EMR}$ take value 0 and the inequality becomes

$$\sum_{a:d_{ia}<d_{ij}} y_a\leqslant q_{ij}-(|\theta_{ib}|+(p-|\theta_{ij}|)^-),$$

i.e.,

$$\sum_{a:d_{ia}<d_{ij}} y_a\leqslant \min\{p,|\theta_{ij}|\}-(p-|\theta_{ij}|))^- -|\theta_{ib}| = |\theta_{ij}|-|\theta_{ib}|.$$

That is to say, the number of plants in the set θ_{ij} is bounded by $|\theta_{ij}| - |\theta_{ib}|$ (which in turn is less than or equal to p). This is a correct valid bound under the assumption that no plant closer to client i than plant b is opened.

To see that the coefficients $|\theta_{ia}| + (p - |\theta_{ij}|)^-$ in $\mathcal{EMR}$ cannot be improved, we show the following result.

Proposition 4.1. *For any $i, j \in A$ and $b \in A$ such that $d_{ib} < d_{ij}$, the maximum value of C which makes the inequality*

$$Cx_{ib} + q_{ij}\sum_{a:d_{ia}\geqslant d_{ij}} x_{ia} + \sum_{a:d_{ia}<d_{ij}} y_a \leqslant q_{ij}, \tag{19}$$

valid for the set

$$\left\{(x,y)\in \mathbb{R}_+^{n^2}\times\mathbb{R}_+^{n} : (2)\text{–}(4),\ (7)\text{–}(9),\ \mathcal{M}\right\} \tag{20}$$

is

$$C = \max\{0, |\theta_{ib}| + (p - |\theta_{ij}|)^-\}.$$

Proof. In order to upperly bound the value of C, we consider all possible points in the set (20).

If $x_{ib} = 0$, coefficient C does not matter. In particular, if $\sum_{a:d_{ia}\geqslant d_{ij}} x_{ia} = 1$, then $x_{ib} = 0$. Therefore we assume in the following that $\sum_{a:d_{ia}\geqslant d_{ij}} x_{ia} = 0$ and $x_{ib} = 1$.

If $x_{ib} = 1$, then due to the CAC $\mathcal{M}$, $y_a = 0\ \forall a : d_{ia} < d_{ib}$. Therefore, (19) reads

$$C + \sum_{a:d_{ib}\leqslant d_{ia}<d_{ij}} y_a \leqslant q_{ij} = \min\{p, |\theta_{ij}|\}.$$

Observe that the number of binary y-variables in this inequality is $|\theta_{ij}| - |\theta_{ib}|$. We consider two cases:

- $p \leqslant |\theta_{ij}|$. Then the right hand side of (19) is p. There are two possibilities:
  [–] If $|\theta_{ij}| - |\theta_{ib}| \leqslant p$, the number of y-variables taking value 1 in the left hand side of the inequality can reach $|\theta_{ij}| - |\theta_{ib}|$, and C cannot be greater than $p - |\theta_{ij}| + |\theta_{ib}|$.
  [–] If $|\theta_{ij}| - |\theta_{ib}| > p$, the number of y-variables taking value 1 in the left hand side of the inequality can reach p, forcing C to be 0.
  Therefore,
  $$C \text{ is } \begin{cases} = 0 & \text{if } |\theta_{ij}| - |\theta_{ib}| > p, \\ \leqslant p - |\theta_{ij}| + |\theta_{ib}| & \text{if } |\theta_{ij}| - |\theta_{ib}| \leqslant p. \end{cases}$$
- $p \geqslant |\theta_{ij}|$. Then the right hand side of (19) is $|\theta_{ij}|$ and the number of y-variables in the left hand side of the inequality can only reach $|\theta_{ij}| - |\theta_{ib}|$. Thus
  $$C \leqslant |\theta_{ib}|.$$
  Considering both cases together we obtain

$$C \text{ is } \begin{cases} = 0 & \text{if } |\theta_{ij}| - |\theta_{ib}| > p, \\ \leqslant |\theta_{ib}| + (p - |\theta_{ij}|)^- & \text{if } |\theta_{ij}| - |\theta_{ib}| \leqslant p. \end{cases} \qquad \square$$

We will prove that constraints $\mathcal{EMR}$ dominate some other sets of constraints and are not dominated by any of them.

Proposition 4.2. *Constraints $\mathcal{EMR}$ dominate constraints $\mathcal{M}$.*

Proof. Each coefficient in the left hand side of $\mathcal{EMR}$ is greater than or equal to the corresponding coefficient in $\mathcal{M}$. $\square$

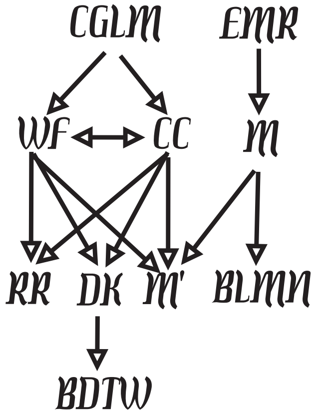

Fig. 2. Relations between constraints.

Example 4.1. Consider an instance with $n = 5$, $p = 3$ and $(d_{ij}) = \begin{pmatrix} 0 & 8 & 6 & 3 & 5 \\ 8 & 0 & 2 & 9 & 3 \\ 6 & 2 & 0 & 7 & 1 \\ 3 & 9 & 7 & 0 & 6 \\ 5 & 3 & 1 & 6 & 0 \end{pmatrix}$ satisfying triangle inequality. A fractional point which satisfies constraints $\mathcal{EMR}$ as well as (2)–(4) is

$$\left(y_j^*\right) = (2/3, 1, 1/3, 2/3, 1/3),\ \left(x_{ij}^*\right) = \begin{pmatrix} 2/3 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 2/3 & 1/3 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}.$$

Furthermore, (x^*,y^*) is an optimal solution of the linear relaxation of the generalized p-median problem, augmented with constraints $\mathcal{EMR}$, given by

$$\begin{aligned} \min\quad & 9x_{12} + 2x_{13} + 8x_{14} + x_{15} + 9x_{21} + 5x_{23} + 6x_{24} \\ & + 10x_{25} + 2x_{31} + 5x_{32} + 7x_{34} + 3x_{35} + \\ & 8x_{41} + 6x_{42} + 7x_{43} + 4x_{45} + x_{51} + 10x_{52} + 3x_{53} + 4x_{54} \\ \text{s.t.}\quad & (2)\text{–}(4), \\ & \mathcal{EMR},\quad 0 \leqslant y_j \leqslant 1 \quad \forall j,\quad x_{ij} \geqslant 0 \quad \forall i,j. \end{aligned}$$

When $i = 3$ and $j = 2$, the left hand side of $\mathcal{RR}$ in (x^*,y^*) takes value $x_{32}^* + y_3^* + y_5^* = 2/3$ and its right hand side takes value $y_2^* = 1$. This constraint is not satisfied by (x^*,y^*). When $i = 3$ and $j = 2$, the left hand side of $\mathcal{BDTW}$ in (x^*,y^*) takes value $7/3 + (M-2)y_2^* = 1/3 + M$.

Therefore constraints $\mathcal{EMR}$ neither dominate constraints $\mathcal{RR}$ nor $\mathcal{BDTW}$.

Example 4.2. Consider the seventh example in Table 2 (row $\mathcal{M}$). When $i = 4$ and $j = 3$, the left hand side of $\mathcal{EMR}$ in (x^*,y^*) takes value $x_{42}^* + y_1^* + y_2^* + y_4^* = 5/2$, greater than the right hand side 2. This constraint is not satisfied by (x^*,y^*).

Therefore constraints $\mathcal{M}$ do not dominate constraints $\mathcal{EMR}$.

These examples, together with the dominance and non-dominance relations established in the previous sections, show that constraints $\mathcal{EMR}$ dominate constraints $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{BLMN}$ and are not dominated by any set of constraints.

In this point, the sets of constraints which are not dominated by any other set are $\mathcal{EMR}$ and $\mathcal{CGLM}$. Fig. 2 illustrates the dominance constraints between sets of inequalities. Here $A \to B$ means A dominates B.

5. Additional properties of some constraints

Next we show that, using some of the CAC, the integrality of the x-variables can be relaxed. Whereas in the most simple and widely studied discrete location problems (i.e., the Simple Plant Location Problem and the p-median Problem) x-integrality is guaranteed in the optimum due to the shape of the objective function, in our case it is a consequence of the set of constraints, regardless of the objective function shape, and is satisfied by all the feasible solutions.

Proposition 5.1. *The following equalities hold:*

$$\left\{(x,y)\in\mathbb{Z}_+^{n^2}\times\mathbb{Z}_+^n:(2)-(4),\mathcal{X}\right\}=\left\{(x,y)\in\mathbb{R}_+^{n^2}\times\mathbb{Z}_+^n:(2)-(4),\mathcal{X}\right\},\tag{21}$$

where

$$\mathcal{X}\in\{\mathcal{RR},\mathcal{WF},\mathcal{CC},\mathcal{DK},\mathcal{CGLM},\mathcal{BDTW},\mathcal{EMR}\}.\tag{22}$$

That is to say, when some of the CAC in the set (22) are added to constraints (2)–(4), integrality of the x-variables can be relaxed.

Proof. Observe that, due to the dominance relationships between CAC, it is only needed to prove equality (21) when $\mathcal{X}\in\{\mathcal{RR},\mathcal{BDTW},\mathcal{EMR}\}$.

Clearly in all cases the set on the left is included in the set on the right hand side. Consider now an element (x,y) in the set on the right hand side with $y_j=1\ \forall j\in A_1$ and $y_j=0\ \forall j\in A_0$, $A_1\cup A_0=A$. Using (2) and (3) it follows

$$\sum_{j\in A_1}x_{ij}=1\quad\forall i\in A.\tag{23}$$

Fix any $i\in A$ and let $j_i:=\operatorname{argmin}\{d_{ij}:j\in A_1\}$ be the closest plant with respect to i. Then

- From $\mathcal{RR}$, $x_{ij_i}\geqslant y_{j_i}=1$, then from (23) $x_{ij_i}=1$ and $x_{ij}=0\ \forall j\neq j_i$, $x\in\mathbb{Z}_+^{n^2}$ and (21) follows.
- From $\mathcal{BDTW}$ and (3) it follows
$$\sum_{a\in A_1}d_{ia}x_{ia}\leqslant d_{ij_i}.\tag{24}$$

All the coefficients in the left hand side are greater than or equal to d_{ij_i}. Then, if $x_{ij}>0$ for some $j\neq j_i$, from (23) the left hand term in (24) would be strictly greater than d_{ij_i}. Therefore, $x_{ij_i}=1$, $x\in\mathbb{Z}_+^{n^2}$ and (21) follows.

- For i and j_i, using (2), (3) and the corresponding constraint $\mathcal{EMR}$ we obtain that
$$\sum_{a:d_{ia}\geqslant d_{ij_i}}x_{ia}=1.$$

Moreover, for i and $\bar{j}_i$ such that $|\theta_{i\bar{j}_i}|-|\theta_{ij_i}|=1\leqslant p$, the left hand side of the corresponding constraint $\mathcal{EMR}$ is

$$\begin{aligned}(|\theta_{ij_i}|+(p-|\theta_{i\bar{j}_i}|)^-)x_{ij_i}+q_{i\bar{j}_i}\sum_{a:d_{ia}\geqslant d_{i\bar{j}_i}}x_{ia}+\sum_{a:d_{ia}<d_{i\bar{j}_i}}y_a&=(|\theta_{ij_i}|\\ +(p-|\theta_{ij_i}|-1)^-)x_{ij_i}-q_{i\bar{j}_i}x_{ij_i}+q_{i\bar{j}_i}\sum_{a:d_{ia}\geqslant d_{ij_i}}x_{ia}&+\sum_{a:d_{ia}<d_{i\bar{j}_i}}y_a,\end{aligned}$$

that is to say, the constraint $\mathcal{EMR}$ is

$$(|\theta_{ij_i}|+(p-|\theta_{ij_i}|-1)^--q_{i\bar{j}_i})x_{ij_i}+1\leqslant 0.\tag{25}$$

If $p\geqslant|\theta_{i\bar{j}_i}|=|\theta_{ij_i}|+1$, then the inequality (25) is

$$(|\theta_{ij_i}|-q_{i\bar{j}_i})x_{ij_i}+1=-x_{ij_i}+1\leqslant 0$$

and then $x_{ij_i}=1$. If $p<|\theta_{i\bar{j}_i}|=|\theta_{ij_i}|+1$, (25) is

$$(|\theta_{ij_i}|+p-|\theta_{ij_i}|-1-p)x_{ij_i}+1=-x_{ij_i}+1\leqslant 0$$

and then $x_{ij_i}=1$. Therefore, (21) follows. □

Note that this result is also valid for the case where the number of facilities to be located is not fixed in advance (except for the case of $\mathcal{EMR}$).

Equality (21) was observed in Gerrard and Church (1996) in the case of constraints $\mathcal{CC}$. The following counterexample shows that the remaining sets of constraints do not present this desirable property, observing again that it is only needed to show counterexamples for the case of $\mathcal{M}$.

Example 5.1. Consider an instance with $n=3$, $p=2$ and $(d_{ij})=\begin{pmatrix}0&3&2\\3&0&1\\2&1&0\end{pmatrix}$. A fractional point which satisfies constraints $\mathcal{M}$ as well as (2)–(4) is $\left(y_j^*\right)=(0,1,1)$, $\left(x_{ij}^*\right)=\begin{pmatrix}0&1/2&1/2\\0&1&0\\0&0&1\end{pmatrix}$. Furthermore, (x^*,y^*) is an optimal solution of the generalized p-median problem, augmented with constraints $\mathcal{M}$ given by

$$\begin{aligned}\min\quad&x_{12}+2x_{13}+x_{21}+3x_{23}+2x_{31}+3x_{32}\\ \text{s.t.}\quad&(2)-(4),\ \mathcal{M},y_j\in\{0,1\}\ \forall j,\quad x_{ij}\geqslant 0\ \forall i,j.\end{aligned}$$

Constraints $\mathcal{RR}$ present an additional property. They make constraints $\sum_{j\in A}x_{ij}\geqslant 1$ and (3) unnecessary. Analogously, constraints $\mathcal{CC}$ make constraints $\sum_{j\in A}x_{ij}\geqslant 1$ unnecessary.

Proposition 5.2. *The following equality holds:*

$$\left\{(x,y)\in\mathbb{R}_+^{n^2}\times\mathbb{Z}_+^n:\sum_{j\in A}x_{ij}\leqslant 1,\quad(4),\quad\mathcal{RR}\right\}$$
$$=\left\{(x,y)\in\mathbb{R}_+^{n^2}\times\mathbb{Z}_+^n:(2)-(4),\quad\mathcal{RR}\right\}.$$

Proof. Fix any $i\in A$. From (4) and since $y_j\in\{0,1\}\ \forall j$, an index $j_i:=\operatorname{argmin}\{d_{ij}:y_j=1\}$ exists. Then, from $\mathcal{RR}$,

$$x_{ij_i}+\sum_{a:d_{ia}<d_{ij_i}}y_a\geqslant y_{j_i}\iff x_{ij_i}\geqslant 1.$$

Now,

- from $\sum_{j\in A}x_{ij}\leqslant 1\ \Rightarrow\ x_{ij_i}=1$ and $x_{ij}=0\ \forall j\neq j_i$. Therefore $x_{ij}=0$ for any j such that $y_j=0$ and (3) follows.
- $\sum_{i\in A}x_{ij}\geqslant x_{ij_i}=1$ and (2) follows. □

It was observed in Gerrard and Church (1996) that constraints (3) can be relaxed in the presence of $\mathcal{RR}$.

Proposition 5.3. *The following equality holds:*

$$\left\{(x,y)\in\mathbb{R}_+^{n^2}\times\mathbb{Z}_+^n:\sum_{j\in A}x_{ij}\leqslant 1,\ (3),\ (4),\ \mathcal{CC}\right\}$$
$$=\left\{(x,y)\in\mathbb{R}_+^{n^2}\times\mathbb{Z}_+^n:(2)-(4),\ \mathcal{CC}\right\}.\tag{26}$$

Proof. Fix any $i\in A$. From (4) and since $y_j\in\{0,1\}\ \forall j$, an index $j_i:=\operatorname{argmin}\{d_{ij}:y_j=1\}$ exists. Then, from $\mathcal{CC}$, $\sum_{a:d_{ia}\leqslant d_{ij_i}}x_{ia}\geqslant y_{j_i}=1\iff$ (2) holds. □

The fact demonstrated in Proposition 5.3 was observed in Gerrard and Church (1996). The rest of the constraints do not present any of these properties. To prove this assertion we present two counterexamples.

Example 5.2. Consider an instance with $n = 3$, $p = 2$ and $(d_{ij}) = \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$. An optimal solution of the problem

$$\begin{array}{ll} \min & 2x_{12} + 6x_{13} + 2x_{21} + x_{23} + 6x_{31} + x_{32} \\ \text{s.t.} & (2),\ (4), \\ & \mathcal{WF},\ \mathcal{CC},\ \mathcal{DK},\ \mathcal{CGLM},\ \mathcal{BLMN},\ \mathcal{BDTW}, \\ & \mathcal{M},\ \mathcal{M}',\ \mathcal{EMR}, y_j \in \{0,1\}\ \forall j, \quad x_{ij} \in \{0,1\}\ \forall i,j \end{array}$$

is $\left(y_j^*\right) = (1,1,0)$, $\left(x_{ij}^*\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which violates constraint $x_{33} \leqslant y_3$.

Therefore constraints (3) cannot be relaxed when one of the sets of constraints $\mathcal{WF}$, $\mathcal{CC}$, $\mathcal{DK}$, $\mathcal{CGLM}$, $\mathcal{BLMN}$, $\mathcal{BDTW}$, $\mathcal{M}$, $\mathcal{M}'$ or $\mathcal{EMR}$ are used as CAC in a formulation.

Example 5.3. Consider an instance with $n = 3$, $p = 2$ and $(d_{ij}) = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$. An optimal solution of the problem

$$\begin{array}{ll} \min & 2x_{12} + 3x_{13} + 2x_{21} + x_{23} + 3x_{31} + x_{32} \\ \text{s.t.} & (3),\ (4),\ \mathcal{WF},\ \mathcal{DK},\ \mathcal{CGLM},\ \mathcal{BLMN},\ \mathcal{BDTW}, \\ & \mathcal{M},\ \mathcal{M}',\ \mathcal{EMR},\ y_j \in \{0,1\}\ \forall j, \quad x_{ij} \in \{0,1\}\ \forall i,j \end{array}$$

is $\left(y_j^*\right) = (1,0,1)$, $\left(x_{ij}^*\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which violates constraints $\sum_{j=1}^{3} x_{ij} \geqslant 1\ \forall i = 1,2,3$.

6. Adaptation to the case of ties between distances

In this section we analyze the previous results to check whether they are still valid when the distance from one client to two or more plants can be equal. A typical situation of that type is when several alternative facility sizes with different capacities and/or costs are possible at the same site (i.e., colocations are possible) and the situation is modeled considering each alternative as a separate potential facility (but with the same associated distances).

In this case constraints $\mathcal{RR}$ fail. To see it, let j_i be again the closest plant with respect to client i but suppose there is a second plant j^i such that $d_{ij_i} = d_{ij^i}$. then, applying $\mathcal{RR}$ to (i,j_i) and (i,j^i), it follows $x_{ij_i} = 1$ and $x_{ij^i} = 1$. This fact was noted in Gerrard and Church (1996) where the authors proposed alternative constraints depending on the number of tied distances. For the sake of brevity we only present the constraints they proposed for the case of two tied distances, i.e., $d_{ij} = d_{ik}$ for some $i, j \neq k \in A$:

$$x_{ij} + \sum_{a \neq j : d_{ia} \leqslant d_{ij}} y_a \geqslant y_j, \quad \mathcal{GC}$$

$$x_{ik} + \sum_{a \neq k : d_{ia} \leqslant d_{ik}} y_a \geqslant y_k, \quad \mathcal{GC}'$$

$$x_{ij} + x_{ik} + \sum_{a : d_{ia} < d_{ij}} y_a \geqslant \frac{1}{2} y_j + \frac{1}{2} y_k. \quad \mathcal{GC}''$$

We see how these constraints work. If $y_j = y_k = 0$, the constraints have no effect. If $y_j = 1$ and $y_k = 0$ (resp. $y_k = 1$ and $y_j = 0$), the effect of $\mathcal{GC}$ (resp. $\mathcal{GC}'$) is the same as in $\mathcal{RR}$ and $\mathcal{GC}'$ has no effect. Finally, if $y_j = y_k = 1$, $\mathcal{GC}$ and $\mathcal{GC}'$ have no effect and $\mathcal{GC}''$ forces i to be assigned to either j, k or a plant closer to client i than plants j and k.

We propose a more simple adaptation of constraints $\mathcal{RR}$ to the case of ties:

$$\sum_{a : d_{ia} = d_{ij}} x_{ia} + \sum_{a : d_{ia} < d_{ij}} y_a \geqslant y_j \quad \forall i,j \in A. \quad \mathcal{RR}_{\mathcal{EMR}}$$

Observe that this inequality is valid for any number of tied distances.

Proposition 6.1. *Constraints $\mathcal{RR}_{\mathcal{EMR}}$ dominate constraints $\mathcal{GC}, \mathcal{GC}'$ and $\mathcal{GC}''$.*

Proof. Fixing i, j and k and writing the two constraints $\mathcal{RR}_{\mathcal{EMR}}$ corresponding to (i,j) and (i,k) we get

$$\sum_{a : d_{ia} = d_{ij}} x_{ia} + \sum_{a : d_{ia} < d_{ij}} y_a \geqslant y_j, \tag{27}$$

$$\sum_{a : d_{ia} = d_{ik}} x_{ia} + \sum_{a : d_{ia} < d_{ik}} y_a \geqslant y_k. \tag{28}$$

- Since –using (3)– $x_{ik} \leqslant y_k$, (27) implies $\mathcal{GC}$.
- Analogously, since $x_{ij} \leqslant y_j$, (28) implies $\mathcal{GC}'$.
- Summing (27) and (28) multiplied by $\frac{1}{2}$, $\mathcal{GC}''$ is obtained. □

It can be easily checked that constraints $\mathcal{WF}$, $\mathcal{CC}$, $\mathcal{DK}$, $\mathcal{CGLM}$, $\mathcal{BLMN}$, $\mathcal{BDTW}$, $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{EMR}$ still work in case of ties. Moreover, the demonstrations of all results in Section 3 which prove relations between these constraints still work in case of ties.

Proposition 3.2 can be straightforwardly adapted to show that the classification of $\mathcal{RR}$ and $\mathcal{RR}_{\mathcal{EMR}}$ with respect to the dominance relations is the same:

Proposition 6.2. *In case of ties, constraints $\mathcal{WF}$ still dominate constraints $\mathcal{RR}_{\mathcal{EMR}}$.*

Proof. Rewriting again $\mathcal{WF}$ as

$$1 - \sum_{a : d_{ia} \leqslant d_{ij}} x_{ia} + y_j \leqslant 1 \iff \sum_{a : d_{ia} = d_{ij}} x_{ia} + \sum_{a : d_{ia} < d_{ij}} x_{ia} \geqslant y_j$$

and now using (3), it follows

$$\sum_{a : d_{ia} = d_{ij}} x_{ia} + \sum_{a : d_{ia} < d_{ij}} y_a \geqslant y_j. \qquad \square$$

All examples used to show non-dominance relations are still valid in the case of ties.

Integrality of x-variables is not yet implied by any of the CAC. We show it in the following example.

Example 6.1. Consider an instance with $n = 3$, $p = 2$ and $(d_{ij}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. An optimal solution of the problem

$$\begin{array}{ll} \min & x_{12} + x_{13} + x_{21} + x_{23} + x_{31} + x_{32} \\ \text{s.t.} & (2)\text{–}(4), \\ & \mathcal{RR}_{\mathcal{EMR}},\ \mathcal{WF}, \mathcal{CC},\ \mathcal{DK},\ \mathcal{CGLM},\ \mathcal{BLMN}, \\ & \mathcal{BDTW},\ \mathcal{M},\ \mathcal{M}', \mathcal{EMR},\ y_j \in \{0,1\}\ \forall j, \quad x_{ij} \geqslant 0\ \forall i,j \end{array}$$

is $\left(y_j^*\right) = (1,1,0)$, $\left(x_{ij}^*\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$, which is not integer.

Nevertheless, in some problems what is important can be not if x-variables take integer values, but if closest assignment is still satisfied by the (non-integer) optimal solution of the problem. For instance, in the p-median problem with ties, if two plants j_1 and j_2 are at the same distance from client i, a solution with $x_{ij_1} = x_{ij_2} = 1/2$ is not a problem since it can be transformed into

Table 3
Summary of results.

Constraints	Dominate	Dominated by	p Required	Valid when ties	Relax (no ties)	Relax (ties)	Fractional CA
$\mathcal{RR}$	-	$\mathcal{WF}$ $\mathcal{CC}$ $\mathcal{CGLM}$	N	N	(2) (8) $\geqslant$	NS	NS
$\mathcal{RR}_{\mathcal{EMR}}$	-	$\mathcal{WF}$ $\mathcal{CC}$ $\mathcal{CGLM}$	N	Y	(2) (8) $\geqslant$	$\geqslant$	Y
$\mathcal{WF}$	$\mathcal{RR}$ $\mathcal{RR}_{\mathcal{EMR}}$ $\mathcal{CC}$ $\mathcal{DK}$ $\mathcal{M}'$ $\mathcal{BDTW}$	$\mathcal{CC}$ $\mathcal{CGLM}$	N	Y	(8)	-	Y
$\mathcal{CC}$	$\mathcal{RR}$ $\mathcal{RR}_{\mathcal{EMR}}$ $\mathcal{WF}$ $\mathcal{DK}$ $\mathcal{M}'$ $\mathcal{BDTW}$	$\mathcal{WF}$ $\mathcal{CGLM}$	N	Y	(8) $\geqslant$	$\geqslant$	Y
$\mathcal{DK}$	$\mathcal{BDTW}$	$\mathcal{WF}$ $\mathcal{CC}$ $\mathcal{CGLM}$	N	Y	(8)	-	Y
$\mathcal{CGLM}$	$\mathcal{RR}$ $\mathcal{RR}_{\mathcal{EMR}}$ $\mathcal{WF}$ $\mathcal{CC}$ $\mathcal{DK}$ $\mathcal{M}'$ $\mathcal{BDTW}$	-	N	Y	(8)	-	Y
$\mathcal{BLMN}$	-	$\mathcal{M}$ $\mathcal{EMR}$	Y	Y	-	-	N
$\mathcal{BDTW}$	-	$\mathcal{WF}$ $\mathcal{CC}$ $\mathcal{DK}$ $\mathcal{CGLM}$	N	Y	(8)	-	Y
$\mathcal{M}$	$\mathcal{BLMN}$ $\mathcal{M}'$	$\mathcal{EMR}$	Y	Y	-	-	N
$\mathcal{M}'$	-	$\mathcal{WF}$ $\mathcal{CC}$ $\mathcal{CGLM}$ $\mathcal{M}$ $\mathcal{EMR}$	N	Y	-	-	N
$\mathcal{EMR}$	$\mathcal{BLMN}$ $\mathcal{M}$ $\mathcal{M}'$	-	Y	Y	(8)	-	N

another solution with the same objective value simply doing $x_{ij_1}=1$ and $x_{ij_2}=0$. We check now which of the sets of constraints keep the closest assignment property in case of ties.

Proposition 6.3. *At any optimal solution* (x^*,y^*,z^*) *of Problem (1)–(7), (9)–(10) (the general problem where the integrality of x-variables has been realxed) with an additional family of constraints* $\mathcal{X}\in\{\mathcal{RR}_{\mathcal{EMR}},\ \mathcal{WF},\ \mathcal{CC},\ \mathcal{DK},\mathcal{CGLM},\mathcal{BDTW}\}$, *it holds*

$$\forall i,j_1,j_2\in A:\ x^*_{ij_1}>0,\ x^*_{ij_2}>0\Rightarrow d_{ij_1}=d_{ij_2}. \tag{29}$$

Proof. Using (2) and (3) it follows

$$\sum_{j\in A_1}x_{ij}=1\ \forall i\in A. \tag{30}$$

Let $j_i\in\operatorname{argmin}\{d_{ij}:j\in A_1\}$ be a closest plant with respect to i. Then

- From either $\mathcal{WF},\ \mathcal{DK}$ or $\mathcal{CGLM},\ x^*_{ij}=0\ \forall j:d_{ij}>d_{ij_i}$. Then, from (30), $\sum_{j\in A_1:d_{ij}=d_{ij_i}}x^*_{ij}=1$ and (29) follows.
- From either $\mathcal{RR}_{\mathcal{EMR}}$ or $\mathcal{CC}, \sum_{j\in A_1:d_{ij}=d_{ij_i}}x^*_{ij}\geqslant 1$ and (29) follows.
- From $\mathcal{BDTW}$ and (3) it follows

$$\sum_{a\in A_1}d_{ia}x^*_{ia}\leqslant d_{ij_i}. \tag{31}$$

All the coefficients in the left hand side are greater than or equal to d_{ij_i}. Then, if $x^*_{ij}>0$ for some j: $d_{ij}>d_{ij_i}$, from (30) the left hand term in (31) would be strictly greater than d_{ij_i}. Therefore, $\sum_{j\in A_1:d_{ij}=d_{ij_i}}x^*_{ij}=1$ and (29) follows. □

However, this property is not satisfied by any of the remaining CAC, as shown in the following example.

Example 6.2. Consider an instance with $n=4$, $p=2$ and $(d_{ij})=\begin{pmatrix}0&2&4&5\\2&0&6&7\\4&6&0&5\\5&7&5&0\end{pmatrix}$. An optimal solution of the problem

$$\begin{aligned}\min\quad & x_{12}+7x_{13}+5x_{14}+x_{21}+5x_{23}+x_{24}+\\ & 7x_{31}+5x_{32}+2x_{34}+5x_{41}+x_{42}+2x_{43}\\ \text{s.t.}\quad & (2)-(4),\\ & \mathcal{BLMN},\ \mathcal{M},\ \mathcal{M}',\mathcal{EMR},\ y_j\in\{0,1\}\ \forall j,\quad x_{ij}\geqslant 0\ \forall i,j\end{aligned}$$

is $\left(y^*_j\right)=(0,1,1,0),\ \left(x^*_{ij}\right)=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&0&1&0\\0&1/2&1/2&0\end{pmatrix}$. Here $x^*_{42}=x^*_{43}=1/2$ but $d_{42}=7\neq d_{43}=5$.

The part of the proof of Proposition 5.2 where it is proved that $\mathcal{RR}$ implies $\sum_{j\in A}x_{ij}\geqslant 1$ (replacing $\mathcal{RR}$ by $\mathcal{RR}_{\mathcal{EMR}}$) and Proposition 5.3 are valid in case of ties. Nevertheless, constraints $\mathcal{RR}_{\mathcal{EMR}}$ do not imply (3) as shown in the following counterexample.

Example 6.3. Consider an instance with $n=3$, $p=1$ and $(d_{ij})=\begin{pmatrix}0&1&1\\1&0&2\\1&2&0\end{pmatrix}$. An optimal solution of the problem

$$\begin{aligned}\min\quad & 4x_{12}+7x_{13}+4x_{21}+4x_{23}+7x_{31}+4x_{32}\\ \text{s.t.}\quad & (2),\ (4),\\ & \mathcal{RR}_{\mathcal{EMR}},\ y_j\in\{0,1\}\ \forall j,\quad x_{ij}\in\{0,1\}\forall i,j\end{aligned}$$

is $\left(y^*_j\right)=(0,0,1),\ \left(x^*_{ij}\right)=\begin{pmatrix}0&1&0\\0&0&1\\0&0&1\end{pmatrix}$, which violates constraint $x_{12}\leqslant y_2$.

7. Conclusions

We have analyzed and completely classified from the point of view of the dominance all the Closest Assignment Constraints in the literature of Discrete Location Theory. We have also proposed a new non dominated set of constraints and shown that some of the CAC present, mainly when there are no ties between distances from the same client, some additional good properties which make unnecessary either integrality of the x-variables or some sets of classical location constraints.

Some of the CAC imply the integrality of the x-variables, reducing the number of integer variables in the formulation by n^2, and some of them avoid the use of constraints $x_{ij}\leqslant y_j$, reducing the number of constraints in the formulation by n^2, making them also valuable.

As a summary of the results obtained in the paper, Table 3 reports, for every family of CAC, (i) which other families it dominates, (ii) which other families dominate it, (iii) whether a fixed number of plants is required to use this CAC (Y) or not (N), (iv) whether it can be used (Y) or not (N) in the presence of ties in some row of the distance matrix, (v) which classical location constraints are implied by the corresponding family of CAC –and could be relaxed— in the absence of ties (here "$\geqslant$" means $\sum_j x_{ij}\geqslant 1$), (vi) which classical location constraints are implied by the family of CAC in the presence of ties, and (vii) whether (Y) or not (N) constraints (6) can be relaxed (even if they are not implied by the CAC) because an integer optimal solution can be trivially derived from the fractional optimal solution of the relaxation (see Proposition 6.3). In the table, "NS" means that the question has no sense in this case.

The constraints which are not dominated by any other set of constraints are those presented in Cánovas et al. (2007), termed $\mathcal{CGLM}$, and the ones newly developed in this paper, termed $\mathcal{EMR}$. $\mathcal{CGLM}$ do not make use of the fixed number of plants and can be incorporated to models without this condition, whereas $\mathcal{EMR}$ can only be used under this hypothesis. Although $\mathcal{CGLM}$ has cardinality $\mathcal{O}(n^3)$, a subset of cardinality $\mathcal{O}(n^2)$ suffices to force closest assignment and still dominates the same subsets of constraints. Combining both sets of inequalities can be an adequate approach for solving CA discrete location problems with the p-median constraint. Designing a separation algorithm inside a branch-and-cut procedure, and computationally checking the improvement of this combination with respect to using each set of constraints separately, is a matter of future research.

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